

Research Article

Reciprocity and Representation Theorems for Flux- and Field-Normalised Decomposed Wave Fields

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We consider wave propagation problems in which there is a preferred direction of propagation. To account for propagation in preferred directions, the wave equation is decomposed into a set of coupled equations for waves that propagate in opposite directions along the preferred axis. This decomposition is not unique. We discuss flux-normalised and field-normalised decomposition in a systematic way, analyse the symmetry properties of the decomposition operators, and use these symmetry properties to derive reciprocity theorems for the decomposed wave fields, for both types of normalisation. Based on the field-normalised reciprocity theorems, we derive representation theorems for decomposed wave fields. In particular, we derive double- and single-sided Kirchhoff-Helmholtz integrals for forward and backward propagation of decomposed wave fields. The single-sided Kirchhoff-Helmholtz integrals for backward propagation of field-normalised decomposed wave fields find applications in reflection imaging, accounting for multiple scattering.

1. Introduction

In many wave propagation problems, it is possible to define a preferred direction of propagation. For example, in ocean acoustics, waves propagate primarily in the horizontal direction in an acoustic wave guide, bounded by the water surface and the ocean bottom. Similarly, in communication engineering, microwaves or optical waves propagate as beams through electromagnetic or optical wave guides. Wave propagation in preferred directions is not restricted to wave guides. For example, in geophysical reflection imaging applications, seismic or electromagnetic waves propagate mainly in the vertical direction (downward and upward) through a laterally unbounded medium.

To account for propagation in preferred directions, the wave equation for the full wave field can be decomposed into a set of coupled equations for waves that propagate in opposite directions along the preferred axis (for example, leftward and rightward in ocean acoustics or downward and upward in reflection imaging). In the literature on electromagnetic wave propagation, these oppositely propagating waves are often called “bidirectional beams” [1, 2] whereas in

the acoustic literature they are usually called “one-way wave fields” [3–7]. In this paper, we use the latter terminology.

There is a vast amount of literature on the analytical and numerical aspects of one-way wave propagation [8–13]. A discussion of this is beyond the scope of this paper. Instead, we concentrate on the choice of the decomposition operator and the consequences for reciprocity and representation theorems.

Decomposition of a wave field into one-way wave fields is not unique. In particular, the amplitudes of the one-way wave fields can be scaled in different ways. In this paper, we distinguish between the so-called “flux-normalised” and “field-normalised” one-way wave fields. The square of the amplitude of a flux-normalised one-way wave field is by definition the power-flux density (or, for quantum-mechanical waves, the probability-flux density) in the direction of preference. Field-normalised one-way wave fields, on the other hand, are scaled such that the sum of the two oppositely propagating components equals the full wave field. These two forms of normalisation have been briefly analysed by de Hoop [14, 15]. From this analysis, it appeared that the operators for flux-normalised decomposition exhibit more

symmetry than the operators for field-normalised decomposition. Exploiting the symmetry of the flux-normalised decomposition operators, the author derived reciprocity and representation theorems for flux-normalised one-way wave fields [16, 17].

The first aim of this paper is to discuss flux-normalised versus field-normalised decomposition in a systematic way. In particular, it will be shown that reciprocity theorems for field-normalised one-way wave fields can be derived in a similar way as those for flux-normalised one-way wave fields, even though the operators for field-normalised decomposition exhibit less symmetry.

The second aim is to discuss representation theorems for field-normalised one-way wave fields in a systematic way. This discussion includes links to “classical” Kirchhoff-Helmholtz integrals for one-way wave fields as well as to recent single-sided representations for backward propagation, used for example in Marchenko imaging [18]. Despite the links to earlier results, the discussed representations are more general. An advantage of the representations for field-normalised one-way wave fields is that a straightforward summation of the one-way wave fields gives the full wave field.

We restrict the discussion to scalar wave fields. In Section 2, we formulate a unified scalar wave equation for acoustic waves, horizontally polarised shear waves, transverse electric and transverse magnetic EM waves, and finally quantum-mechanical waves. Next, we reformulate the unified wave equation into a matrix-vector form, discuss symmetry properties of the operator matrix, and use this to derive reciprocity theorems in matrix-vector form. In Section 3, we decompose the matrix-vector wave equation into a coupled system of equations for oppositely propagating one-way wave fields. We separately consider flux normalisation and field normalisation and derive reciprocity theorems for one-way wave fields, using both normalisations. In Section 4, we extensively discuss representation theorems for field-normalised one-way wave fields and indicate applications. We end with conclusions in Section 5.

2. Unified Wave Equation and Its Symmetry Properties

2.1. Unified Scalar Wave Equation. Using a unified notation, wave propagation in a lossless medium (or, for quantum-mechanical waves, in a lossless potential) is governed by the following two equations in the space-frequency domain:

$$-i\omega\alpha P + \partial_j Q_j = B, \quad (1)$$

$$-i\omega\beta Q_j + \partial_j P = C_j. \quad (2)$$

Here, i is the imaginary unit and ω the angular frequency (in this paper, we consider positive frequencies only). Operator ∂_j stands for the spatial differential operator $\partial/\partial x_j$, and Einstein’s summation convention applies to repeated subscripts. $P(\mathbf{x}, \omega)$ and $Q_j(\mathbf{x}, \omega)$ are space- and frequency-dependent wave field quantities, $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$

TABLE 1: Quantities in unified equations (1) and (2).

	P	Q_j	α	β	B	C_j
Acoustic waves (3D)	p	v_j	κ	ρ	q	f_j
SH waves (2D)	v_2	$-\tau_{2j}$	ρ	$\frac{1}{\mu}$	f_2	$2h_{2j}$
TE waves (2D)	E_2	$-\epsilon_{2jk}H_k$	ϵ	μ	$-J_2^e$	$\epsilon_{2jk}J_k^m$
TM waves (2D)	H_2	$\epsilon_{2jk}E_k$	μ	ϵ	$-J_2^m$	$-\epsilon_{2jk}J_k^e$
Quantum waves (3D)	Ψ	$\frac{2\hbar}{mi}\partial_j\Psi$	$4 - \frac{4V}{\hbar\omega}$	$\frac{m}{2\hbar\omega}$		

are real-valued space-dependent parameters, and $B(\mathbf{x}, \omega)$ and $C_j(\mathbf{x}, \omega)$ are space- and frequency-dependent source distributions. Parameters α and β are both assumed to be positive; hence, metamaterials are not considered in this paper. All quantities are specified in Table 1 for different wave phenomena and are discussed in more detail below. As indicated in the first column of Table 1, we consider 3D and 2D wave problems. For the 3D situation, $\mathbf{x} = (x_1, x_2, x_3)$ is the 3D Cartesian coordinate vector and lowercase Latin subscripts take on the values 1, 2, and 3. For the 2D situation, $\mathbf{x} = (x_1, x_3)$ is the 2D Cartesian coordinate vector and lowercase Latin subscripts take on the values 1 and 3 only.

The unified boundary conditions at an interface between two media with different parameters state that P and $n_j Q_j$ are continuous over the interface. Here, n_j represents the components of the normal vector $\mathbf{n} = (n_1, n_2, n_3)$ at the interface for the 3D situation or $\mathbf{n} = (n_1, n_3)$ for the 2D situation.

We discuss the quantities in Table 1 in more detail. The quantities in row 1, associated to 3D acoustic wave propagation in a lossless fluid medium, are acoustic pressure $p(\mathbf{x}, \omega)$, particle velocity $v_j(\mathbf{x}, \omega)$, compressibility $\kappa(\mathbf{x})$, mass density $\rho(\mathbf{x})$, volume-injection rate density $q(\mathbf{x}, \omega)$, and external force density $f_j(\mathbf{x}, \omega)$. For 2D horizontally polarised shear waves in a lossless solid medium, we have in row 2 horizontal particle velocity $v_2(\mathbf{x}, \omega)$, shear stress $\tau_{2j}(\mathbf{x}, \omega)$, mass density $\rho(\mathbf{x})$, shear modulus $\mu(\mathbf{x})$, external horizontal force density $f_2(\mathbf{x}, \omega)$, and external shear deformation rate density $h_{2j}(\mathbf{x}, \omega)$. Rows 3 and 4 contain the quantities for 2D electromagnetic wave propagation, with TE standing for transverse electric and TM for transverse magnetic. The quantities are electric field strength $E_k(\mathbf{x}, \omega)$, magnetic field strength $H_k(\mathbf{x}, \omega)$, permittivity $\epsilon(\mathbf{x})$, permeability $\mu(\mathbf{x})$, external electric current density $J_k^e(\mathbf{x}, \omega)$, and external magnetic current density $J_k^m(\mathbf{x}, \omega)$. Furthermore, ϵ_{ijk} is the alternating tensor (or Levi-Civita tensor), with $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$, and all other components being zero. In row 5, the quantities related to 3D quantum-mechanical wave propagation are wave function $\Psi(\mathbf{x}, \omega)$, potential $V(\mathbf{x})$, particle mass m , and $\hbar = h/2\pi$, with h Planck’s constant.

By eliminating Q_j from equations (1) and (2), we obtain the unified scalar wave equation

$$\beta\partial_j\left(\frac{1}{\beta}\partial_j P\right) + k^2 P = \beta\partial_j\left(\frac{1}{\beta}C_j\right) + i\omega\beta B, \quad (3)$$

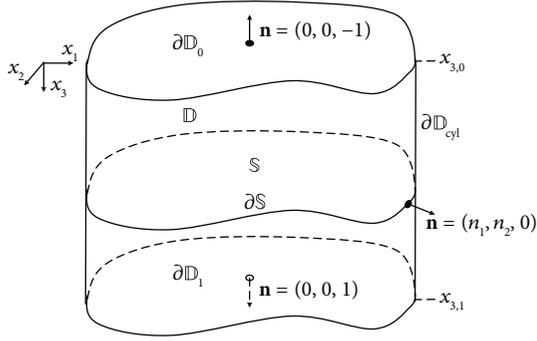


FIGURE 1: Configuration with the x_3 -direction as the preferred direction. In the lateral direction, this configuration can be bounded (for wave guides) or unbounded (for example, for geophysical reflection imaging applications). For the 2D situation, the configuration is a cross section of the 3D situation for $x_2 = 0$.

with wave number k defined via

$$k^2 = \alpha\beta\omega^2. \quad (4)$$

2.2. Unified Wave Equation in Matrix-Vector Form. We define a configuration with a preferred direction and reorganise equations (1) and (2) accordingly.

Consider a 3D spatial domain \mathbb{D} , enclosed by surface $\partial\mathbb{D}$. This surface consists of two planar surfaces $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_1$ perpendicular to the x_3 -axis and a cylindrical surface $\partial\mathbb{D}_{\text{cyl}}$ with its axis parallel to the x_3 -axis, see Figure 1. The surfaces $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_1$ are situated at $x_3 = x_{3,0}$ and $x_3 = x_{3,1}$, respectively, with $x_{3,1} > x_{3,0}$. In general, these surfaces do not coincide with physical boundaries. Surface \mathbb{S} in Figure 1 is a cross section of \mathbb{D} at arbitrary x_3 . The parameters $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are piecewise continuous smoothly varying functions in \mathbb{D} , with discontinuous jumps only at interfaces that are perpendicular to the x_3 -axis (hence, P and Q_3 are continuous over the interfaces). In the lateral direction, the domain \mathbb{D} can be bounded or unbounded. When \mathbb{D} is laterally bounded, the configuration in Figure 1 represents a wave guide. For this situation, we assume that homogeneous Dirichlet or Neumann boundary conditions apply, i.e., $P = Q_3 = 0$ or $n_\nu \partial_\nu P = n_\nu \partial_\nu Q_3 = 0$ at $\partial\mathbb{D}_{\text{cyl}}$, where lowercase Greek subscripts take on the values 1 and 2. When \mathbb{D} is laterally unbounded (for example, for reflection imaging applications), the cylindrical surface $\partial\mathbb{D}_{\text{cyl}}$ has an infinite radius and we assume that P and Q_3 have “sufficient decay” at infinity. For the 2D situation, the configuration is a cross section of the 3D situation for $x_2 = 0$ and lowercase Greek subscripts take on the value 1 only.

We reorganise equations (1) and (2) into a matrix-vector wave equation which acknowledges the x_3 -direction as the direction of preference. By eliminating the lateral components Q_1 and Q_2 (or, for 2D wave problems, the lateral component Q_1), we obtain [8, 15, 19–21]

$$\partial_3 \mathbf{q} = \mathcal{A} \mathbf{q} + \mathbf{d}, \quad (5)$$

where wave vector \mathbf{q} and source vector \mathbf{d} are defined as

$$\mathbf{q} = \begin{pmatrix} P \\ Q_3 \end{pmatrix}, \quad (6)$$

$$\mathbf{d} = \begin{pmatrix} C_3 \\ B_0 \end{pmatrix},$$

with

$$B_0 = B + \frac{1}{i\omega} \partial_\nu \frac{1}{\beta} C_\nu \quad (7)$$

and operator matrix \mathcal{A} defined as

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{A}_{12} \\ \mathcal{A}_{21} & 0 \end{pmatrix}, \quad (8)$$

with

$$\mathcal{A}_{12} = i\omega\beta, \quad (9)$$

$$\mathcal{A}_{21} = i\omega\alpha - \frac{1}{i\omega} \partial_\nu \frac{1}{\beta} \partial_\nu. \quad (10)$$

The notation in the right-hand side of equations (7) and (10) should be understood in the sense that differential operators act on all factors to the right of it. Hence, operator $\partial_\nu (1/\beta) \partial_\nu$, applied via equation (5) to P , stands for $\partial_\nu ((1/\beta) \partial_\nu P)$.

Note that the quantities contained in the wave vector \mathbf{q} are continuous over interfaces perpendicular to the x_3 -axis. Moreover, these quantities constitute the power-flux density (or, for quantum-mechanical waves, the probability-flux density) in the x_3 -direction via

$$j = \frac{1}{4} \{P^* Q_3 + Q_3^* P\}, \quad (11)$$

where the asterisk denotes complex conjugation.

2.3. Symmetry Properties of the Operator Matrix. We discuss the symmetry properties of the operator matrix \mathcal{A} . First, consider a general operator \mathcal{U} (which can be a scalar or a matrix), containing space-dependent parameters and differential operators ∂_ν . We introduce the transpose operator \mathcal{U}^t via the following integral property:

$$\int_{\mathbb{S}} (\mathcal{U}f)^t g \, d\mathbf{x}_L = \int_{\mathbb{S}} f^t (\mathcal{U}^t g) \, d\mathbf{x}_L. \quad (12)$$

Here, \mathbf{x}_L is the lateral coordinate vector, with $\mathbf{x}_L = (x_1, x_2)$ for 3D and $\mathbf{x}_L = x_1$ for 2D wave problems. \mathbb{S} denotes an integration surface perpendicular to the x_3 -axis at arbitrary x_3 , with edge $\partial\mathbb{S}$, see Figure 1. The quantities $f(\mathbf{x}_L)$ and $g(\mathbf{x}_L)$ are space-dependent test functions (scalars or vectors). When these functions are vectors, f^t is the transpose of f ; when they are scalars, f^t is equal to f . When \mathbb{S} is bounded,

homogeneous Dirichlet or Neumann conditions are imposed at $\partial\mathbb{S}$. When \mathbb{S} is unbounded, $\partial\mathbb{S}$ has an infinite radius and $f(\mathbf{x}_L)$ and $g(\mathbf{x}_L)$ are assumed to have sufficient decay along \mathbb{S} towards infinity. Operator \mathcal{U} is said to be symmetric when $\mathcal{U}^\dagger = \mathcal{U}$ and skew-symmetric when $\mathcal{U}^\dagger = -\mathcal{U}$. For the special case that $\mathcal{U} = \partial_\nu$, equation (12) implies $\partial_\nu^\dagger = -\partial_\nu$ (via integration by parts along \mathbb{S}). Hence, operator ∂_ν is skew-symmetric.

We introduce the adjoint operator \mathcal{U}^\dagger (i.e., the complex conjugate transpose of \mathcal{U}) via the integral property

$$\int_{\mathbb{S}} (\mathcal{U}f)^\dagger g \, d\mathbf{x}_L = \int_{\mathbb{S}} f^\dagger (\mathcal{U}^\dagger g) \, d\mathbf{x}_L. \quad (13)$$

When the test functions are vectors, f^\dagger is the complex conjugate transpose of f ; when they are scalars, f^\dagger is the complex conjugate of f . Operator \mathcal{U} is said to be Hermitian (or self-adjoint) when $\mathcal{U}^\dagger = \mathcal{U}$ and skew-Hermitian when $\mathcal{U}^\dagger = -\mathcal{U}$. For the operators \mathcal{A}_{12} and \mathcal{A}_{21} , defined in equations (9) and (10), we find $\mathcal{A}_{12}^\dagger = \mathcal{A}_{12}$, $\mathcal{A}_{21}^\dagger = \mathcal{A}_{21}$, $\mathcal{A}_{12}^\dagger = -\mathcal{A}_{12}$, and $\mathcal{A}_{21}^\dagger = -\mathcal{A}_{21}$. Hence, operators \mathcal{A}_{12} and \mathcal{A}_{21} are symmetric and skew-Hermitian. With these relations, we find for the operator matrix \mathcal{A}

$$\mathcal{A}^\dagger \mathbf{N} = -\mathbf{N} \mathcal{A}, \quad (14)$$

$$\mathcal{A}^\dagger \mathbf{K} = -\mathbf{K} \mathcal{A}, \quad (15)$$

with

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (16)$$

$$\mathbf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that, using the expressions for \mathbf{q} and \mathbf{K} in equations (6) and (16), we can rewrite equation (11) for the power-flux density (or, for quantum-mechanical waves, the probability-flux density) as

$$j = \frac{1}{4} \mathbf{q}^\dagger \mathbf{K} \mathbf{q}. \quad (17)$$

2.4. Reciprocity Theorems. We derive reciprocity theorems between two independent solutions of wave equation (5) for the configuration of Figure 1. We consider two states A and B , characterised by wave vectors $\mathbf{q}_A(\mathbf{x}, \omega)$ and $\mathbf{q}_B(\mathbf{x}, \omega)$, obeying wave equation (5), with source vectors $\mathbf{d}_A(\mathbf{x}, \omega)$ and $\mathbf{d}_B(\mathbf{x}, \omega)$. In domain \mathbb{D} , the parameters α and β , and hence the matrix operator \mathcal{A} , are chosen the same in the two states (outside $\partial\mathbb{D}$ they may be different in the two states). Consider the quantity $\partial_3(\mathbf{q}_A^\dagger \mathbf{N} \mathbf{q}_B)$ in domain \mathbb{D} . Applying the product rule for differentiation, using equation (5) for both states, integrating the result over \mathbb{D} and applying the theorem of Gauss yields

$$\int_{\mathbb{D}} \left(((\mathcal{A} \mathbf{q}_A)^\dagger + \mathbf{d}_A^\dagger) \mathbf{N} \mathbf{q}_B + \mathbf{q}_A^\dagger \mathbf{N} (\mathcal{A} \mathbf{q}_B + \mathbf{d}_B) \right) d\mathbf{x} \quad (18)$$

$$= \int_{\partial\mathbb{D}} \mathbf{q}_A^\dagger \mathbf{N} \mathbf{q}_B n_3 d\mathbf{x}.$$

Here, n_3 is the component parallel to the x_3 -axis of the outward pointing normal vector on $\partial\mathbb{D}$, with $n_3 = -1$ at $\partial\mathbb{D}_0$, $n_3 = +1$ at $\partial\mathbb{D}_1$, and $n_3 = 0$ at $\partial\mathbb{D}_{\text{cyl}}$, see Figure 1. In the following, the integral on the right-hand side is restricted to the horizontal surfaces $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_1$, which together are denoted by $\partial\mathbb{D}_{0,1}$. The integral on the left-hand side can be written as $\int_{\mathbb{D}} (\dots) d\mathbf{x} = \int_{x_{3,0}}^{x_{3,1}} dx_3 \int_{\mathbb{S}} (\dots) d\mathbf{x}_L$. Using equation (12) for the integral along \mathbb{S} and symmetry property (14), it follows that the two terms in equation (18) containing operator \mathcal{A} cancel each other. Hence, we are left with

$$\int_{\mathbb{D}} (\mathbf{d}_A^\dagger \mathbf{N} \mathbf{q}_B + \mathbf{q}_A^\dagger \mathbf{N} \mathbf{d}_B) d\mathbf{x} = \int_{\partial\mathbb{D}_{0,1}} \mathbf{q}_A^\dagger \mathbf{N} \mathbf{q}_B n_3 d\mathbf{x}_L. \quad (19)$$

This is a convolution-type reciprocity theorem [22–24], because products like $\mathbf{q}_A^\dagger(\mathbf{x}, \omega) \mathbf{N} \mathbf{q}_B(\mathbf{x}, \omega)$ in the frequency domain correspond to convolutions in the time domain. A more familiar form is obtained by substituting the expressions for \mathbf{q} , \mathbf{d} , and \mathbf{N} (equations (6) and (16)), choosing $C_j = 0$ and using equation (2) to eliminate Q_3 , which gives

$$\int_{\mathbb{D}} (-B_A P_B + P_A B_B) d\mathbf{x} = \int_{\partial\mathbb{D}_{0,1}} \frac{1}{i\omega\beta} (P_A \partial_3 P_B - (\partial_3 P_A) P_B) n_3 d\mathbf{x}_L. \quad (20)$$

Next, consider the quantity $\partial_3(\mathbf{q}_A^\dagger \mathbf{K} \mathbf{q}_B)$ in domain \mathbb{D} . Following the same steps as above, using equations (13) and (15) instead of (12) and (14), we obtain

$$\int_{\mathbb{D}} (\mathbf{d}_A^\dagger \mathbf{K} \mathbf{q}_B + \mathbf{q}_A^\dagger \mathbf{K} \mathbf{d}_B) d\mathbf{x} = \int_{\partial\mathbb{D}_{0,1}} \mathbf{q}_A^\dagger \mathbf{K} \mathbf{q}_B n_3 d\mathbf{x}_L. \quad (21)$$

This is a correlation-type reciprocity theorem [25], because products like $\mathbf{q}_A^\dagger(\mathbf{x}, \omega) \mathbf{K} \mathbf{q}_B(\mathbf{x}, \omega)$ in the frequency domain correspond to correlations in the time domain. Substituting the expressions for \mathbf{q} , \mathbf{d} , and \mathbf{K} and choosing $C_j = 0$ yield the more familiar form

$$\int_{\mathbb{D}} (B_A^* P_B + P_A^* B_B) d\mathbf{x} = \int_{\partial\mathbb{D}_{0,1}} \frac{1}{i\omega\beta} (P_A^* \partial_3 P_B - (\partial_3 P_A)^* P_B) n_3 d\mathbf{x}_L. \quad (22)$$

We obtain a special case by choosing states A and B identical. Dropping the subscripts A and B in equations (21) and (22) and multiplying both sides of these equations by $1/4$ give

$$\frac{1}{4} \int_{\mathbb{D}} (\mathbf{d}^\dagger \mathbf{K} \mathbf{q} + \mathbf{q}^\dagger \mathbf{K} \mathbf{d}) d\mathbf{x} = \frac{1}{4} \int_{\partial\mathbb{D}_{0,1}} \mathbf{q}^\dagger \mathbf{K} \mathbf{q} n_3 d\mathbf{x}_L, \quad (23)$$

$$\frac{1}{4} \int_{\mathbb{D}} (B^* P + P^* B) dx = \frac{1}{4} \int_{\partial \mathbb{D}_{0,1}} \frac{1}{i\omega\beta} (P^* \partial_3 P - (\partial_3 P)^* P) n_3 dx_L, \quad (24)$$

respectively. These equations quantify conservation of power (or, for quantum-mechanical waves, probability).

3. Decomposed Wave Equation and Its Symmetry Properties

3.1. General Decomposition of the Matrix-Vector Wave Equation. To facilitate the decomposition of the matrix-vector wave equation (equation (5)), we recast the operator matrix \mathcal{A} into a somewhat different form. To this end, we introduce an operator \mathcal{H}_2 , according to

$$\mathcal{H}_2 = -i\omega \sqrt{\beta} \mathcal{A}_{21} \sqrt{\beta} = k^2 + \sqrt{\beta} \partial_v \frac{1}{\beta} \partial_v \sqrt{\beta}, \quad (25)$$

with operator \mathcal{A}_{21} defined in equation (10) and wavenumber k in equation (4). Operator \mathcal{H}_2 can be rewritten as a Helmholtz operator [14, 21]

$$\mathcal{H}_2 = k_s^2 + \partial_v \partial_v, \quad (26)$$

with the scaled wavenumber k_s defined as [26]

$$k_s^2 = k^2 - \frac{3(\partial_v \beta)(\partial_v \beta)}{4\beta^2} + \frac{(\partial_v \partial_v \beta)}{2\beta}. \quad (27)$$

Note that $\mathcal{H}_2^\dagger = \mathcal{H}_2$ and $\mathcal{H}_2^\dagger = \mathcal{H}_2$; hence, operator \mathcal{H}_2 is symmetric and self-adjoint and its spectrum is real-valued (with positive and negative eigenvalues). Using equation (25), we rewrite operator matrix \mathcal{A} , defined in equation (8), as

$$\mathcal{A} = \begin{pmatrix} 0 & i\omega\beta \\ -\frac{1}{i\omega\sqrt{\beta}} \mathcal{H}_2 \frac{1}{\sqrt{\beta}} & 0 \end{pmatrix}. \quad (28)$$

Next, we decompose this operator matrix as follows

$$\mathcal{A} = \mathcal{L} \mathcal{H} \mathcal{L}^{-1}, \quad (29)$$

with

$$\mathcal{H} = \begin{pmatrix} i\mathcal{H}_1 & 0 \\ 0 & -i\mathcal{H}_1 \end{pmatrix}, \quad (30)$$

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_1 \\ \mathcal{L}_2 & -\mathcal{L}_2 \end{pmatrix}, \quad (31)$$

$$\mathcal{L}^{-1} = \frac{1}{2} \begin{pmatrix} \mathcal{L}_1^{-1} & \mathcal{L}_2^{-1} \\ \mathcal{L}_1^{-1} & -\mathcal{L}_2^{-1} \end{pmatrix}. \quad (32)$$

Operators \mathcal{H}_1 , \mathcal{L}_1 , and \mathcal{L}_2 are pseudodifferential operators [7, 8, 14, 16, 21, 27–30]. The decomposition expressed by equation (29) is not unique; hence, different choices for operators \mathcal{H}_1 , \mathcal{L}_1 , and \mathcal{L}_2 are possible. We discuss two of these choices in detail in the next two sections. Here, we derive some general relations that are independent of these choices.

By substituting equations (28), (30), (31), and (32) into equation (29), we obtain the following relations

$$\omega\beta = \mathcal{L}_1 \mathcal{H}_1 \mathcal{L}_2^{-1}, \quad (33)$$

$$\frac{1}{\omega\sqrt{\beta}} \mathcal{H}_2 \frac{1}{\sqrt{\beta}} = \mathcal{L}_2 \mathcal{H}_1 \mathcal{L}_1^{-1}. \quad (34)$$

We introduce a decomposed field vector \mathbf{p} and a decomposed source vector \mathbf{s} via

$$\mathbf{q} = \mathcal{L} \mathbf{p}, \quad \mathbf{p} = \mathcal{L}^{-1} \mathbf{q}, \quad (35)$$

$$\mathbf{d} = \mathcal{L} \mathbf{s}, \quad \mathbf{s} = \mathcal{L}^{-1} \mathbf{d}, \quad (36)$$

where

$$\mathbf{p} = \begin{pmatrix} P^+ \\ P^- \end{pmatrix}, \quad (37)$$

$$\mathbf{s} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}.$$

Substitution of equations (29), (35), and (36) into the matrix-vector wave equation (5) yields

$$\partial_3 \mathbf{p} = (\mathcal{H} - \mathcal{L}^{-1} \partial_3 \mathcal{L}) \mathbf{p} + \mathbf{s}. \quad (38)$$

Substituting equations (30), (31), (32), and (37) into equation (38) gives

$$\partial_3 \begin{pmatrix} P^+ \\ P^- \end{pmatrix} = \begin{pmatrix} i\mathcal{H}_1 & 0 \\ 0 & -i\mathcal{H}_1 \end{pmatrix} \begin{pmatrix} P^+ \\ P^- \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \mathcal{L}_1^{-1} & \mathcal{L}_2^{-1} \\ \mathcal{L}_1^{-1} & -\mathcal{L}_2^{-1} \end{pmatrix} \cdot \begin{pmatrix} \partial_3 \mathcal{L}_1 & \partial_3 \mathcal{L}_1 \\ \partial_3 \mathcal{L}_2 & -\partial_3 \mathcal{L}_2 \end{pmatrix} \begin{pmatrix} P^+ \\ P^- \end{pmatrix} + \begin{pmatrix} S^+ \\ S^- \end{pmatrix}. \quad (39)$$

This is a system of coupled one-way wave equations. From the first term on the right-hand side, it follows that the one-way wave fields P^+ and P^- propagate in the positive and negative x_3 -direction, respectively. The second term on the right-hand side accounts for coupling between P^+ and P^- . The last term on the right-hand side contains sources S^+ and S^- which emit waves in the positive and negative x_3 -direction, respectively.

We conclude this section by substituting equations (35) and (36) into equations (19), (21), and (23). Using equations (12) and (13) for the integration along the lateral coordinates, this yields

$$\begin{aligned} & \int_{\mathbb{D}} (\mathbf{s}_A^t \mathcal{L}^t \mathbf{N} \mathcal{L} \mathbf{p}_B + \mathbf{p}_A^t \mathcal{L}^t \mathbf{N} \mathcal{L} \mathbf{s}_B) \, \mathbf{d}\mathbf{x} \\ &= \int_{\partial \mathbb{D}_{0,1}} \mathbf{p}_A^t \mathcal{L}^t \mathbf{N} \mathcal{L} \mathbf{p}_B n_3 \, \mathbf{d}\mathbf{x}_L, \end{aligned} \quad (40)$$

$$\begin{aligned} & \int_{\mathbb{D}} (\mathbf{s}_A^\dagger \mathcal{L}^\dagger \mathbf{K} \mathcal{L} \mathbf{p}_B + \mathbf{p}_A^\dagger \mathcal{L}^\dagger \mathbf{K} \mathcal{L} \mathbf{s}_B) \, \mathbf{d}\mathbf{x} \\ &= \int_{\partial \mathbb{D}_{0,1}} \mathbf{p}_A^\dagger \mathcal{L}^\dagger \mathbf{K} \mathcal{L} \mathbf{p}_B n_3 \, \mathbf{d}\mathbf{x}_L, \end{aligned} \quad (41)$$

$$\begin{aligned} & \frac{1}{4} \int_{\mathbb{D}} (\mathbf{s}^\dagger \mathcal{L}^\dagger \mathbf{K} \mathcal{L} \mathbf{p} + \mathbf{p}^\dagger \mathcal{L}^\dagger \mathbf{K} \mathcal{L} \mathbf{s}) \, \mathbf{d}\mathbf{x} \\ &= \frac{1}{4} \int_{\partial \mathbb{D}_{0,1}} \mathbf{p}^\dagger \mathcal{L}^\dagger \mathbf{K} \mathcal{L} \mathbf{p} n_3 \, \mathbf{d}\mathbf{x}_L. \end{aligned} \quad (42)$$

These equations form the basis for reciprocity theorems for the decomposed field and source vectors \mathbf{p} and \mathbf{s} in the next two sections.

3.2. Flux-Normalised Decomposition and Reciprocity Theorems. The first choice of operators \mathcal{H}_1 , \mathcal{L}_1 , and \mathcal{L}_2 obeying equations (33) and (34) is [14–16]

$$\mathcal{H}_1 = \mathcal{H}_2^{1/2}, \quad (43)$$

$$\mathcal{L}_1 = (\omega/2)^{1/2} \beta^{1/2} \mathcal{H}_1^{-1/2}, \quad (44)$$

$$\mathcal{L}_2 = (2\omega)^{-1/2} \beta^{-1/2} \mathcal{H}_1^{1/2}. \quad (45)$$

Operator \mathcal{H}_1 , which is the square root of the Helmholtz operator \mathcal{H}_2 , is commonly known as the square root operator [3, 4, 8]. Like the Helmholtz operator \mathcal{H}_2 , the square root operator \mathcal{H}_1 is a symmetric operator [16], hence $\mathcal{H}_1^t = \mathcal{H}_1$. For the adjoint square root operator, we have $\mathcal{H}_1^\dagger = (\mathcal{H}_1^t)^* = \mathcal{H}_1^*$. The spectrum of \mathcal{H}_1 is real-valued for propagating waves and imaginary-valued for evanescent waves. Hence, unlike the Helmholtz operator, the square root operator is not self-adjoint. If we neglect evanescent waves, we may approximate the adjoint square root operator as $\mathcal{H}_1^\dagger \approx \mathcal{H}_1$. Similar relations hold for the square root of the square root operator and its inverse; hence, $(\mathcal{H}_1^{\pm 1/2})^t = \mathcal{H}_1^{\pm 1/2}$, and neglecting evanescent waves, $(\mathcal{H}_1^{\pm 1/2})^\dagger \approx \mathcal{H}_1^{\pm 1/2}$. From here onward, we replace \approx by $=$ when the only approximation is the negligence of evanescent waves. Using these symmetry relations for \mathcal{H}_1 and equations (16), (31), (44), and (45), we obtain

$$\mathcal{L}^t \mathbf{N} \mathcal{L} = -\mathbf{N}, \quad (46)$$

and neglecting evanescent waves,

$$\mathcal{L}^\dagger \mathbf{K} \mathcal{L} = \mathbf{J}, \quad (47)$$

with

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (48)$$

Hence, equations (40), (41), and (42) simplify to

$$\begin{aligned} & - \int_{\mathbb{D}} (\mathbf{s}_A^t \mathbf{N} \mathbf{p}_B + \mathbf{p}_A^t \mathbf{N} \mathbf{s}_B) \, \mathbf{d}\mathbf{x} = - \int_{\partial \mathbb{D}_{0,1}} \mathbf{p}_A^t \mathbf{N} \mathbf{p}_B n_3 \, \mathbf{d}\mathbf{x}_L, \\ & \int_{\mathbb{D}} (\mathbf{s}_A^\dagger \mathbf{J} \mathbf{p}_B + \mathbf{p}_A^\dagger \mathbf{J} \mathbf{s}_B) \, \mathbf{d}\mathbf{x} = \int_{\partial \mathbb{D}_{0,1}} \mathbf{p}_A^\dagger \mathbf{J} \mathbf{p}_B n_3 \, \mathbf{d}\mathbf{x}_L, \\ & \frac{1}{4} \int_{\mathbb{D}} (\mathbf{s}^\dagger \mathbf{J} \mathbf{p} + \mathbf{p}^\dagger \mathbf{J} \mathbf{s}) \, \mathbf{d}\mathbf{x} = \frac{1}{4} \int_{\partial \mathbb{D}_{0,1}} \mathbf{p}^\dagger \mathbf{J} \mathbf{p} n_3 \, \mathbf{d}\mathbf{x}_L. \end{aligned} \quad (49)$$

By substituting the expressions for \mathbf{p} , \mathbf{s} , \mathbf{N} , and \mathbf{J} (equations (37), (16), and (48)), we obtain

$$\begin{aligned} & \int_{\mathbb{D}} (-S_A^+ P_B^- + S_A^- P_B^+ - P_A^+ S_B^- + P_A^- S_B^+) \, \mathbf{d}\mathbf{x} \\ &= \int_{\partial \mathbb{D}_{0,1}} (-P_A^+ P_B^- + P_A^- P_B^+) n_3 \, \mathbf{d}\mathbf{x}_L, \end{aligned} \quad (50)$$

$$\begin{aligned} & \int_{\mathbb{D}} (S_A^{+*} P_B^+ - S_A^{-*} P_B^- + P_A^{+*} S_B^+ - P_A^{-*} S_B^-) \, \mathbf{d}\mathbf{x} \\ &= \int_{\partial \mathbb{D}_{0,1}} (P_A^{+*} P_B^+ - P_A^{-*} P_B^-) n_3 \, \mathbf{d}\mathbf{x}_L, \end{aligned} \quad (51)$$

$$\begin{aligned} & \frac{1}{4} \int_{\mathbb{D}} (S^{+*} P^+ - S^{-*} P^- + P^{+*} S^+ - P^{-*} S^-) \, \mathbf{d}\mathbf{x} \\ &= \frac{1}{4} \int_{\partial \mathbb{D}_{0,1}} (|P^+|^2 - |P^-|^2) n_3 \, \mathbf{d}\mathbf{x}_L. \end{aligned} \quad (52)$$

Note that, since the right-hand side of equation (52) is equal to the right-hand side of equation (24), it quantifies the power flux (or the probability flux for quantum-mechanical waves) through the surface $\partial \mathbb{D}_{0,1}$. Therefore, we call P^+ and P^- flux-normalised one-way wave fields. Consequently, equations (50) and (51) are reciprocity theorems of the convolution type and correlation type, respectively, for flux-normalised one-way wave fields. These theorems have been derived previously [16] and have found applications in advanced wave field imaging methods for active and passive data [31–42].

3.3. Field-Normalised Decomposition and Reciprocity Theorems. The second choice of operators \mathcal{H}_1 , \mathcal{L}_1 , and \mathcal{L}_2 obeying equations (33) and (34) is [21]

$$\mathcal{H}_1 = \beta^{1/2} \mathcal{H}_2^{1/2} \beta^{-1/2}, \quad (53)$$

$$\mathcal{L}_1 = 1, \quad (54)$$

$$\mathcal{L}_2 = (\omega\beta)^{-1} \mathcal{H}_1. \quad (55)$$

Only the Helmholtz operator \mathcal{H}_2 is the same as in the previous section (it is defined in equation (26)). The

operators \mathcal{H}_1 , \mathcal{L}_1 , and \mathcal{L}_2 are different from those in the previous section, but for convenience, we use the same symbols. Using $\mathbf{q} = \mathcal{L}\mathbf{p}$ (equation (35)) and equations (6), (31), (37), and (54), we find

$$P = P^+ + P^-. \quad (56)$$

Hence, P^+ and P^- have the same physical dimension as the full field variable P (which is defined in Table 1 for different wave phenomena). Therefore, we call P^+ and P^- field-normalised one-way wave fields (for convenience, we use the same symbols as in the previous section).

The square root operator $\mathcal{H}_2^{1/2}$ is symmetric, but \mathcal{H}_1 defined in equation (53) is not. From this equation, it easily follows that \mathcal{H}_1 premultiplied by β^{-1} is symmetric, hence

$$\left(\frac{1}{\beta}\mathcal{H}_1\right)^t = \frac{1}{\beta}\mathcal{H}_1, \quad (57)$$

and neglecting evanescent waves,

$$\left(\frac{1}{\beta}\mathcal{H}_1\right)^\dagger = \frac{1}{\beta}\mathcal{H}_1. \quad (58)$$

Using these symmetry relations for $(1/\beta)\mathcal{H}_1$ and equations (16), (31), (54), and (55), we obtain

$$\begin{aligned} \mathcal{L}^t \mathbf{N} \mathcal{L} &= \begin{pmatrix} 0 & -2\mathcal{L}_2 \\ 2\mathcal{L}_2 & 0 \end{pmatrix} = -\mathbf{N} \left(\frac{2}{\omega\beta}\mathcal{H}_1\right) \\ &= -\left(\frac{2}{\omega\beta}\mathcal{H}_1\right)^t \mathbf{N}, \end{aligned} \quad (59)$$

and neglecting evanescent waves,

$$\begin{aligned} \mathcal{L}^\dagger \mathbf{K} \mathcal{L} &= \begin{pmatrix} 2\mathcal{L}_2 & 0 \\ 0 & -2\mathcal{L}_2 \end{pmatrix} = \mathbf{J} \left(\frac{2}{\omega\beta}\mathcal{H}_1\right) \\ &= \left(\frac{2}{\omega\beta}\mathcal{H}_1\right)^\dagger \mathbf{J}. \end{aligned} \quad (60)$$

Using this in equations (40) and (41) yields

$$\begin{aligned} &-\int_{\mathbb{D}} \left[\mathbf{s}_A^t \left(\frac{2}{\omega\beta}\mathcal{H}_1\right)^t \mathbf{N} \mathbf{p}_B + \mathbf{p}_A^t \mathbf{N} \left(\frac{2}{\omega\beta}\mathcal{H}_1\right) \mathbf{s}_B \right] \mathbf{d}\mathbf{x} \\ &= -\int_{\partial\mathbb{D}_{0,1}} \mathbf{p}_A^t \left(\frac{2}{\omega\beta}\mathcal{H}_1\right)^t \mathbf{N} \mathbf{p}_B n_3 \mathbf{d}\mathbf{x}_L, \\ &\int_{\mathbb{D}} \left[\mathbf{s}_A^\dagger \left(\frac{2}{\omega\beta}\mathcal{H}_1\right)^\dagger \mathbf{J} \mathbf{p}_B + \mathbf{p}_A^\dagger \mathbf{J} \left(\frac{2}{\omega\beta}\mathcal{H}_1\right) \mathbf{s}_B \right] \mathbf{d}\mathbf{x} \\ &= \int_{\partial\mathbb{D}_{0,1}} \mathbf{p}_A^\dagger \left(\frac{2}{\omega\beta}\mathcal{H}_1\right)^\dagger \mathbf{J} \mathbf{p}_B n_3 \mathbf{d}\mathbf{x}_L. \end{aligned} \quad (61)$$

By substituting the expressions for \mathbf{p} , \mathbf{s} , \mathbf{N} , and \mathbf{J} (equations (37), (16), and (48)), using equations (12) and (13), we obtain

$$\begin{aligned} &-\int_{\mathbb{D}} \frac{2}{\omega\beta} \left((\mathcal{H}_1 S_A^+) P_B^- - (\mathcal{H}_1 S_A^-) P_B^+ \right. \\ &\quad \left. + P_A^+ (\mathcal{H}_1 S_B^-) - P_A^- (\mathcal{H}_1 S_B^+) \right) \mathbf{d}\mathbf{x} \\ &= -\int_{\partial\mathbb{D}_{0,1}} \frac{2}{\omega\beta} \left((\mathcal{H}_1 P_A^+) P_B^- - (\mathcal{H}_1 P_A^-) P_B^+ \right) n_3 \mathbf{d}\mathbf{x}_L, \end{aligned} \quad (62)$$

$$\begin{aligned} &\int_{\mathbb{D}} \frac{2}{\omega\beta} \left((\mathcal{H}_1 S_A^+) P_B^- - (\mathcal{H}_1 S_A^-) P_B^+ \right. \\ &\quad \left. + P_A^+ (\mathcal{H}_1 S_B^-) - P_A^- (\mathcal{H}_1 S_B^+) \right) \mathbf{d}\mathbf{x} \\ &= \int_{\partial\mathbb{D}_{0,1}} \frac{2}{\omega\beta} \left((\mathcal{H}_1 P_A^+) P_B^- - (\mathcal{H}_1 P_A^-) P_B^+ \right) n_3 \mathbf{d}\mathbf{x}_L. \end{aligned} \quad (63)$$

We aim to remove the operator \mathcal{H}_1 from these equations. From equations (39) and (54), we obtain

$$\partial_3 P^+ = +i\mathcal{H}_1 P^+ - \frac{1}{2} (\mathcal{L}_2^{-1} \partial_3 \mathcal{L}_2) (P^+ - P^-) + S^+, \quad (64)$$

$$\partial_3 P^- = -i\mathcal{H}_1 P^- + \frac{1}{2} (\mathcal{L}_2^{-1} \partial_3 \mathcal{L}_2) (P^+ - P^-) + S^-, \quad (65)$$

with \mathcal{L}_2 defined in equation (55). Assuming that in state A the derivatives in the x_3 -direction of the parameters α and β at $\partial\mathbb{D}_{0,1}$ vanish and there are no sources at $\partial\mathbb{D}_{0,1}$, we find from equations (64) and (65)

$$\partial_3 P_A^\pm = \pm i\mathcal{H}_1 P_A^\pm \text{ at } \partial\mathbb{D}_{0,1}. \quad (66)$$

Below we use this to remove \mathcal{H}_1 from the right-hand sides of equations (62) and (63). Next, we aim to remove \mathcal{H}_1 from the left-hand sides of these equations. From $\mathbf{s} = \mathcal{L}^{-1} \mathbf{d}$ (equation (36)) and equations (6), (32), (37), (54), and (55), we find

$$S^\pm = \pm \frac{1}{2} \left(\frac{1}{\omega\beta}\mathcal{H}_1\right)^{-1} B_0 + \frac{1}{2} C_3, \quad (67)$$

or

$$\pm \frac{2}{\omega\beta} \mathcal{H}_1 S^\pm = B_0 \pm \frac{1}{\omega\beta} \mathcal{H}_1 C_3. \quad (68)$$

We define new decomposed sources B_0^+ and B_0^- , according to

$$B_0^\pm = B_0 \pm \frac{1}{\omega\beta} \mathcal{H}_1 C_3 = \pm \frac{2}{\omega\beta} \mathcal{H}_1 S^\pm. \quad (69)$$

Using equations (66) and (69) in the right- and left-hand sides of equations (62) and (63), we obtain

$$\int_{\mathbb{D}} (-B_{0,A}^+ P_B^- - B_{0,A}^- P_B^+ + P_A^+ B_{0,B}^- + P_A^- B_{0,B}^+) d\mathbf{x} \\ = \int_{\partial\mathbb{D}_{0,1}} \frac{-2}{i\omega\beta} ((\partial_3 P_A^+) P_B^- + (\partial_3 P_A^-) P_B^+) n_3 d\mathbf{x}_L, \quad (70)$$

$$\int_{\mathbb{D}} (B_{0,A}^{+*} P_B^+ + B_{0,A}^{-*} P_B^- + P_A^{+*} B_{0,B}^+ + P_A^{-*} B_{0,B}^-) d\mathbf{x} \\ = \int_{\partial\mathbb{D}_{0,1}} \frac{-2}{i\omega\beta} ((\partial_3 P_A^+)^* P_B^+ + (\partial_3 P_A^-)^* P_B^-) n_3 d\mathbf{x}_L. \quad (71)$$

Equations (70) and (71) are reciprocity theorems of the convolution type and correlation type, respectively, for field-normalised one-way wave fields. These theorems are modifications of previously obtained results [43, 44]. The main modification is that we applied decomposition at both sides of the equations instead of at the right-hand sides only. Moreover, in the present derivation, the condition for the validity of equation (66) is only imposed for state A. In the next section, we use equations (70) and (71) to derive representation theorems for field-normalised one-way wave fields and we indicate applications.

4. Field-Normalised Representation Theorems

4.1. Green's Functions. Representation theorems are obtained by substituting Green's functions in reciprocity theorems. Our aim is to introduce one-way Green's functions, to be used in the reciprocity theorems for field-normalised one-way wave fields (equations (70) and (71)). First, we introduce the full Green's function $G(\mathbf{x}, \mathbf{x}_A, \omega)$ as a solution of the unified wave equation (3) for a unit monopole point source at \mathbf{x}_A , with $B(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_A)$ and $C_j(\mathbf{x}, \omega) = 0$. Hence,

$$\beta \partial_j \left(\frac{1}{\beta} \partial_j G \right) + k^2 G = i\omega \beta \delta(\mathbf{x} - \mathbf{x}_A). \quad (72)$$

As boundary condition, we impose the radiation condition (i.e., outward propagating waves at infinity). Next, we introduce one-way Green's function as solutions of the coupled one-way equations (64) and (65) for a unit monopole point source at \mathbf{x}_A . Hence, we choose again $B(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_A)$ and $C_j(\mathbf{x}, \omega) = 0$. Using equations (69) and (7), we define decomposed sources as $B_0^\pm = B^\pm = \pm 2\mathcal{L}_2 S^\pm$, with \mathcal{L}_2 defined in equation (55), or

$$S^\pm(\mathbf{x}, \omega) = \pm \frac{1}{2} \mathcal{L}_2^{-1} B^\pm(\mathbf{x}, \omega) = \pm \frac{1}{2} \mathcal{L}_2^{-1} B(\mathbf{x}, \omega) \\ = \pm \frac{1}{2} \mathcal{L}_2^{-1} \delta(\mathbf{x} - \mathbf{x}_A). \quad (73)$$

We consider two sets of one-way Green's functions. For the first set, we choose a point source $S^+(\mathbf{x}, \omega) = (1/2)\mathcal{L}_2^{-1} B^+(\mathbf{x}, \omega)$, with $B^+(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_A)$, which emits waves from \mathbf{x}_A in the positive x_3 -direction, and we set $S^-(\mathbf{x}, \omega)$

equal to zero. Hence, for this first set, one-way equations (64) and (65) become

$$\partial_3 G^{+,+} = +i\mathcal{H}_1 G^{+,+} - \frac{1}{2} (\mathcal{L}_2^{-1} \partial_3 \mathcal{L}_2) (G^{+,+} - G^{-,+}) \\ + \frac{1}{2} \mathcal{L}_2^{-1} \delta(\mathbf{x} - \mathbf{x}_A), \quad (74)$$

$$\partial_3 G^{-,+} = -i\mathcal{H}_1 G^{-,+} + \frac{1}{2} (\mathcal{L}_2^{-1} \partial_3 \mathcal{L}_2) (G^{+,+} - G^{-,+}). \quad (75)$$

Here, $G^{\pm,+}$ stands for $G^{\pm,+}(\mathbf{x}, \mathbf{x}_A, \omega)$. The second superscript (+) indicates that the source at \mathbf{x}_A emits waves in the positive x_3 -direction. The first superscript (\pm) denotes the propagation direction at \mathbf{x} . For the second set of one-way Green's functions, we choose a point source $S^-(\mathbf{x}, \omega) = -(1/2)\mathcal{L}_2^{-1} B^-(\mathbf{x}, \omega)$, with $B^-(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_A)$, which emits waves from \mathbf{x}_A in the negative x_3 -direction, and we set $S^+(\mathbf{x}, \omega)$ equal to zero. Hence, for this second set, one-way equations (64) and (65) become

$$\partial_3 G^{+,-} = +i\mathcal{H}_1 G^{+,-} - \frac{1}{2} (\mathcal{L}_2^{-1} \partial_3 \mathcal{L}_2) (G^{+,-} - G^{-,-}), \quad (76)$$

$$\partial_3 G^{-,-} = -i\mathcal{H}_1 G^{-,-} + \frac{1}{2} (\mathcal{L}_2^{-1} \partial_3 \mathcal{L}_2) (G^{+,-} - G^{-,-}) \\ - \frac{1}{2} \mathcal{L}_2^{-1} \delta(\mathbf{x} - \mathbf{x}_A). \quad (77)$$

Here, $G^{\pm,-}$ stands for $G^{\pm,-}(\mathbf{x}, \mathbf{x}_A, \omega)$, with the second superscript (-) indicating that the source at \mathbf{x}_A emits waves in the negative x_3 -direction. Like for the full Green's function $G(\mathbf{x}, \mathbf{x}_A, \omega)$, we impose radiation conditions for both sets of one-way Green's functions.

To find a relation between the full Green's function and the one-way Green's functions, we evaluate $\beta \partial_3 (1/\beta) \partial_3 (G^{+,+} + G^{-,+} + G^{+,-} + G^{-,-})$ using equations (74), (75), (76), (77), (25), (53), and (55). This gives equation (72), with G replaced by $G^{+,+} + G^{-,+} + G^{+,-} + G^{-,-}$. Since the full Green's function and the one-way Green's functions obey the same radiation conditions, we thus find

$$G = G^{+,+} + G^{-,+} + G^{+,-} + G^{-,-}. \quad (78)$$

This very simple relation is a consequence of the field-normalised decomposition, introduced in Section 3.3.

4.2. Source-Receiver Reciprocity. We derive source-receiver reciprocity relations for the field-normalised one-way Green's functions introduced in the previous section. To this end, we make use of the reciprocity theorem of the convolution type for field-normalised one-way wave fields (equation (70)). This theorem was derived for the configuration of Figure 1, assuming that in domain \mathbb{D} , the parameters α and β are the same in the two states (see Section 2.4). Outside \mathbb{D} , these parameters may be different in the two states. For the Green's state, we choose the parameters for $x_3 \leq x_{3,0}$ and for $x_3 \geq x_{3,1}$ independent of the x_3 -coordinate, according to $\alpha(\mathbf{x}_L)$ and

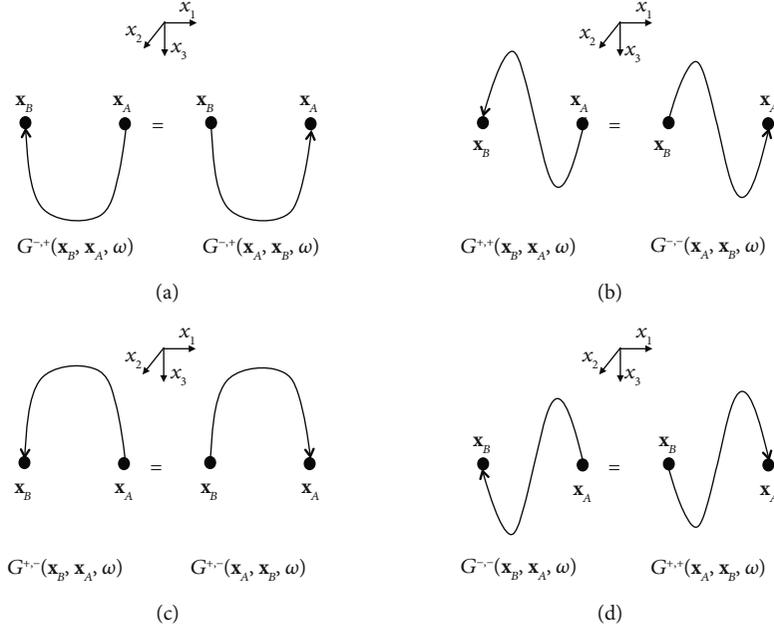


FIGURE 2: Visualisation of the source-receiver reciprocity relations for the field-normalised one-way Green's functions, formulated by equations (80), (81), (82), and (83). The “rays” in this and subsequent figures are strong simplifications of the complete one-way wave fields, which include primary and multiple scattering.

$\beta(\mathbf{x}_1)$. Hence, if we let Green's state (with a point source at \mathbf{x}_A in \mathbb{D}) take the role of state A, then the condition for the validity of equation (66) is fulfilled. Moreover, Green's functions are purely outward propagating at $\partial\mathbb{D}_{0,1}$ (because outside \mathbb{D} no scattering occurs along the x_3 -coordinate). Hence, $G^{+,\pm}(\mathbf{x}, \mathbf{x}_A, \omega) = 0$ at $\partial\mathbb{D}_0$ and $G^{-,\pm}(\mathbf{x}, \mathbf{x}_A, \omega) = 0$ at $\partial\mathbb{D}_1$. We let a second Green's state (with a point source at \mathbf{x}_B in \mathbb{D} and the same parameters α and β as in state A, inside as well as outside \mathbb{D}) take the role of state B. Hence, $G^{+,\pm}(\mathbf{x}, \mathbf{x}_B, \omega) = 0$ at $\partial\mathbb{D}_0$ and $G^{-,\pm}(\mathbf{x}, \mathbf{x}_B, \omega) = 0$ at $\partial\mathbb{D}_1$. With only outward propagating waves at $\partial\mathbb{D}_{0,1}$, the surface integral on the right-hand side of equation (70) vanishes. Hence, taking into account that $B_0^\pm = B^\pm$ (since $C_j = 0$), equation (70) simplifies to

$$\int_{\mathbb{D}} (-B_A^+ P_B^- - B_A^- P_B^+ + P_A^+ B_B^- + P_A^- B_B^+) d\mathbf{x} = 0. \quad (79)$$

First, we consider sources emitting waves in the positive x_3 -direction in both Green's states, hence $B_A^+ = \delta(\mathbf{x} - \mathbf{x}_A)$, $B_A^- = 0$, $P_A^+ = G^{\pm,+}(\mathbf{x}, \mathbf{x}_A, \omega)$, $B_B^+ = \delta(\mathbf{x} - \mathbf{x}_B)$, $B_B^- = 0$, and $P_B^\pm = G^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega)$. Substituting this into equation (79) yields

$$G^{-,+}(\mathbf{x}_B, \mathbf{x}_A, \omega) = G^{-,+}(\mathbf{x}_A, \mathbf{x}_B, \omega), \quad (80)$$

see Figure 2(a). Next, we replace the source in state B by one emitting waves in the negative x_3 -direction, hence $B_B^+ = 0$, $B_B^- = \delta(\mathbf{x} - \mathbf{x}_B)$, and $P_B^\pm = G^{\pm,-}(\mathbf{x}, \mathbf{x}_B, \omega)$. This gives

$$G^{+,-}(\mathbf{x}_B, \mathbf{x}_A, \omega) = G^{-,+}(\mathbf{x}_A, \mathbf{x}_B, \omega), \quad (81)$$

see Figure 2(b). By replacing also the source in state A by one emitting waves in the negative x_3 -direction, according to $B_A^+ = 0$, $B_A^- = \delta(\mathbf{x} - \mathbf{x}_A)$, and $P_A^\pm = G^{\pm,-}(\mathbf{x}, \mathbf{x}_A, \omega)$, we obtain

$$G^{+,-}(\mathbf{x}_B, \mathbf{x}_A, \omega) = G^{+,-}(\mathbf{x}_A, \mathbf{x}_B, \omega), \quad (82)$$

see Figure 2(c). Finally, changing the source in state B back to the one emitting waves in the positive x_3 -direction yields

$$G^{-,+}(\mathbf{x}_B, \mathbf{x}_A, \omega) = G^{+,-}(\mathbf{x}_A, \mathbf{x}_B, \omega), \quad (83)$$

see Figure 2(d).

Source-receiver reciprocity relations similar to equations (80), (81), (82), and (83) were previously derived for flux-normalised one-way Green's functions [17], except that two of those relations involve a change of sign when interchanging the source and the receiver. The absence of sign changes in equations (80), (81), (82), and (83) is due to the definition of B_0^\pm in equation (69). Moreover, unlike the flux-normalised reciprocity relations, the field-normalised source-receiver reciprocity relations of equations (80), (81), (82), and (83) have a very straightforward relation with the well-known source-receiver reciprocity relation for the full Green's function. By separately summing the left- and right-hand sides of equations (80), (81), (82), and (83) and using equation (78), we simply obtain

$$G(\mathbf{x}_B, \mathbf{x}_A, \omega) = G(\mathbf{x}_A, \mathbf{x}_B, \omega). \quad (84)$$

4.3. Kirchhoff-Helmholtz Integrals for Forward Propagation. We derive Kirchhoff-Helmholtz integrals of the convolution type for field-normalised one-way wave fields. For state B, we

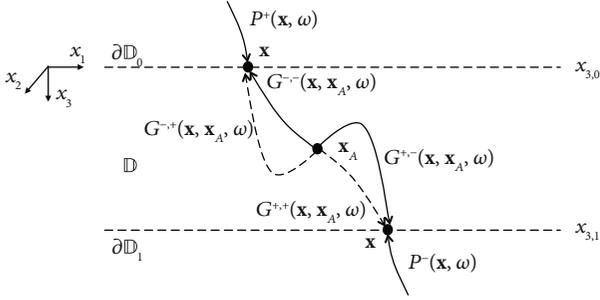


FIGURE 3: Visualisation of the different terms in the field-normalised one-way Kirchhoff-Helmholtz integral for forward propagation, formulated by equation (87). The solid Green's functions contribute to $P^+(\mathbf{x}_A, \omega)$, the dashed Green's functions to $P^-(\mathbf{x}_A, \omega)$.

consider the decomposed actual field, with sources only outside \mathbb{D} ; hence, $B_{0,B}^\pm = 0$ in \mathbb{D} and $P_B^\pm = P^\pm(\mathbf{x}, \omega)$. The parameters α and β are the actual parameters inside as well as outside \mathbb{D} . For state A, we choose the Green's state with a unit point source at \mathbf{x}_A in \mathbb{D} . The parameters α and β in \mathbb{D} are the same as those in state B, but for $x_3 \leq x_{3,0}$ and for $x_3 \geq x_{3,1}$, they are chosen independent of the x_3 -coordinate. Hence, the condition for the validity of equation (66) is fulfilled. First, we consider a source in state A which emits waves in the positive x_3 -direction, hence $B_A^+ = \delta(\mathbf{x} - \mathbf{x}_A)$, $B_A^- = 0$, and $P_A^\pm = G^{\pm,+}(\mathbf{x}, \mathbf{x}_A, \omega)$. Substituting all this into equation (70) (with $B_{0,A}^\pm = B_A^\pm$) gives

$$P^-(\mathbf{x}_A, \omega) = \int_{\partial\mathbb{D}_{0,1}} \frac{2}{i\omega\beta(\mathbf{x})} \left((\partial_3 G^{+,+}(\mathbf{x}, \mathbf{x}_A, \omega)) P^-(\mathbf{x}, \omega) + (\partial_3 G^{-,+}(\mathbf{x}, \mathbf{x}_A, \omega)) P^+(\mathbf{x}, \omega) \right) n_3 d\mathbf{x}_L. \quad (85)$$

Next, we replace the source in state A by one which emits waves in the negative x_3 -direction, hence $B_A^+ = 0$, $B_A^- = \delta(\mathbf{x} - \mathbf{x}_A)$, and $P_A^\pm = G^{\pm,-}(\mathbf{x}, \mathbf{x}_A, \omega)$. Equation (70) thus gives

$$P^+(\mathbf{x}_A, \omega) = \int_{\partial\mathbb{D}_{0,1}} \frac{2}{i\omega\beta(\mathbf{x})} \left((\partial_3 G^{+,-}(\mathbf{x}, \mathbf{x}_A, \omega)) P^-(\mathbf{x}, \omega) + (\partial_3 G^{-,-}(\mathbf{x}, \mathbf{x}_A, \omega)) P^+(\mathbf{x}, \omega) \right) n_3 d\mathbf{x}_L. \quad (86)$$

Recall that $\partial\mathbb{D}_{0,1}$ consists of $\partial\mathbb{D}_0$ (with $n_3 = -1$) and $\partial\mathbb{D}_1$ (with $n_3 = +1$), see Figure 1. Since $G^{+,\pm}(\mathbf{x}, \mathbf{x}_A, \omega) = 0$ at $\partial\mathbb{D}_0$ and $G^{-,\pm}(\mathbf{x}, \mathbf{x}_A, \omega) = 0$ at $\partial\mathbb{D}_1$ (because outside \mathbb{D} no scattering occurs along the x_3 -coordinate in state A), the first term under the integral in equations (85) and (86) gives a contribution only at $\partial\mathbb{D}_1$ and the second term only at $\partial\mathbb{D}_0$.

Hence,

$$P^\pm(\mathbf{x}_A, \omega) = \int_{\partial\mathbb{D}_0} \frac{-2}{i\omega\beta(\mathbf{x})} (\partial_3 G^{-,\mp}(\mathbf{x}, \mathbf{x}_A, \omega)) P^\pm(\mathbf{x}, \omega) d\mathbf{x}_L + \int_{\partial\mathbb{D}_1} \frac{2}{i\omega\beta(\mathbf{x})} (\partial_3 G^{+,\mp}(\mathbf{x}, \mathbf{x}_A, \omega)) P^\pm(\mathbf{x}, \omega) d\mathbf{x}_L. \quad (87)$$

Note that there is no contribution from $P^-(\mathbf{x}, \omega)$ at $\partial\mathbb{D}_0$ nor from $P^+(\mathbf{x}, \omega)$ at $\partial\mathbb{D}_1$, see Figure 3.

We conclude this section by considering a special case. Suppose the source of the actual field (state B) is located at \mathbf{x}_B in the half-space $x_3 < x_{3,0}$. Then, by taking $x_{3,1} \rightarrow \infty$, the field P^- at $\partial\mathbb{D}_1$ vanishes. This leaves the single-sided representation

$$P^\pm(\mathbf{x}_A, \mathbf{x}_B, \omega) = \int_{\partial\mathbb{D}_0} \frac{-2}{i\omega\beta(\mathbf{x})} (\partial_3 G^{-,\mp}(\mathbf{x}, \mathbf{x}_A, \omega)) P^\pm(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L. \quad (88)$$

Note that we included the source coordinate vector \mathbf{x}_B in the argument list of $P^\pm(\mathbf{x}_A, \mathbf{x}_B, \omega)$. This representation is an extension of a previously derived result [43], in which the fields were decomposed at $\partial\mathbb{D}_0$ but not at \mathbf{x}_A . It describes forward propagation of the one-way field $P^+(\mathbf{x}, \mathbf{x}_B, \omega)$ from the surface $\partial\mathbb{D}_0$ to \mathbf{x}_A (with \mathbf{x}_A and \mathbf{x}_B defined at opposite sides of $\partial\mathbb{D}_0$). In the following two sections, we discuss representations for backward propagation of one-way wave fields.

4.4. Kirchhoff-Helmholtz Integrals for Backward Propagation (Double-Sided).

We derive Kirchhoff-Helmholtz integrals of the correlation type for field-normalised one-way wave fields. For state B, we consider the decomposed actual field, with a point source at \mathbf{x}_B and source spectrum $s(\omega)$. The parameters α and β are the actual parameters inside as well as outside \mathbb{D} . For state A, we choose the Green's state with a unit point source at \mathbf{x}_A in \mathbb{D} . The parameters α and β in \mathbb{D} are the same as those in state B, but for $x_3 \leq x_{3,0}$ and for $x_3 \geq x_{3,1}$, they are chosen independent of the x_3 -coordinate. Hence, the condition for the validity of equation (66) is fulfilled. First, we consider sources emitting waves in the positive x_3 -direction in both states, hence $B_A^+ = \delta(\mathbf{x} - \mathbf{x}_A)$, $B_A^- = 0$, $P_A^\pm = G^{\pm,+}(\mathbf{x}, \mathbf{x}_A, \omega)$, $B_B^+ = \delta(\mathbf{x} - \mathbf{x}_B)s(\omega)$, $B_B^- = 0$, and $P_B^\pm = P^{\pm,+}(\mathbf{x}, \mathbf{x}_B, \omega)$. Substituting this into equation (71) (with $B_{0,A}^\pm = B_A^\pm$ and $B_{0,B}^\pm = B_B^\pm$) gives

$$P^{+,+}(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi(\mathbf{x}_B) \{G^{+,+}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}^* s(\omega) = \int_{\partial\mathbb{D}_{0,1}} \frac{-2}{i\omega\beta(\mathbf{x})} \left(\{ \partial_3 G^{+,+}(\mathbf{x}, \mathbf{x}_A, \omega) \}^* P^{+,+}(\mathbf{x}, \mathbf{x}_B, \omega) + \{ \partial_3 G^{-,+}(\mathbf{x}, \mathbf{x}_A, \omega) \}^* P^{-,+}(\mathbf{x}, \mathbf{x}_B, \omega) \right) n_3 d\mathbf{x}_L, \quad (89)$$

where χ is the characteristic function of the domain \mathbb{D} . It is defined as

$$\chi(\mathbf{x}_B) = \begin{cases} 1, & \text{for } \mathbf{x}_B \text{ in } \mathbb{D}, \\ \frac{1}{2}, & \text{for } \mathbf{x}_B \text{ on } \partial\mathbb{D}_{0,1}, \\ 0, & \text{for } \mathbf{x}_B \text{ outside } \mathbb{D}. \end{cases} \quad (90)$$

Since $G^{+,\pm}(\mathbf{x}, \mathbf{x}_A, \omega) = 0$ at $\partial\mathbb{D}_0$ and $G^{-,\pm}(\mathbf{x}, \mathbf{x}_A, \omega) = 0$ at $\partial\mathbb{D}_1$ (because outside \mathbb{D} no scattering occurs along the x_3 -coordinate in state A), the first term under the integral

in equation (89) gives a contribution only at $\partial\mathbb{D}_1$ and the second term only at $\partial\mathbb{D}_0$. Hence,

$$\begin{aligned} & P^{+,+}(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi(\mathbf{x}_B) \{G^{+,+}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}^* s(\omega) \\ &= \int_{\partial\mathbb{D}_0} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^{-,+}(\mathbf{x}, \mathbf{x}_A, \omega)\}^* P^{-,+}(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L \\ &\quad - \int_{\partial\mathbb{D}_1} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^{+,+}(\mathbf{x}, \mathbf{x}_A, \omega)\}^* P^{+,+}(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L. \end{aligned} \quad (91)$$

Next, we replace the source in state B by one emitting waves in the negative x_3 -direction, hence $B_B^+ = 0$, $B_B^- = \delta(\mathbf{x} - \mathbf{x}_B)s(\omega)$ and $P_B^\pm = P^{\pm,-}(\mathbf{x}, \mathbf{x}_B, \omega)$. This gives

$$\begin{aligned} & P^{+,-}(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi(\mathbf{x}_B) \{G^{-,+}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}^* s(\omega) \\ &= \int_{\partial\mathbb{D}_0} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^{-,+}(\mathbf{x}, \mathbf{x}_A, \omega)\}^* P^{-,-}(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L \\ &\quad - \int_{\partial\mathbb{D}_1} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^{+,+}(\mathbf{x}, \mathbf{x}_A, \omega)\}^* P^{+,-}(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L. \end{aligned} \quad (92)$$

By replacing also the source in state A by one emitting waves in the negative x_3 -direction, according to $B_A^+ = 0$, $B_A^- = \delta(\mathbf{x} - \mathbf{x}_A)$, and $P_A^\pm = G^{\pm,-}(\mathbf{x}, \mathbf{x}_A, \omega)$, we obtain

$$\begin{aligned} & P^{-,-}(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi(\mathbf{x}_B) \{G^{-,-}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}^* s(\omega) \\ &= \int_{\partial\mathbb{D}_0} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^{-,-}(\mathbf{x}, \mathbf{x}_A, \omega)\}^* P^{-,-}(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L \\ &\quad - \int_{\partial\mathbb{D}_1} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^{+,-}(\mathbf{x}, \mathbf{x}_A, \omega)\}^* P^{+,-}(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L. \end{aligned} \quad (93)$$

Finally, changing the source in state B back to the one emitting waves in the positive x_3 -direction yields

$$\begin{aligned} & P^{-,+}(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi(\mathbf{x}_B) \{G^{+,-}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}^* s(\omega) \\ &= \int_{\partial\mathbb{D}_0} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^{-,-}(\mathbf{x}, \mathbf{x}_A, \omega)\}^* P^{-,+}(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L \\ &\quad - \int_{\partial\mathbb{D}_1} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^{+,-}(\mathbf{x}, \mathbf{x}_A, \omega)\}^* P^{-,+}(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L. \end{aligned} \quad (94)$$

Equation (93) is an extension of a previously derived result [44], in which the fields were decomposed at $\partial\mathbb{D}_{0,1}$ but not at \mathbf{x}_A and \mathbf{x}_B . Equations (91), (92), and (94) are further variations. Equation (94) is visualised in Figure 4. Together, these equations describe backward propagation of the one-way wave fields $P^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega)$ from $\partial\mathbb{D}_0$ and $P^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega)$ from $\partial\mathbb{D}_1$ to \mathbf{x}_A . Except for some special cases, the integrals along $\partial\mathbb{D}_1$ do not vanish by taking $x_{3,1} \rightarrow \infty$. Hence, unlike the forward propagation representation (87), the double-sided backward propagation representations (91), (92), (93), and (94) in general do not simplify to

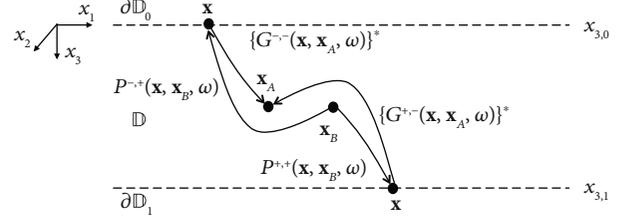


FIGURE 4: Visualisation of the different terms in the field-normalised one-way Kirchhoff-Helmholtz integral for backward propagation, formulated by equation (94).

single-sided representations. In the next section, we discuss an alternative method to derive single-sided representations for backward propagation.

We conclude this section by considering a special case. Suppose that in state B the parameters α and β are the same as in state A not only in \mathbb{D} but also outside \mathbb{D} . Then, $P^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega) = G^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega)s(\omega)$ for all \mathbf{x} . Substituting this into representations (91), (92), (93), and (94), summing the left- and right-hand sides of these representations separately and dividing both sides by $s(\omega)$, using equations (78) and (84) and assuming that \mathbf{x}_B is located in \mathbb{D} , we obtain

$$\begin{aligned} G_h(\mathbf{x}_A, \mathbf{x}_B, \omega) &= \int_{\partial\mathbb{D}_0} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^-(\mathbf{x}, \mathbf{x}_A, \omega)\}^* G^-(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L \\ &\quad - \int_{\partial\mathbb{D}_1} \frac{2}{i\omega\beta(\mathbf{x})} \{\partial_3 G^+(\mathbf{x}, \mathbf{x}_A, \omega)\}^* G^+(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L, \end{aligned} \quad (95)$$

where the so-called homogeneous Green's function $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega)$ is defined as

$$\begin{aligned} G_h(\mathbf{x}_A, \mathbf{x}_B, \omega) &= G(\mathbf{x}_A, \mathbf{x}_B, \omega) + G^*(\mathbf{x}_A, \mathbf{x}_B, \omega) \\ &= 2\mathcal{R}\{G(\mathbf{x}_A, \mathbf{x}_B, \omega)\}, \end{aligned} \quad (96)$$

(with \mathcal{R} denoting the real part) and where $G^\pm(\mathbf{x}, \mathbf{x}_A, \omega) = G^{\pm,+}(\mathbf{x}, \mathbf{x}_A, \omega) + G^{\pm,-}(\mathbf{x}, \mathbf{x}_A, \omega)$ (and a similar expression for $G^\pm(\mathbf{x}, \mathbf{x}_B, \omega)$). Equation (95) is akin to the well-known representation for the homogeneous Green's function [45, 46], but with decomposed Green's functions under the integrals. The simple relation between representations (91), (92), (93), and (94) on the one hand and the homogeneous Green's function representation (95) on the other hand is a consequence of the field-normalised decomposition, introduced in Section 3.3.

4.5. Kirchhoff-Helmholtz Integrals for Backward Propagation (Single-Sided). The complex-conjugated Green's functions $\{\partial_3 G^{\pm,\pm}(\mathbf{x}, \mathbf{x}_A, \omega)\}^*$ under the integrals in equations (91), (92), (93), and (94) can be seen as focusing functions, which focus the wave fields $P^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega)$ onto a focal point \mathbf{x}_A . However, this focusing process requires that these wave fields are available at two boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_1$, enclosing the focal point \mathbf{x}_A . Here, we discuss single-sided field-normalised focusing functions $f_1^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$ and we use these in modifications of reciprocity theorems (70) and (71) to

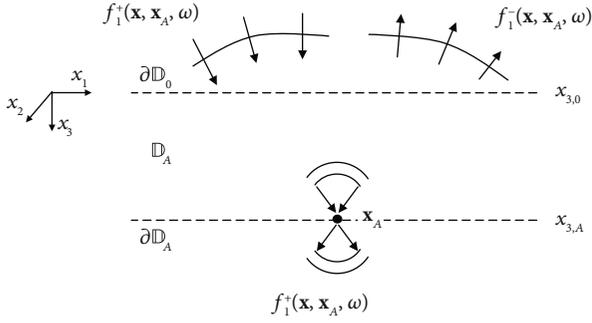


FIGURE 5: Configuration for the derivation of the single-sided Kirchhoff-Helmholtz integrals for backward propagation.

derive single-sided Kirchhoff-Helmholtz integrals for backward propagation.

We start by defining a new domain \mathbb{D}_A , enclosed by two surfaces $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_A$ perpendicular to the x_3 -axis at $x_3 = x_{3,0}$ and $x_3 = x_{3,A}$, respectively, with $x_{3,A} > x_{3,0}$, see Figure 5. Hence, $\partial\mathbb{D}_A$ is chosen such that it contains the focal point \mathbf{x}_A . The two surfaces $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_A$ are together denoted by $\partial\mathbb{D}_{0,A}$. The focusing functions $f_1^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$, which will play the role of state A in the reciprocity theorems, obey the one-way wave equations (64) and (65) (but without the source terms S^\pm), with parameters α and β in \mathbb{D}_A equal to those in the actual state B , and independent of the x_3 -coordinate for $x_3 \leq x_{3,0}$ and for $x_3 \geq x_{3,A}$. Hence, the condition for the validity of equation (66) is fulfilled. Analogous to equation (56), the field-normalised focusing functions $f_1^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$ are related to the full focusing function $f_1(\mathbf{x}, \mathbf{x}_A, \omega)$, according to

$$f_1(\mathbf{x}, \mathbf{x}_A, \omega) = f_1^+(\mathbf{x}, \mathbf{x}_A, \omega) + f_1^-(\mathbf{x}, \mathbf{x}_A, \omega). \quad (97)$$

The focusing function $f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)$ is incident to the domain \mathbb{D}_A from the half-space $x_3 < x_{3,0}$ (see Figure 5). It propagates and scatters in the inhomogeneous domain \mathbb{D}_A , focuses at \mathbf{x}_A on surface $\partial\mathbb{D}_A$, and continues as $f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)$ in the half-space $x_3 > x_{3,A}$. The back-scattered field leaves \mathbb{D}_A via surface $\partial\mathbb{D}_0$ and continues as $f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)$ in the half-space $x_3 < x_{3,0}$. The focusing conditions at the focal plane $\partial\mathbb{D}_A$ are [18]

$$[\partial_3 f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)]_{x_3=x_{3,A}} = \frac{1}{2} i\omega\beta(\mathbf{x}_A)\delta(\mathbf{x}_L - \mathbf{x}_{L,A}), \quad (98)$$

$$[\partial_3 f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)]_{x_3=x_{3,0}} = 0. \quad (99)$$

Here, $\mathbf{x}_{L,A}$ denotes the lateral coordinates of \mathbf{x}_A . The operators ∂_3 and the factor $(1/2)i\omega\beta(\mathbf{x}_A)$ are not necessary to define the focusing conditions but are chosen for later convenience. To avoid instability, evanescent waves are excluded from the focusing functions. This implies that the delta function in equation (98) should be interpreted as a spatially band-limited delta function. Note that the sifting property of the delta function, $h(\mathbf{x}_{L,A}) = \int_{\mathbb{S}} \delta(\mathbf{x}_L - \mathbf{x}_{L,A})h(\mathbf{x}_L)d\mathbf{x}_L$, remains

valid for a spatially band-limited delta function, assuming $h(\mathbf{x}_L)$ is also spatially band-limited.

We now derive single-sided Kirchhoff-Helmholtz integrals for backward propagation. We consider the reciprocity theorems for field-normalised one-way wave fields (equations (70) and (71)), with \mathbb{D} and $\partial\mathbb{D}_{0,1}$ replaced by \mathbb{D}_A and $\partial\mathbb{D}_{0,A}$, respectively. For state A , we consider the focusing functions discussed above; hence, $B_A^+(\mathbf{x}, \omega) = B_A^-(\mathbf{x}, \omega) = 0$ and $P_A^\pm(\mathbf{x}, \omega) = f_1^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$. For state B , we consider the decomposed actual field, with a point source at \mathbf{x}_B in the half-space $x_3 > x_{3,0}$ and source spectrum $s(\omega)$. The parameters α and β in state B are the actual parameters inside as well as outside $\partial\mathbb{D}_{0,A}$. First, we consider a source in state B which emits waves in the positive x_3 -direction, hence $B_B^+(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_B)s(\omega)$, $B_B^-(\mathbf{x}, \omega) = 0$, and $P_B^\pm(\mathbf{x}, \omega) = P^{\pm,+}(\mathbf{x}, \mathbf{x}_B, \omega)$. Substituting all this into equations (70) and (71) (with $B_0^\pm = B^\pm$), using equations (98) and (99) in the integrals along $\partial\mathbb{D}_A$, gives

$$\begin{aligned} P^{-,+}(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi_A(\mathbf{x}_B)f_1^-(\mathbf{x}_B, \mathbf{x}_A, \omega)s(\omega) \\ = \int_{\partial\mathbb{D}_0} \frac{2}{i\omega\beta(\mathbf{x})} \left((\partial_3 f_1^+(\mathbf{x}, \mathbf{x}_A, \omega))P^{-,+}(\mathbf{x}, \mathbf{x}_B, \omega) \right. \\ \left. + (\partial_3 f_1^-(\mathbf{x}, \mathbf{x}_A, \omega))P^{+,+}(\mathbf{x}, \mathbf{x}_B, \omega) \right) d\mathbf{x}_L, \end{aligned} \quad (100)$$

$$\begin{aligned} P^{+,+}(\mathbf{x}_A, \mathbf{x}_B, \omega) - \chi_A(\mathbf{x}_B)\{f_1^+(\mathbf{x}_B, \mathbf{x}_A, \omega)\}^*s(\omega) \\ = \int_{\partial\mathbb{D}_0} \frac{-2}{i\omega\beta(\mathbf{x})} \left(\{\partial_3 f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)\}^*P^{+,+}(\mathbf{x}, \mathbf{x}_B, \omega) \right. \\ \left. + \{\partial_3 f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)\}^*P^{-,+}(\mathbf{x}, \mathbf{x}_B, \omega) \right) d\mathbf{x}_L, \end{aligned} \quad (101)$$

where χ_A is the characteristic function of the domain \mathbb{D}_A . It is defined by equation (90), with \mathbb{D} and $\partial\mathbb{D}_{0,1}$ replaced by \mathbb{D}_A and $\partial\mathbb{D}_{0,A}$, respectively. Next, we replace the source in state B by one which emits waves in the negative x_3 -direction, hence $B_B^+(\mathbf{x}, \omega) = 0$, $B_B^-(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_B)s(\omega)$, and $P_B^\pm(\mathbf{x}, \omega) = P^{\pm,-}(\mathbf{x}, \mathbf{x}_B, \omega)$. This gives

$$\begin{aligned} P^{-,-}(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi_A(\mathbf{x}_B)f_1^+(\mathbf{x}_B, \mathbf{x}_A, \omega)s(\omega) \\ = \int_{\partial\mathbb{D}_0} \frac{2}{i\omega\beta(\mathbf{x})} \left((\partial_3 f_1^+(\mathbf{x}, \mathbf{x}_A, \omega))P^{-,-}(\mathbf{x}, \mathbf{x}_B, \omega) \right. \\ \left. + (\partial_3 f_1^-(\mathbf{x}, \mathbf{x}_A, \omega))P^{+,-}(\mathbf{x}, \mathbf{x}_B, \omega) \right) d\mathbf{x}_L, \end{aligned} \quad (102)$$

$$\begin{aligned} P^{+,-}(\mathbf{x}_A, \mathbf{x}_B, \omega) - \chi_A(\mathbf{x}_B)\{f_1^-(\mathbf{x}_B, \mathbf{x}_A, \omega)\}^*s(\omega) \\ = \int_{\partial\mathbb{D}_0} \frac{-2}{i\omega\beta(\mathbf{x})} \left(\{\partial_3 f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)\}^*P^{+,-}(\mathbf{x}, \mathbf{x}_B, \omega) \right. \\ \left. + \{\partial_3 f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)\}^*P^{-,-}(\mathbf{x}, \mathbf{x}_B, \omega) \right) d\mathbf{x}_L. \end{aligned} \quad (103)$$

Equations (100), (101), (102), and (103) are single-sided representations for backward propagation of the one-way wave fields $P^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega)$ from $\partial\mathbb{D}_0$ to \mathbf{x}_A . Similar results have been previously obtained [47, 48], but without decomposition at \mathbf{x}_B . An advantage of these equations over equations (91), (92), (93), and (94) is that the backward propagated fields $P^{\pm,\pm}(\mathbf{x}_A, \mathbf{x}_B, \omega)$ are expressed entirely in terms of integrals along the surface $\partial\mathbb{D}_0$.

Single-sided representations containing the field-normalised focusing functions $f_1^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$ find applications, for example, in reflection imaging methods, which account for multiple scattering. In these methods, the focusing functions are retrieved from the reflection response at the surface $\partial\mathbb{D}_0$, using the Marchenko method [18, 49–51].

We conclude this section by considering a special case. Suppose that in state B the parameters α and β are the same as in the Green's state. Then, $P^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega) = G^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega)s(\omega)$ for all \mathbf{x} . Moreover, $P^{\pm,\pm}(\mathbf{x}, \mathbf{x}_B, \omega) = 0$ for \mathbf{x} at $\partial\mathbb{D}_0$. Substituting this into representations (100), (101), (102), and (103), summing the left- and right-hand sides of these representations separately, dividing both sides by $s(\omega)$ and using equation (97), we obtain

$$\begin{aligned} & G(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi_A(\mathbf{x}_B)2i\mathcal{I}\{f_1(\mathbf{x}_B, \mathbf{x}_A, \omega)\} \\ &= \int_{\partial\mathbb{D}_0} \frac{2}{i\omega\beta(\mathbf{x})} \partial_3(f_1^+(\mathbf{x}, \mathbf{x}_A, \omega) \\ &\quad - \{f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)\}^*) G^-(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}_L, \end{aligned} \quad (104)$$

(with \mathcal{I} denoting the imaginary part), where $G^-(\mathbf{x}, \mathbf{x}_B, \omega) = G^{-,+}(\mathbf{x}, \mathbf{x}_B, \omega) + G^{-,-}(\mathbf{x}, \mathbf{x}_B, \omega)$. Taking the real part of both sides gives

$$\begin{aligned} G_h(\mathbf{x}_A, \mathbf{x}_B, \omega) &= \mathcal{R} \int_{\partial\mathbb{D}_0} \frac{4}{i\omega\beta(\mathbf{x})} \partial_3(f_1^+(\mathbf{x}, \mathbf{x}_A, \omega) \\ &\quad - \{f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)\}^*) G^-(\mathbf{x}, \mathbf{x}_B, \omega) d\mathbf{x}, \end{aligned} \quad (105)$$

where $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega)$ is the homogeneous Green's function, defined in equation (96). Unlike in equation (95), here the homogeneous Green's function is represented by a single integral along the surface $\partial\mathbb{D}_0$, containing field normalised one-way focusing and Green's functions.

5. Conclusions

We have considered flux-normalised and field-normalised decomposition of scalar wave fields into coupled one-way wave fields. The operators for field-normalised decomposition exhibit less symmetry than those for flux-normalised decomposition. Nevertheless, we have shown that reciprocity theorems can be derived for field-normalised one-way wave fields in a similar way as those for flux-normalised one-way wave fields. An additional condition for the reciprocity theorems for field-normalised one-way wave fields is that in one of the states the derivatives in the x_3 -direction of the parameters α and β vanish at the boundary of the considered domain. This condition is easily fulfilled when one of the states is a Green's function or a focusing function, for which the parameters α and β can be freely chosen at and outside the boundary of the domain.

We have used the reciprocity theorems for field-normalised one-way wave fields as a starting point for deriving representation theorems for field-normalised one-way wave fields in a systematic way. We obtained representations for forward and for backward propagations of one-way wave fields. These representations account for multiple scattering.

Whereas the Kirchhoff-Helmholtz integrals for forward propagation can be easily transformed into single-sided representations, this transformation is less straightforward for the Kirchhoff-Helmholtz integrals for backward propagation. By replacing the Green's functions by focusing functions, we obtained single-sided representations for backward propagation of field-normalised one-way wave fields. These representations are particularly useful to retrieve wave fields in the interior of a domain in situations where measurements can be carried out only at a single surface. An important application is reflection imaging, accounting for multiple scattering.

Data Availability

No datasets have been used for this study.

Conflicts of Interest

The author declares that he has no competing interests.

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References

- [1] H. J. W. M. Hoekstra, "On beam propagation methods for modelling in integrated optics," *Optical and Quantum Electronics*, vol. 29, no. 2, pp. 157–171, 1997.
- [2] M. J. N. Van Stralen, H. Blok, and M. V. De Hoop, "Design of sparse matrix representations for the propagator used in the BPM and directional wave field decomposition," *Optical and Quantum Electronics*, vol. 29, no. 2, pp. 179–197, 1997.
- [3] J. F. Claerbout, "Toward a unified theory of reflector mapping," *Geophysics*, vol. 36, no. 3, pp. 467–481, 1971.
- [4] A. J. Berkhouit, *Seismic Migration. Imaging of Acoustic Energy by Wave Field Extrapolation. A. Theoretical Aspects*, Elsevier, 1982.
- [5] J. J. McCoy and L. N. Frazer, "Pseudodifferential operators, operator orderings, marching algorithms and path integrals for one-way equations," *Wave Motion*, vol. 9, no. 5, pp. 413–427, 1987.
- [6] O. Holberg, "Towards optimum one-way wave propagation," *Geophysical Prospecting*, vol. 36, no. 2, pp. 99–114, 1988.
- [7] L. Fishman, "One-way wave propagation methods in direct and inverse scalar wave propagation modeling," *Radio Science*, vol. 28, no. 5, pp. 865–876, 1993.
- [8] L. Fishman and J. J. McCoy, "Derivation and application of extended parabolic wave theories. I. The factorized Helmholtz equation," *Journal of Mathematical Physics*, vol. 25, no. 2, pp. 285–296, 1984.
- [9] L. Fishman and J. J. McCoy, "Derivation and application of extended parabolic wave theories. II. Path integral representations," *Journal of Mathematical Physics*, vol. 25, no. 2, pp. 297–308, 1984.
- [10] L. Halpern and L. N. Trefethen, "Wide-angle one-way wave equations," *The Journal of the Acoustical Society of America*, vol. 84, no. 4, pp. 1397–1404, 1988.

- [11] V. H. Weston, "Wave splitting and the reflection operator for the wave equation in \mathbb{R}^3 ," *Journal of Mathematical Physics*, vol. 30, no. 11, pp. 2545–2562, 1989.
- [12] L. Fishman, "Exact and approximate solutions of the Helmholtz, Weyl composition equation in scalar wave propagation," *Wave Motion*, vol. 14, no. 3, pp. 205–224, 1991.
- [13] J. L. T. Grimbergen, F. J. Dessing, and K. Wapenaar, "Modal expansion of one-way operators in laterally varying media," *Geophysics*, vol. 63, no. 3, pp. 995–1005, 1998.
- [14] M. V. de Hoop, "Directional decomposition of transient acoustic wave fields," Ph.D. thesis, Delft University of Technology, 1992.
- [15] M. V. de Hoop, "Generalization of the Bremmer coupling series," *Journal of Mathematical Physics*, vol. 37, no. 7, pp. 3246–3282, 1996.
- [16] C. P. A. Wapenaar and J. L. T. Grimbergen, "Reciprocity theorems for one-way wavefields," *Geophysical Journal International*, vol. 127, no. 1, pp. 169–177, 1996.
- [17] C. P. A. Wapenaar, "One-way representations of seismic data," *Geophysical Journal International*, vol. 127, no. 1, pp. 178–188, 1996.
- [18] K. Wapenaar, J. Thorbecke, J. van der Neut, F. Broggin, E. Slob, and R. Snieder, "Marchenko imaging," *Geophysics*, vol. 79, no. 3, pp. WA39–WA57, 2014.
- [19] J. P. Coron, "Bremmer series that correct parabolic approximations," *Journal of Mathematical Analysis and Applications*, vol. 50, no. 2, pp. 361–372, 1975.
- [20] B. Ursin, "Review of elastic and electromagnetic wave propagation in horizontally layered media," *Geophysics*, vol. 48, no. 8, pp. 1063–1081, 1983.
- [21] C. P. A. Wapenaar and A. J. Berkhou, *Elastic Wave Field Extrapolation*, Elsevier, Amsterdam, 1989.
- [22] J. T. Fokkema and P. M. van den Berg, *Seismic Applications of Acoustic Reciprocity*, Elsevier, Amsterdam, 1993.
- [23] A. T. de Hoop, *Handbook of Radiation and Scattering of Waves*, Academic Press, London, 1995.
- [24] J. D. Achenbach, *Reciprocity in Elastodynamics*, Cambridge University Press, Cambridge, 2003.
- [25] N. N. Bojarski, "Generalized reaction principles and reciprocity theorems for the wave equations, and the relationship between the time-advanced and time-retarded fields," *The Journal of the Acoustical Society of America*, vol. 74, no. 1, pp. 281–285, 1983.
- [26] L. M. Brekhovskikh, *Waves in Layered Media*, Academic Press, New York, NY, USA, 1960.
- [27] M. V. de Hoop and A. T. de Hoop, "Elastic wave up/down decomposition in inhomogeneous and anisotropic media: an operator approach and its approximations," *Wave Motion*, vol. 20, no. 1, pp. 57–82, 1994.
- [28] L. Fishman, M. V. de Hoop, and M. J. N. van Stralen, "Exact constructions of square-root Helmholtz operator symbols: the focusing quadratic profile," *Journal of Mathematical Physics*, vol. 41, no. 7, pp. 4881–4938, 2000.
- [29] C. J. Thomson, "Seismic transmission operator reciprocity – I: a direct derivation showing the role of elastic impedance operator symmetry," *Geophysical Journal International*, vol. 202, no. 2, pp. 1064–1070, 2015.
- [30] C. J. Thomson, "Seismic transmission operator reciprocity – II: impedance-operator symmetry via elastic lateral modes," *Geophysical Journal International*, vol. 202, no. 2, pp. 1071–1087, 2015.
- [31] K. Wapenaar, J. Thorbecke, and D. Draganov, "Relations between reflection and transmission responses of three-dimensional inhomogeneous media," *Geophysical Journal International*, vol. 156, no. 2, pp. 179–194, 2004.
- [32] M. R. Kumar and M. G. Bostock, "Transmission to reflection transformation of teleseismic wavefields," *Journal of Geophysical Research*, vol. 111, no. B8, article B08306, 2006.
- [33] C. Fan, G. L. Pavlis, A. B. Weglein, and B. G. Nita, "Removing free-surface multiples from teleseismic transmission and constructed reflection responses using reciprocity and the inverse scattering series," *Geophysics*, vol. 71, no. 4, pp. SI71–SI78, 2006.
- [34] A. B. Weglein, B. G. Nita, K. A. Innanen et al., "Using the inverse scattering series to predict the wavefield at depth and the transmitted wavefield without an assumption about the phase of the measured reflection data or back propagation in the overburden," *Geophysics*, vol. 71, no. 4, pp. SI125–SI137, 2006.
- [35] E. Slob, "Interferometry by deconvolution of multicomponent multioffset GPR data," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 47, no. 3, pp. 828–838, 2009.
- [36] E. Ruigrok and K. Wapenaar, "Global-phase seismic interferometry unveils P-wave reflectivity below the Himalayas and Tibet," *Geophysical Research Letters*, vol. 39, no. 11, article L11303, 2012.
- [37] K. Wapenaar, J. Thorbecke, J. van der Neut, F. Broggin, E. Slob, and R. Snieder, "Green's function retrieval from reflection data, in absence of a receiver at the virtual source position," *The Journal of the Acoustical Society of America*, vol. 135, no. 5, pp. 2847–2861, 2014.
- [38] E. Slob, K. Wapenaar, F. Broggin, and R. Snieder, "Seismic reflector imaging using internal multiples with Marchenko-type equations," *Geophysics*, vol. 79, no. 2, pp. S63–S76, 2014.
- [39] M. Ravasi, I. Vasconcelos, A. Kritski, A. Curtis, C. A. da Costa Filho, and G. A. Meles, "Target-oriented Marchenko imaging of a North Sea field," *Geophysical Journal International*, vol. 205, no. 1, pp. 99–104, 2016.
- [40] M. Ravasi, "Rayleigh-Marchenko redatuming for target-oriented, true-amplitude imaging," *Geophysics*, vol. 82, no. 6, pp. S439–S452, 2017.
- [41] P. Elison, D. J. van Manen, J. O. A. Robertsson, M. S. Dukalski, and K. de Vos, "Marchenko-based immersive wave simulation," *Geophysical Journal International*, vol. 215, no. 2, pp. 1118–1131, 2018.
- [42] M. Staring, R. Pereira, H. Douma, J. van der Neut, and K. Wapenaar, "Source-receiver Marchenko redatuming on field data using an adaptive double-focusing method," *Geophysics*, vol. 83, no. 6, pp. S579–S590, 2018.
- [43] A. J. Berkhou and C. P. A. Wapenaar, "One-way versions of the Kirchhoff integral," *Geophysics*, vol. 54, no. 4, pp. 460–467, 1989.
- [44] C. P. A. Wapenaar, G. L. Peels, V. Budejicky, and A. J. Berkhou, "Inverse extrapolation of primary seismic waves," *Geophysics*, vol. 54, no. 7, pp. 853–863, 1989.
- [45] R. P. Porter, "Diffraction-limited, scalar image formation with holograms of arbitrary shape," *Journal of the Optical Society of America*, vol. 60, no. 8, pp. 1051–1059, 1970.
- [46] M. L. Oristaglio, "An inverse scattering formula that uses all the data," *Inverse Problems*, vol. 5, no. 6, pp. 1097–1105, 1989.
- [47] K. Wapenaar, J. Thorbecke, J. van der Neut, E. Slob, and R. Snieder, "Review paper: virtual sources and their responses,

part II: data-driven single-sided focusing,” *Geophysical Prospecting*, vol. 65, no. 6, pp. 1430–1451, 2017.

- [48] J. Van der Neut, J. L. Johnson, K. van Wijk, S. Singh, E. Slob, and K. Wapenaar, “A Marchenko equation for acoustic inverse source problems,” *The Journal of the Acoustical Society of America*, vol. 141, no. 6, pp. 4332–4346, 2017.
- [49] S. Singh and R. Snieder, “Source-receiver Marchenko redatuming: obtaining virtual receivers and virtual sources in the subsurface,” *Geophysics*, vol. 82, no. 3, pp. Q13–Q21, 2017.
- [50] C. A. da Costa Filho, G. A. Meles, A. Curtis, M. Ravasi, and A. Kritski, “Imaging strategies using focusing functions with applications to a North Sea field,” *Geophysical Journal International*, vol. 213, no. 1, pp. 561–573, 2018.
- [51] K. Wapenaar, J. Brackenhoff, and J. Thorbecke, “Green’s theorem in seismic imaging across the scales,” *Solid Earth*, vol. 10, no. 2, pp. 517–536, 2019.