Acoustic and Elastic Kirchhoff Integrals for Primary Wave Field Extrapolation

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Summary

The conventional Kirchhoff integral is based on the two-way wave equation. It formulates how the acoustic pressure in a point A inside a closed surface S can be calculated when the acoustic wave field is known on S. In this paper we decompose the two-way wave field on S into one-way wave fields. Next, by applying the one-way wave equations on S, the conventional Kirchhoff integral is considerably simplified. The resulting expressions describe forward extrapolation of primary waves through arbitrarily inhomogeneous fluids. The theory is also extended for the full elastic case, yielding expressions for forward extrapolation of primary P- and S-waves through arbitrarily inhomogeneous, anisotropic solids. Surprisingly, these full elastic one-way Kirchhoff integrals have the same simple form as the acoustic one-way Kirchhoff integral. For sampled wave fields, the one-way Kirchhoff integrals (acoustic and full elastic) can be represented as a matrix operator acting on a data vector. This matrix operator and its inverse play an important role in a new full elastic seismic processing scheme as proposed by Berkhout and Wapenaar (1988b).

Introduction

In the seismic practice, migration and inversion are largely carried out by one-way algorithms. Two main reasons can be mentioned: 1) In one-way algorithms source strength and source signature need *not* to be known, 2) in one-way algorithms the Green's function is relatively *simple* as primary energy is considered only. As a consequence, one-way algorithms are robust and economically attractive. In this paper the theoretical basis of one-way algorithms, the one-way version of the Kirchhoff integral, is considered for acoustic and elastic data.

Acoustic Kirchhoff integral for two-way wave field extrapolation

Consider the configuration in Figure 1, which represents an arbitrarily inhomogeneous acoustic volume V, enclosed by a surface S with an outward pointing normal vector $\vec{n}$. Assuming that the sources for the acoustic wave field P are outside S, then the <u>two-way</u> wave field at A inside S can be obtained from the two-way wave field on S by means of the Kirchhoff integral:

$$P_A = -\oint_S \left[P \frac{1}{\rho} \nabla G - G \frac{1}{\rho} \nabla P \right] . \vec{n}\, dS , \quad (1)$$

where the Green's function G represents a two-way wave field related to a source at A. For the seismic situation this closed surface integral is often replaced by an integral over a flat surface S_o at depth z_o. In that case the inhomogeneous acoustic volume V covers the entire lower half-space $z \geq z_o$. To simplify the Kirchhoff integral the two-way Green's functions are generally chosen such that either $\partial G/\partial z$ or G vanishes at z_o. This can be accomplished by choosing for the Green's function a rigid or a free surface at z_o. This approach has practical value only for media with small contrasts so that the two-way Green's functions do not contain significant multiple reflections. Here we follow an entirely different approach. We assume that at z_o the acoustic wave field may be written as the sum of downgoing and upgoing waves, according to

$$P = P^+ + P^- \text{ at } z_o, \quad (2a)$$

where P^+ represents the incident wave field (including higher order terms), related to sources above z_o, and where P^- represents the scattered wave field (including higher order terms), related to inhomogeneities below z_o, see also Figure 2a. Similarly, for the Green's function we assume

$$G = G^+ + G^- \text{at } z_o, \quad (2b)$$

where G^- represents the incident Green's wave field (including higher order terms), related to a Green's source at A below z_o, and where G^+ represents the scattered Green's wave field, (including higher order terms) related to inhomogeneities above z_o, see also Figure 2b.

With these assumptions, the Kirchhoff integral may be written as

$$P_A = \iint_{-\infty}^{+\infty} \Big[(P^+ + P^-) \frac{1}{\rho} \frac{\partial}{\partial z} (G^+ + G^-) - (G^+ + G^-) \frac{1}{\rho} \frac{\partial}{\partial z} (P^+ + P^-) \Big]_{z_o} dx\, dy . \quad (3)$$

In this paper we derive simple expressions for <u>primary</u> wave field extrapolation through inhomogeneous acoustic media, using equation (3), with the <u>one-way</u> Green's functions, as a starting point.

Following a similar approach, we also derive simple expressions for extrapolation of primary P- and S-waves through inhomogeneous, anisotropic elastic media.

Acoustic Kirchhoff integrals for primary wave field extrapolation.

Using the one-way wave equations for downgoing and upgoing waves, equation (3) can be simplified considerably. A general derivation is presented by Berkhout and Wapenaar (1988a). Here we make the simplifying assumption that the medium parameters ρ (mass density) and c (propagation velocity) do not vary laterally at z_o. Consider the 2-D version of Parseval's theorem

$$\iint_{-\infty}^{+\infty} A(x,y)\, B(x,y)\, dx\, dy \; = \; \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{+\infty} \tilde{A}(k_x,k_y)\, \tilde{B}(-k_x,-k_y)\, dk_x\, dk_y \; . \quad (4)$$

Applying this theorem to equation (3) and substituting the one-way wave equations

$$\frac{\partial}{\partial z}(\tilde{P}^+ + \tilde{P}^-) = -jk_z(\tilde{P}^+ - \tilde{P}^-) \quad \text{at } z_o \quad (5a)$$

and

$$\frac{\partial}{\partial z}(\tilde{G}^+ + \tilde{G}^-) = -jk_z(\tilde{G}^+ - \tilde{G}^-) \quad \text{at } z_o \quad (5b)$$

with

$$k_z = \sqrt{\frac{\omega^2}{c^2(z_o)} - k_x^2 - k_y^2} \quad , \quad (5c)$$

yields

$$P_A = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{+\infty} \Big[-(\tilde{P}^+ + \tilde{P}^-)\frac{1}{\rho}\, jk_z\,(\tilde{G}_o^+ - \tilde{G}_o^-) + (\tilde{G}_o^+ + \tilde{G}_o^-)\frac{1}{\rho}\, jk_z\,(\tilde{P}^+ - \tilde{P}^-)\Big]_{z_o} dk_x\, dk_y \; , \quad (6a)$$

where the sub-script "o" denotes that k_x and k_y have been replaced by $-k_x$ and $-k_y$, respectively. This equation can be simplified to

$$P_A = 2\left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{+\infty} \Big[\tilde{P}^+ \frac{1}{\rho}\, jk_z \tilde{G}_o^- - \tilde{P}^- \frac{1}{\rho}\, jk_z \tilde{G}_o^+ \Big]_{z_o} dk_x\, dk_y \; , \quad (6b)$$

or, applying once more Parseval's theorem

$$P_A = 2 \iint_{-\infty}^{+\infty} \Big[P^+ \frac{1}{\rho}\frac{\partial G^-}{\partial z} + P^- \frac{1}{\rho}\frac{\partial G^+}{\partial z} \Big]_{z_o} dx\,dy \; . \quad (6c)$$

Note the interesting property that this simplified Kirchhoff integral contains only two of the eight terms of Kirchhoff integral (3). The second term in (6c) may be omitted if we choose for the Green's function a reflection free upper half space, so that $G^+ = 0$ at z_o. This choice is always justified, even when the actual upper half space is not reflection free: outside volume V (the lower half space) P and G need not necessarily satisfy the same wave equations. Finally, when we take for G^- only the primary arrivals at z_o, than (6c) may be rewritten as

$$P_A^+ = 2 \iint_{-\infty}^{+\infty} \Big[P^+ \frac{1}{\rho}\frac{\partial G^-}{\partial z} \Big]_{z_o} dx\, dy \; . \quad (7)$$

This integral describes forward extrapolation of the primary downgoing wave P^+ from surface z_o, through an arbitrarily inhomogeneous medium, to any point A below z_o.

Elastic Kirchhoff integrals for primary wave field extrapolation

Consider the elastic Kirchhoff integral for inhomogeneous, anisotropic solids

$$\Omega_A = \iint_{-\infty}^{+\infty} \Big[\vec{\Gamma}_z \cdot \vec{U} - \vec{G} \cdot \vec{\tau}_z \Big]_{z_o} dx\, dy \; . \quad (8)$$

Here $\vec{U}$ and $\vec{\tau}_z$ represent the two-way elastic wave field at z_o in terms of the displacement and the traction, respectively. The sources for $\vec{U}$ and $\vec{\tau}_z$ are above z_o. $\vec{G}$ and $\vec{\Gamma}_z$ represent the two-way Green's wave field at z_o in terms of the Green's displacement and the Green's traction, respectively. The source for $\vec{G}$ and $\vec{\Gamma}_z$ is at point A below z_o. In the literature this Green's source represents a unit body force at A, acting in the x, y or z direction. Accordingly, Ω_A represents the x, y or z component of the displacement $\vec{U}$ at A (see for example Aki and Richards, 1980). Wapenaar and Haimé (1988) extend this concept. They show that when the Green's source represents a P-wave source or an S-wave source at A, then Ω_A represents a P-wave potential or an S-wave potential for the elastic wave field at A. Let us now, in analogy with (2a) and (2b), express the elastic wave field and the Green's wave field at z_o in terms of one-way potentials for downgoing and upgoing P- and S-waves, according to

P- and S-waves, according to

$$\vec{U} = \frac{1}{\rho\,\omega^2}\left[\nabla(\phi^+ + \phi^-) + \nabla\times(\vec{\psi}^+ + \vec{\psi}^-)\right] \quad \text{at } z_o \tag{9a}$$

and

$$\vec{G} = \frac{1}{\rho\,\omega^2}\left[\nabla(\phi_G^+ + \phi_G^-) + \nabla\times(\vec{\psi}_G^+ + \vec{\psi}_G^-)\right] \quad \text{at } z_o \tag{9b}$$

Substitution in (8) yields a Kirchhoff integral with thirty-two terms. When we follow the same procedure as in the acoustic case we find that equation (8) may be rewritten as (in analogy with equation (7)):

$$\Omega_A^+ = \frac{2}{\omega^2}\iint_{-\infty}^{+\infty}\left[\phi^+ \frac{1}{\rho}\frac{\partial\phi_G^-}{\partial z} + \vec{\psi}^+\cdot\left(\frac{1}{\rho}\frac{\partial\vec{\psi}_G^-}{\partial z}\right)\right]_{z_o} dxdy \tag{10}$$

Note the interesting property that this Kirchhoff integral contains only two of the thirty-two terms of Kirchhoff integral (8). From this equation two simplified versions may be derived. When both the elastic wave field and the Green's wave field are radiated by P-wave sources, then the second term in (10) may be neglected, hence

$$\phi_A^+ \approx \frac{2}{\omega^2}\iint_{-\infty}^{+\infty}\left[\phi^+ \frac{1}{\rho}\frac{\partial\phi_G^-}{\partial z}\right]_{z_o} dxdy \tag{11a}$$

This integral describes forward extrapolation of the primary downgoing P-wave potential from surface z_o, through an arbitrarily inhomogeneous, anisotropic solid, to any point A below z_o. On the other hand, when both the elastic wave field and the Green's wave field are radiated by, for example, SV-wave sources, then the first term in (10a) may be neglected, hence

$$\psi_{y,A}^+ \approx \frac{2}{\omega^2}\iint_{-\infty}^{+\infty}\left[\vec{\psi}^+\cdot\left(\frac{1}{\rho}\frac{\partial\vec{\psi}_G^-}{\partial z}\right)\right]_{z_o} dxdy \tag{11b}$$

This integral describes forward extrapolation of the primary downgoing S-wave potential from surface z_o, through an arbitrarily inhomogeneous, anisotropic solid, to any point A below z_o. Both in (11a) and (11b), the errors are of the same order as the negligence of multiply converted waves.

A matrix notation for sampled wave fields

In practice we are always dealing with sampled wave fields. This means that one-way Kirchhoff integrals (7) for the acoustic case as well as (11a) and (11b) for the full elastic case may be replaced by summations.

In the following we consider the 2-D situation and we choose for A any point at depth level z_A below z_o. Then the summations can be elegantly represented by the following matrix notation:

$$\vec{P}^+(z_A) = W^+(z_A, z_o)\vec{P}^+(z_o)\ . \tag{12a}$$

The elements of vector $\vec{P}^+(z_o)$ represent the (monochromatic) sampled downgoing wave field P^+, ϕ^+ or Ψ_y^+ at z_o. Similarly, the elements of $\vec{P}^+(z_A)$ represent the sampled downgoing wave field P^+, ϕ^+ or Ψ_y^+ at z_A. The elements of the i'th row of matrix $W^+(z_A, z_o)$ represent the sampled Green's wave field $(2\,\Delta x\,\rho^{-1})\partial_z G^-$ or
$(2\,\Delta x\,\rho^{-1}\omega^{-2})\partial_z\phi_G^-$ or
$(2\,\Delta x\,\rho^{-1}\omega^{-2})\partial_z\psi^-_{G,y}$ at z_o related to a Green's source at (x_i, z_A).

Hence, the simple matrix product (12a) may represent forward extrapolation from depth z_o to depth z_A of either

- the primary downgoing acoustic wave field P^+, or
- the primary downgoing P-wave potential ϕ^+, or
- the primary downgoing SV-wave potential Ψ_y^+.

A similar expression can be found for forward extrapolation from z_A to z_o of primary upgoing waves P^-, ϕ^- or Ψ_y^-:

$$\vec{P}^-(z_o) = W^-(z_o, z_A)\vec{P}^-(z_A)\ . \tag{12b}$$

In the same notation, the expressions for inverse extrapolation read

$$\vec{P}^+(z_o) = F^+(z_o, z_A)\ \vec{P}^+(z_A) \tag{13a}$$

and

$$\vec{P}^-(z_A) = F^-(z_A, z_o)\ \vec{P}^-(z_o)\ , \tag{13b}$$

where

$$F^+(z_o, z_A) \triangleq \left[W^+(z_A, z_o)\right]^{-1} \approx \left[W^-(z_o, z_A)\right]^* \tag{14a}$$

and

$$F^-(z_A, z_o) \triangleq \left[W^-(z_o, z_A)\right]^{-1} \approx \left[W^+(z_A, z_o)\right]^*\ , \tag{14b}$$

where * denotes complex conjugation.

Discussion and conclusions

By splitting the wave fields at z_o into downgoing and upgoing waves we have derived one-way versions of the Kirchhoff integrals, both for inhomogeneous acoustic media and for inhomogeneous, anisotropic elastic media. In the acoustic case the one-way Kirchhoff integral describes forward extrapolation of the primary acoustic pressure field; in the full elastic case the one-way Kirchhoff integrals describe forward extrapolation of either the primary P-wave potential or the primary S-wave potential. It is interesting to note that the elastic one-way Kirchhoff integrals (11a) and (11b) are not more complicated than the acoustic one-way Kirchhoff integral (7). Of course the Green's functions are defined differently in (7), (11a) or (11b). In practice they should be found by means of numerical modeling. Examples of acoustic and elastic extrapolation of primary waves through inhomogeneous media will be discussed during the presentation.

We discussed an elegant matrix representation for both forward and inverse extrapolation of sampled downgoing and upgoing waves. The notation holds for inhomogeneous acoustic media as well as for inhomogeneous, anisotropic full elastic media. These simple matrix equations play a key role in the new full elastic seismic processing scheme, as proposed by Berkhout and Wapenaar (1988b).

References

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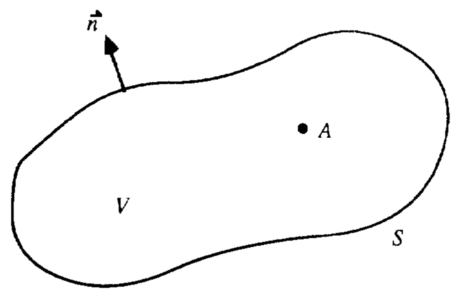

FIG. 1. Kirchhoff integral (1) states that the *two-way* wave field at A inside closed surface S can be computed from two-way wave field on S, provided volume V enclosed by S is source free.

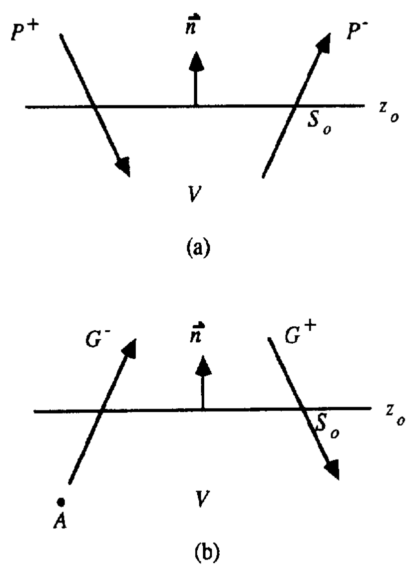

FIG. 2. In this paper, we derive *one-way* versions of Kirchhoff integral. Underlying principle is that we decompose two-way wave fields at z_0 into downgoing and upgoing waves. (a) Situation shown for acoustic pressure (sources above z_0), (b) situation shown for Green's wave field (source at A below z_0).