Seismic Modeling and Migration 7: Theories and Techniques

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Wavefield extrapolation using the wavelet transform

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Summary

A (spatial) wavelet transform is applied to the seismic data model in the space domain, such that a data model in the wavelet domain is obtained. The wavelet transform is a mathematical tool, which transforms a signal to the mixed space-wavenumber domain. Because. of their compact support wavelets are cut out for analyzing finite-length apertures. Moreover, wavelets can zoom in on local high wavenumber aspects.

Subsequently a redatuming scheme (which is the heart of our migration scheme) is derived in the wavelet domain, corresponding to the redatuming scheme in the space domain. The important difference between the two domains is the fact that in the space domain redaturning can be carried out per point source experiment and in the wavelet domain redatuming can be carried out per scale experiment. Redatuming carried out per scale experiment is very advantageous. A remarkably small part of the extrapolation operators already reveals the correct structural information.

Introduction

Our departure point is a wave equation based data model in the discrete space domain. In its simplest form (i.e. after preprocessing) it is given by the following matrix equation [1]

$$\underset{\sim}{\boldsymbol{P}}^- = \underset{\sim}{\boldsymbol{W}}^- \underset{\sim}{\boldsymbol{R}} \underset{\sim}{\boldsymbol{W}}^+ \underset{\sim}{\boldsymbol{S}}^+ \tag{1}$$

where $\underset{\sim}{\boldsymbol{S}}^+$ contains the downgoing source wavefields at the surface, $\underset{\sim}{\boldsymbol{W}}^+$ describes downward propagation into the subsurface, $\underset{\sim}{\boldsymbol{R}}$ describes reflection in the subsurface (at one depth level), $\underset{\sim}{\boldsymbol{W}}^-$ describes upward propagation to the surface and, finally, $\underset{\sim}{\boldsymbol{P}}^-$ contains the upgoing wavefields registered at the surface. Equation (1) is a monochromatic description of the seismic response of a single reflecting depth level; the different elements in the matrices correspond to different lateral positions. The matrix multiplications in equation (1) represent generalized convolutions along the lateral space coordinates. In a generalized convolution the convolution kernel changes during the convolution process due to lateral variations of the subsurface parameters.

An image of the subsurface consists of the correct estimate of $\underset{\sim}{\boldsymbol{R}}$ (phase and preferably also amplitude). $\underset{\sim}{\boldsymbol{R}}$ is directly related to the velocity and density variations of the subsurface. So, if one knows the reflectivity at all depth levels, then one has an image of the earth. Inversion with respect to the reflectivity $\underset{\sim}{\boldsymbol{R}}$ of the data model of equation (1) gives

$$\underset{\sim}{\boldsymbol{R}} = [\underset{\sim}{\boldsymbol{W}}^-]^H \underset{\sim}{\boldsymbol{P}}^- [\underset{\sim}{\boldsymbol{W}}^+]^H \tag{2}$$

where we have used the modified matched filter approach [1, 8], where H denotes complex conjugation and transposition, and where we have assumed the source wavefields matrix $\underset{\sim}{\boldsymbol{S}}^+$ to represent a series of normalized dipole sources, i.e. an identity matrix. Redatuming equation (2) is the heart of our migration scheme. It says that the reflectivity at a certain depth level can be found by correcting the data $\underset{\sim}{\boldsymbol{P}}^-$ for the propagation through the overburden between the acquisition level of $\underset{\sim}{\boldsymbol{P}}^-$ and the new acquisition level of $\underset{\sim}{\boldsymbol{R}}$.

It is possible and advantageous [4] to transform equations (1) and (2) to the wavenumber domain, even if one is dealing with a heterogeneous macro model. The transformation to the wavenumber domain is naturally suggested by the fact that for laterally homogeneous macro models wavefield extrapolation in the wavenumber domain is described by a diagonal matrix [5]. Generalization of this concept for heterogeneous macro models leads to band matrices describing wavefield extrapolation in the subsurface.

The two afore mentioned data models are based on global solutions of the wave equation. In the spatial description of the data angular information is not available; in the wavenumber description of the data spatial information is not available. Intermediate space-wavenumber data models that are based on transient wave phenomena may be more appropriate, because of the finite size of macro models generally used. A beautiful transform that can be used to arrive at an intermediate description is the (spatial) wavelet transform. The wavelet transform is usually called a space-resolution transform rather than a space-wavenumber transform. It divides the data model of equation (1) in different resolutions or scales. A scale or resolution corresponds to a certain wavenumber interval. Each scale or resolution step corresponds to a bisection of the wavenumber interval. The division into scales facilitates zooming in on local high wavenumber aspects. (See [6] for a general introduction.)

The wavelet transform creates the opportunity to do wavefield extrapolation or redatuming in a very elegant stepwise approach. We start with a rough scale approximation of the extrapolation operators. In the rough scales the structural information of the macro model is already taken into account. The rough scales make up a small part of the extrapolation operators in the wavelet domain. So, redatuming can be carried out fast. Moreover, it gives the possibility to update macro models in an iterative way. This means that the correctness of the macro model can be checked upon in an early stage. By adding detail of the extrapolation operators the lateral resolution can be improved.

In the next section wavefield extrapolation in the space domain will be introduced. Next, the basic mathematical aspects of the wavelet transform will be introduced. Subsequently, the wavelet transform will be applied to the wavefield extrapolation operator. In the examples it will be shown that wavefield extrapolation in the wavelet domain is carried out elegantly and efficiently.

Wavefield extrapolation in the space domain

Consider a 2-D monochromatic downgoing acoustic wavefield $P^+(x, z_0, \omega)$, registered as a function of the horizontal coordinate x at depth level z_0 and frequency w. Downward extrapolation from depth level z_0 to depth level z_m is mathematically described by the generalized convolution integral [1]

$$P^+(x', z_m, \omega) = \int_{-\infty}^{\infty} W^+(x', z_m; x, z_0, \omega) P^+(x, z_0, \omega) dx, \tag{3}$$

with

$$W^+(x', z_m; x, z_0, \omega) = 2\frac{\partial G^+(x', z_m; x, z = z_0, \omega)}{\partial z}. \tag{4}$$

Here, $P^+(x', z_m, \omega)$ represents the extrapolated downgoing wavefield at position (x', z_m) and $G^+(x', z_m; x, z = z_0, \omega)$ represents the downgoing part of the Green's wavefield at (x', z_m) related to a monopole at (x, z_0). Hence, the extrapolation operator $W^+(x', z_m; x, z_0, \omega)$ may be seen as the downgoing response at (x', z_m) of a dipole at

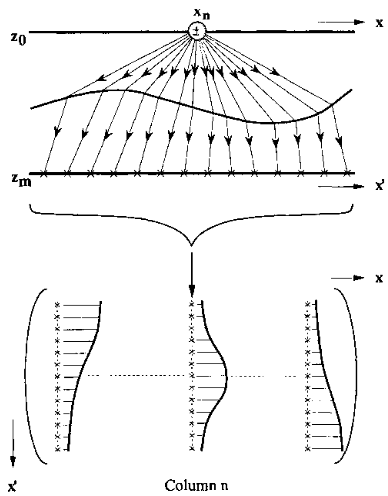

Figure 1: The extrapolation matrix in the space-frequency domain. Column n contains the (monochromatic) discretixed response as a function of x' at depth z_m for a dipole source at (x_n, z_0) (only the modulus is shown; the actual matrix is complex valued)

(x, z_0).
For discretized wavefields (discretization interval Ax) equation (3) can be rewritten in matrix notation, according to

$$\boldsymbol{P}^{+}(z_m) = \underset{\sim}{\boldsymbol{W}}^{+}(z_m; z_0)\boldsymbol{P}^{+}(z_0). \qquad (5)$$

Here, $\boldsymbol{P}^{+}(z_0)$ and $\boldsymbol{P}^{+}(z_m)$ are column vectors, containing the (monochromatic) discretized wavefields at depth levels z_0 and z_m, respectively. $\underset{\sim}{\boldsymbol{W}}^{+}(z_m; z_0)$ is an extrapolation matrix; column n contains the (monochromatic) discretized response $\Delta x W^{+}(x', z_m; x_n, z_0, \omega)$ as a function of x' at depth z_m for a dipole source at (x,, z_0), see Figure 1. In practice this matrix is constructed by forward modelling of dipole wavefields through a macro model of the subsurface between depth levels z_0 and z_m and by storing the results for a given frequency into the columns of the matrix.

The wavelet transform

The continuous wavelet transform is defined by

$$\check{f}(a, b) = \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{|a|}} g\left(\frac{x-b}{a}\right) dx \qquad a \neq 0. \qquad (6)$$

$\check{f}(a, b)$ is the inner product of $f(x)$ and the function $g^{(a,b)}$ (x) which is a shifted and dilated version of the function g(x)

$$g^{(a,b)}(x) = \frac{1}{\sqrt{|a|}} g\left(\frac{x-b}{a}\right). \qquad (7)$$

The parameter b gives the position of the wavelet, while the dilation parameter a governs its frequency. For $|a| \ll 1$, $g^{(a,b)}$ (x) is a highly concentrated version of g(x) which means that it extracts the high frequency content of the signal $f(x)$. For $|a| \gg 1$, the wavelet $g^{(a,b)}(x)$ is very much spread out: it extracts the low frequency content of the signal $f(x)$.
Discretization with respect to the parameters a, b such that $a = 2^m$ and $b = 2^m n$ is particularly interesting. Each step in m corresponds to a bisection of the frequency content. The wavelet transform then elegantly fits into the theory of multiresolut ion analysis [7]. The idea of multiresolution analysis is to write a function $f(x)$ as a limit of successive approximations each of which is a smoothed version of $f(x)$, by using more and more concentrated smoothing functions. The successive approximations thus use a different resolution, whence the name multiresolution analysis. The difference between two successive approximations is the detail at a certain resolution. This detail is exactly a wavelet transform for a certain value of m. The theory of multiresolution analysis and the discovery of orthogonal compactly supported wavelets [3] made it possible to implement the wavelet transform as an O(N) algorithm. In its discrete implementation the wavelet transform can be interpreted in the following way

$$\{f_i\}, i = 1, \cdots, \mathrm{N} \xrightarrow{\underset{\sim}{\boldsymbol{\Gamma}}_w} \{a^J, d^J, d^{J-1}, d^{J-2}, \cdots, d^2, d^1\}. \qquad (8)$$

An operator $\underset{\sim}{\boldsymbol{\Gamma}}_w$ divides the discretized version of the signal $f(x)$ in a coarse approximation a^J (consisting of 2 points), a coarse scale detail d^J (2 points) and subsequent finer details d^{J-1}(4 points), . . . , d^1 (N/2 points), where d^1 is the finest detail. The total amount of points does not change and an exact reconstruction of the original data is feasible.

Wavefield extrapolation in the wavelet domain

The transformation of the wavefield extrapolation equation **(5)** in the space domain to the wavelet domain is now realized in the following way. Define the forward and inverse wavelet transform in the matrix notation respectively by

$$\check{\boldsymbol{P}}^{+}(z_0) = \underset{\sim}{\boldsymbol{\Gamma}}_w \boldsymbol{P}^{+}(z_0) \quad \text{and} \quad \boldsymbol{P}^{+}(z_0) = \underset{\sim}{\boldsymbol{\Gamma}}_w^T \check{\boldsymbol{P}}^{+}(z_0) \qquad (9)$$

in which $\underset{\sim}{\boldsymbol{\Gamma}}_w$ is the wavelet transform operator and $\underset{\sim}{\boldsymbol{\Gamma}}_w^T$ is the transposed wavelet transform operator, which equals the inverse wavelet transform operator. Application of equations (9) to equation (5) yields

$$\check{\boldsymbol{P}}^{+}(z_m) = \check{\underset{\sim}{\boldsymbol{W}}}^{+}(z_m; z_0)\check{\boldsymbol{P}}^{+}(z_0) \qquad (10)$$

where

$$\check{\underset{\sim}{\boldsymbol{W}}}^{+}(z_m; z_0) = \underset{\sim}{\boldsymbol{\Gamma}}_w \underset{\sim}{\boldsymbol{W}}^{+}(z_m; z_0)\underset{\sim}{\boldsymbol{\Gamma}}_w^T. \qquad (11)$$

Here, the wavefield extrapolation operator $\underset{\sim}{\boldsymbol{W}}^{+}(z_m; z_0)$ is transformed both at the receiver side (with $\underset{\sim}{\boldsymbol{\Gamma}}_w$) and at the source side (with $\underset{\sim}{\boldsymbol{\Gamma}}_w^T$) independently. Such a transform is generally called the standard wavelet transform of a 2-D operator [2]. The global structure of $\check{\underset{\sim}{\boldsymbol{W}}}^{+}(z_m; z_0)$ for a homogeneous medium is shown in Figure 2. The full matrix consists of 256 x 256 points. It describes extrapolation in the wavelet domain from depth level z_0 = 0 m to z_m = 800 m. So, it is a non-recursive extrapolation operator. In wavelet terms it consists of a coarse approximation $(a_r^J - a_s^J)$, which gives the response of coarse approximation sources measured by coarse approximation receivers (a 2 x 2 matrix!), and it consists further of all combinations of approximations and details of sources and receivers. It is important to notice that the 2-D wavelet transform divides an operator into an approximation and a number of details. The coarse approximation, which consists of aspects with a low wavenumber content, can now be used to get a first impression of the quality of the macro model.
The forward and inverse wavelet transform have been defined and now we can write the data model of equation **(1)** in the wavelet domain by applying the operators $\underset{\sim}{\boldsymbol{\Gamma}}_w$ and $\underset{\sim}{\boldsymbol{\Gamma}}_w^T$, which yields

$$\check{\underset{\sim}{\boldsymbol{P}}}^{-}(z_0) = \check{\underset{\sim}{\boldsymbol{W}}}^{-}(z_0; z_m)\check{\underset{\sim}{\boldsymbol{R}}}(z_m)\check{\underset{\sim}{\boldsymbol{W}}}^{+}(z_m; z_0)\check{\underset{\sim}{\boldsymbol{S}}}^{+}(z_0). \qquad (12)$$

With this data model in the wavelet domain, we can derive a redatuming scheme in the wavelet domain, which is comparable with the scheme of equation (2)

$$\underline{\hat{R}}(z_m) = \left[\underline{\hat{W}}^{-}(z_0; z_m)\right]^H \underline{\hat{P}}^{-}(z_0) \left[\underline{\hat{W}}^{+}(z_m; z_0)\right]^H. \qquad (13)$$

The important difference with redatuming in the space domain is the fact that in the wavelet domain redatuming may be carried out per scale experiment, whereas in the space domain redatuming may be carried out per point source experiment. In the examples it will be shown that the division into scale experiments is very advantageous.

Examples

In this section the two following aspects of processing in the wavelet domain will be illustrated. Firstly, it will be shown that inverse extrapolation in the wavelet domain gives exactly the same inversely extrapolated wavefield as in the space domain. This feature proves that the wavelet domain is not an approximation, but merely another description of the data. Secondly, it will be shown that only a small part of the extrapolation operators in the wavelet domain Need to be used to get a good image of the subsurface. This feature Illustrates the potential power of the wavelet domain in relation to fast migration algorithms and in relation to macro model estimation.

We have chosen to work with a homogeneous macro model with a source structure consisting of two small exploding reflectors (Figure 3a). The corresponding response at z_0 is shown in Figure 3b. Inverse extrapolation of the response at z_0 to z_m with the full inverse extrapolation operators in the wavelet domain leads to exactly the same wavefield as with the operators in the space domain (Figure 4a). So, no approximations are involved in going from the space domain to the wavelet domain.

The division into scales makes a stepwise approach possible. Each resolution covers the whole aperture (with less points at coarser scales). In the space domain such an elegant division is not possible. A first idea of the quality of a macro model can be obtained by using the coarsest approximation of the extrapolation operators in the wavelet domain (this means a 2 x 2 matrix) in the inverse extrapolation process. In the case of a correct macro model the correct structural information is already obtained (Figure 4b) with such a small part of the operators. The amplitude cross section is worse, because of the fact that at coarse scales the lateral resolution is small. Only the low wavenumber aspects have been taken into account in the coarse scales. By adding detail of the wavefield extrapolation operators the lateral resolution improves (Figure 4c-f). For a 32 x 32 matrix in the wavelet domain (the coarsest approximation and 4 details) the inversely extrapolated wavefield is nearly equal to that obtained with the full operators (256 x 256 matrices) in the space domain.

Conclusions

Two data models have been generated: in the space domain and in the wavelet domain. The redatumed result is independent of the do main chosen. In the wavelet domain redatuming can be carried out per scale experiment. In the example it is shown that this is very advantageous. The coarsest scale gives the structural information of the macro model, while the details improve the lateral resolution.

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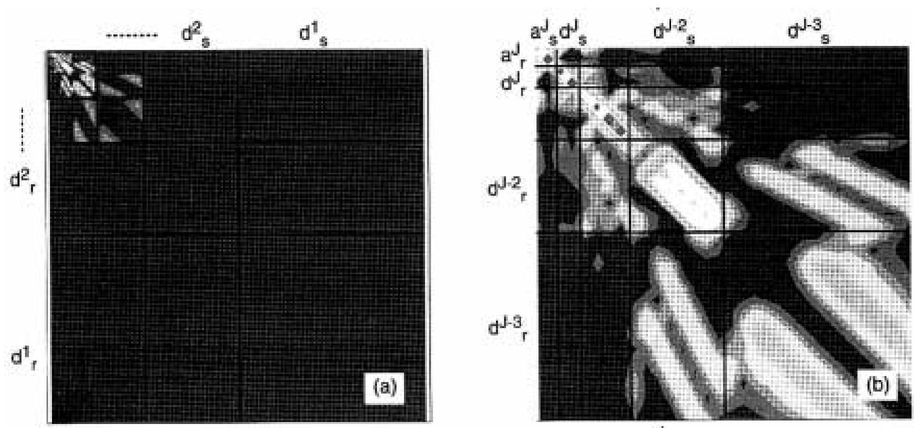

Figure 2: The wavefield extrapolation operator for a homogeneous medium in the wavelet domain for one frequency component. The subscripts "s" and "r" denote the source and receiver side respectively. a) the full matrix; b) a magnification of the top left part of a).

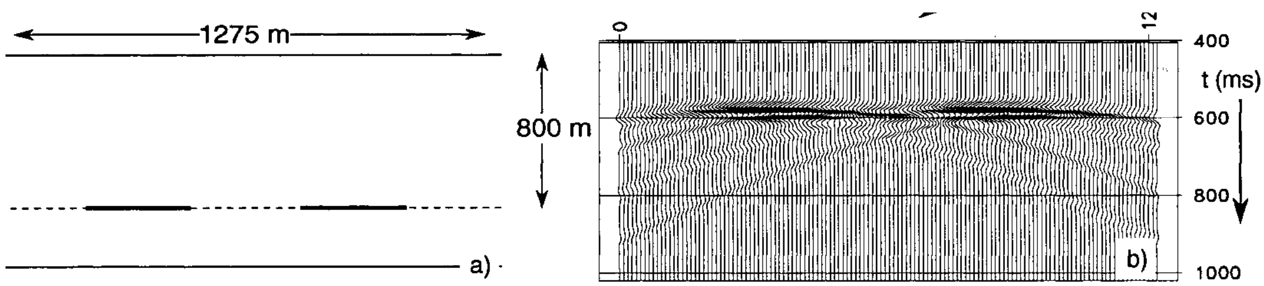

Figure 3: a) Homogeneous macro model with the source positions; b) The response at the surface.

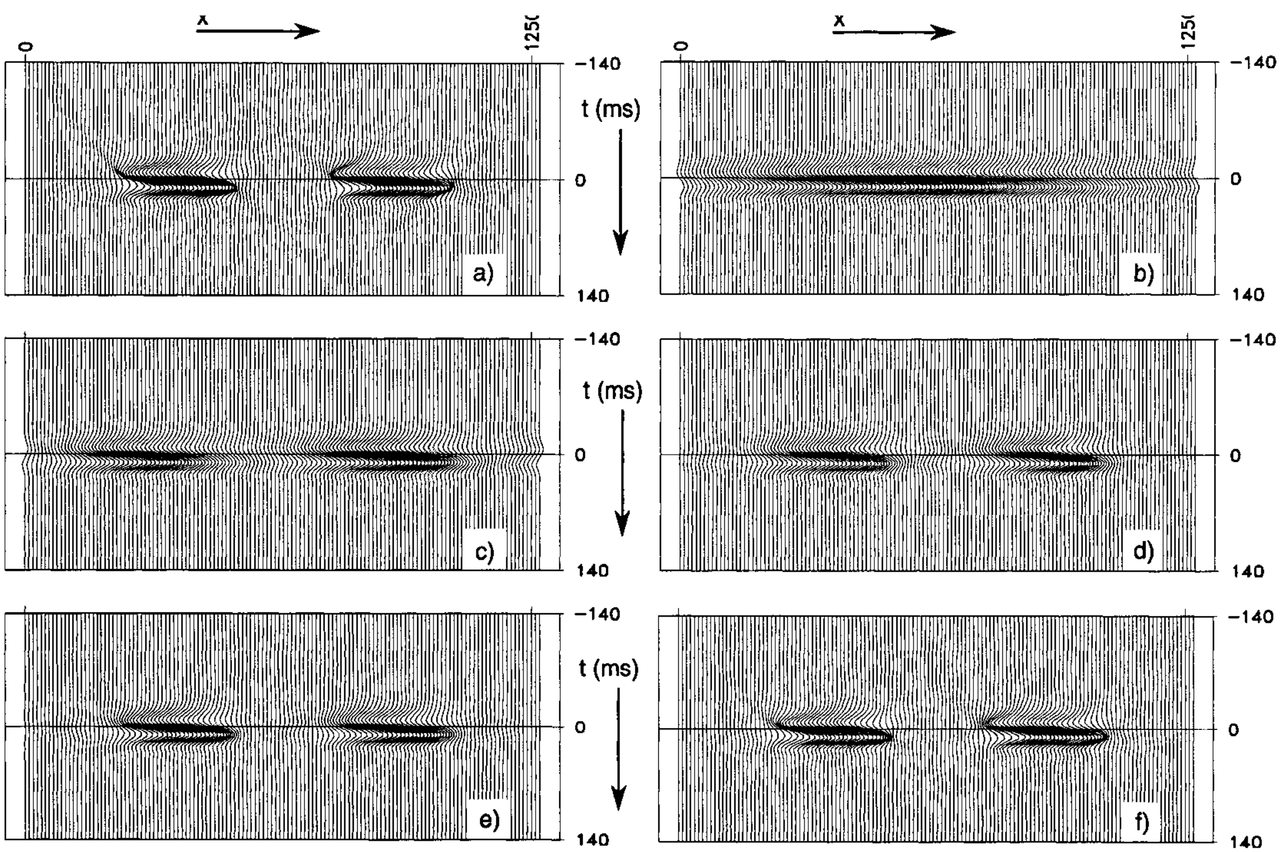

Figure 4: Inversely extrapolated wavefields with different parts of the wavelet transformed wavefield extrapolation operators. a) the full operator (256×256 matrix) in the space and in the wavelet domain; b) the coarsest approximation in the wavelet domain (2×2 matrix); c) the coarsest approximation and the coarsest detail (4×4 matrix); d) the coarsest approximation and the two coarsest details (8×8 matrix); e) the coarsest approximation and the three coarsest details (16×16 matrix); f) the coarsest approximation and the four coarsest details (32×32 matrix). Note that the structural information is obtained already in the coarsest approximation and that the details improve the resolution.