A Frequency-Domain Formulation of Linearized Inversion Problems

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SUMMARY

A review is given on linear trace inversion, seismic migration and nonlinear multi-dimensional inversion. Basic expressions are given for the inversion operators involved. In particular, spectral versions of those expressions are discussed. It is argued that a spectral presentation is invaluable for the evaluation of inversion results.
The spectral role of "initial estimates" in the inversion process is discussed. It is shown that for spectral components of the seismic data with a high signal-to- noise ratio inversion results are largely determined by the data; for spectral components of the seismic data with a low signal-to-noise ratio inversion results are largely determined by the initial estimate. It is concluded that in practical situations the amount of detail that can be extracted from seismic data depends on the maximum available wave numbers k_x, k_y and k_z; the amount of trend that can be extracted from seismic data depends on the minimum available wave number k_z. Detail and trend outside the above limits can never be extracted from the seismic data by (multi-dimensional) linearized inversion techniques, whether they are carried out in a recursive or nonrecursive way, but should be included as prior information in the initial estimate.
During the presentation, the importance of the data acquisition geometry will be illustrated with examples.

INTRODUCTION

Today, inversion is the most important subject in seismic research and development. Seismic inversion procedures have been proposed in terms of deconvolution, trace inversion, migration and inverse scattering. The algorithms being used may be linear or nonlinear. A very important aspect in seismic inversion is the uniqueness of the inversion result. Particularly, in recursive techniques the question whether we converge to the correct solution or not formulates one of the most difficult and important problems.

In this paper the solution of linear and nonlinear inversion is critically investigated. It is argued that spectral properties of measurements are an essential issue in the evaluation of solutions of linearized inversion problems.

INVERSION FOR THE TEMPORAL WAVELET

Consider the discrete time model of a seismic trace

$$\vec{p} = \mathbf{S}\vec{r} + \vec{n}, \tag{1}$$

where $\vec{r}$ = reflectivity vector,
$\mathbf{S}$ = seismic wavelet matrix,
$\vec{n}$ = noise vector,
$\vec{p}$ = data vector (in terms of pressure).

Using the data vector $\vec{p}$ and an initial estimate of $\vec{r}$, indicated by $\vec{r}_0$, we want to find a linear combination of $\vec{p}$ and $\vec{r}_0$ that determines $\vec{r}$ in some optimum sense:

$$\langle\vec{r}\rangle = \mathbf{F}\vec{p} + \mathbf{G}\vec{r}_0 \tag{2a}$$

or making use of (1),

$$\langle\vec{r}\rangle = [\mathbf{F}\,\mathbf{S}]\vec{r} + \mathbf{F}\vec{n} + \mathbf{G}\vec{r}_0$$

or, defining $\mathbf{H} = \mathbf{F}\,\mathbf{S}$,

$$\langle\vec{r}\rangle = \mathbf{H}\vec{r} + \mathbf{G}\vec{r}_0 + \mathbf{F}\vec{n}. \tag{2b}$$

Using the least squares criterion, it can be easily shown that

$$\mathbf{F} = \left[\mathbf{S}^T\mathbf{C}_{n,n}^{-1}\mathbf{S} + \mathbf{C}_{r,r}^{-1}\right]^{-1}\mathbf{S}^T\mathbf{C}_{n,n}^{-1} \tag{3a}$$

and

$$\mathbf{G} = \left[\mathbf{S}^T\mathbf{C}_{n,n}^{-1}\mathbf{S} + \mathbf{C}_{r,r}^{-1}\right]^{-1}\mathbf{C}_{r,r}^{-1} \tag{3b}$$

or, in stationary situations,

$$\mathbf{F} = \left[\mathbf{S}^T\mathbf{S} + \mathbf{C}_{n,n}\mathbf{C}_{r,r}^{-1}\right]^{-1}\mathbf{S}^T \tag{3c}$$

$$\mathbf{G} = \left[\mathbf{S}^T\mathbf{S} + \mathbf{C}_{n,n}\mathbf{C}_{r,r}^{-1}\right]^{-1}\mathbf{C}_{n,n}\mathbf{C}_{r,r}^{-1}\ . \tag{3d}$$

From (3) we may derive the important relation

$$\mathbf{G} = \mathbf{I} - \mathbf{FS}.$$

Hence using $\vec{p}_0 = \mathbf{S}\vec{r}_0$, the least-squares solution of (2) can be formulated as

$$<\vec{r}> = \vec{r}_0 + \mathbf{F}(\vec{p} - \vec{p}_0) \quad (4a)$$

and

$$<\Delta\vec{r}> = \underbrace{\mathbf{G}(\vec{r}_0 - \vec{r})}_{\text{bias}} + \underbrace{\mathbf{F}\vec{n}.}_{\text{variance}} \quad (4b)$$

If $\vec{r}$ is defined on a regular grid, expressions (4) can also be formulated in the frequency domain

$$F(\omega) = \frac{S^*(\omega)}{|S(\omega)|^2 + R'_{n,n}(\omega)}\ ;\ G(\omega) = \frac{R'_{n,n}(\omega)}{|S(\omega)|^2 + R'_{n,n}(\omega)} \quad (5)$$

where $R'_{n,n}(\omega) = R_{n,n}(\omega)/R_{r,r}(\omega)$ and $G(\omega) = 1 - F(\omega)S(\omega)$.
In expressions (5) $S(\omega)$ is the Fourier spectrum of the seismic wavelet, $R_{n,n}(\omega)$ is the noise power spectrum and $R_{r,r}(\omega)$ is the reflectivity power spectrum.
Using (5) we may write

$$<R(\omega)> = \frac{|S(\omega)|^2 R(\omega) + R'_{n,n}(\omega) R_0(\omega) + S^*(\omega) N(\omega)}{|S(\omega)|^2 + R'_{n,n}(\omega)} \quad (6a)$$

$$\Delta R(\omega) = \frac{R_{n,n}(\omega)[R_0(\omega) - R(\omega)] + S^*(\omega) N(\omega)}{|S(\omega)|^2 + R'_{n,n}(\omega)}\ . \quad (6b)$$

From expressions (6) we may conclude that (fig. 1):

1. $<R(\omega)> \to R(\omega)$ for $R_{n,n}(\omega) << |S(\omega)|^2 R_{r,r}(\omega)$
   (solution is determined by the data)

2. $<R(\omega)> \to R_0(\omega)$ for $R_{n,n}(\omega) >> |S(\omega)|^2 R_{r,r}(\omega)$
   (solution is determined by the initial estimate).

Hence for frequencies with $S(\omega)=o$ the data has no influence on the solution. Note that the low frequency part of range I defines the missing trend information in a seismic trace (along the time axis); the high frequency part of range I defines the missing detail (along the time axis).

III. INVERSION FOR THE SPATIAL WAVELET

Consider the monochromatic discrete space model of zero-offset seismic data (2D version):

$$\vec{P}(z_0) = S \sum_{i=1}^{M} \mathbf{W}(z_0, z_i)\vec{R}(z_i) + \vec{N}(z_0), \quad (7)$$

where $\vec{R}$ = zero-offset reflectivity vector,
$\mathbf{W}$ = zero-offset propagation matrix,
S = Fourier component of seismic wavelet,
$\vec{N}$ = noise vector,
$\vec{P}$ = zero-offset data vector.

Assuming the wavelet Fourier component unity ($S(\omega_i)=1$) and looking for the moment at the response from one depth level only, expression (7) can be reformulated as

$$\vec{P}(z_0) = \mathbf{W}(z_0, z_m)\vec{R}(z_m) + \vec{N}(z_0). \quad (8)$$

Note the complete similarity with discrete time model (1). Hence, the least-squares solution is given by

$$<\vec{R}(z_m)> = \vec{R}_0(z_m) + \mathbf{F}(z_m, z_0)[\vec{P}(z_0) - \vec{P}_0(z_0)] \quad (9)$$

with

$$\mathbf{F} = \left[\mathbf{W}^T \mathbf{C}_{N,N}^{-1} \mathbf{W} + \mathbf{C}_{R,R}^{-1}\right]^{-1} \mathbf{W}^T \mathbf{C}_{N,N}^{-1}\ . \quad (10)$$

In stationary situations $\mathbf{W}$ becomes a space-invariant convolution operator along the x-coordinate and expression (10) may be formulated in the wave number domain

$$\tilde{F}(k_x, \omega_i) = \frac{\tilde{W}^*(k_x, \omega_i)}{|\tilde{W}(k_x, \omega_i)|^2 + \tilde{R}'_{N,N}(k_x, \omega_i)}\ , \quad (11)$$

where $\tilde{R}'_{N,N} = \tilde{R}_{N,N}/\tilde{R}_{R,R}$.

For solution (9) we may write

$$<\tilde{R}> = \frac{|\tilde{W}|^2 \tilde{R} + \tilde{R}'_{N,N}\tilde{R}_0 + \tilde{W}^* \tilde{N}}{|\tilde{W}|^2 + \tilde{R}'_{N,N}}\ . \quad (12)$$

From this expression we may conclude (fig. 2):

1. $<\tilde{R}(k_x,\omega_i)> \rightarrow \tilde{R}(k_x,\omega_i)$ for

$$\tilde{R}_{N,N}(k_x,\omega_i) << |\tilde{W}(k_x,\omega_i)|^2 \tilde{R}_{R,R}(k_x,\omega_i)$$

(solution is determined by the data)

2. $<\tilde{R}(k_x,\omega_i)> \rightarrow \tilde{R}_0(k_x,\omega_i)$ for

$$\tilde{R}_{N,N}(k_x,\omega_i) >> |\tilde{W}(k_x,\omega_i)|^2 \tilde{R}_{R,R}(k_x,\omega_i).$$

(solution is determined by the initial estimate)

Hence for wave numbers with $\tilde{W}(k_x,\omega_i)=o$ the data has no influence on the solution. Note that for frequency component ω_i wave number range I defines the missing lateral detail in the seismic data.

IV. TRACE INVERSION AND MIGRATION

Consider again the model for zero-offset data

$$\vec{P}(z_0) = S \sum_{m=1}^{M} \mathbf{W}(z_0,z_m)\vec{R}(z_m)+\vec{N}(z_0). \qquad (13)$$

Now, if we want to apply broad-band linear inversion for both temporal wavelet S and spatial wavelet $\mathbf{W}$ for all depth levels then figure 3 shows the three areas that contribute differently to the solution (compare with figures 1 and 2). Note that $|k_z|< k_{min}\cos\alpha_{max}$ defines the missing trend and $(|k_z|>k_{max}, |k_x|> k_{max}\sin\alpha_{max}$ defines the missing detail in the seismic data.

In conclusion, any information in the inversion result outside the area $(k_{min}\cos\alpha_{max}, k_{max})$ and $(-k_{max}\sin\alpha_{max}, k_{max}\sin\alpha_{max})$ is fully determined by the prior knowledge and not by the seismic data.

V. NONLINEAR INVERSION

In its most general form, seismic measurements may be represented by the nonlinear discrete model

$$\vec{P} = \vec{M}(\vec{\lambda})+\vec{N}, \qquad (13)$$

where $\vec{\lambda}$ defines the parameters of nonlinear model M, $\vec{N}$ represents the deviations from model M ("the noise") and $\vec{P}$ equals the measured seismic data.

For a given initial estimate $\vec{\lambda}_0$, the simulated seismic data is given by

$$\vec{P}_0 = \vec{M}(\vec{\lambda}_0). \qquad (14)$$

Hence, we may write for the residue

$$\vec{P}-\vec{P}_0 = [\vec{M}(\vec{\lambda}) - \vec{M}(\vec{\lambda}_0)] + \vec{N}. \qquad (15)$$

If we linearize around $\vec{\lambda}_0$,

$$\vec{M}(\vec{\lambda}) = \vec{M}(\vec{\lambda}_0) + \mathbf{A}_0 \Delta\vec{\lambda}_0, \qquad (16a)$$

the residue may be rewritten as

$$\vec{P}-\vec{P}_0 = \mathbf{A}_0 \Delta\vec{\lambda}_0 + \vec{N}. \qquad (16b)$$

Note that coefficient matrix $\mathbf{A}_0$ contains all partial derivatives $\partial P_0(i)/\partial\lambda_0(j)$. Note also the similarity of (16b) with linear expressions (1) and (7).

Hence, the optimum least-squares estimate of parameter update $\Delta\vec{\lambda}_0$ is given by

$$\Delta\vec{\lambda}_0 = \mathbf{F}_0(\vec{P}-\vec{P}_0), \qquad (17a)$$

where

$$\mathbf{F}_0 = \left[\mathbf{A}_0^T \mathbf{C}_{N,N}^{-1} \mathbf{A}_0 + \mathbf{C}_{\lambda,\lambda}^{-1}\right]^{-1} \mathbf{A}_0^T \mathbf{C}_{N,N}^{-1}. \qquad (17b)$$

Hence, we may conclude that by "piece-wise linearization" a recursive inverse procedure may be formulated (n=o,1,...):

$$\vec{\lambda}_{n+1} = \vec{\lambda}_n + \mathbf{F}_n(\vec{P}-\vec{P}_n) \qquad (18a)$$

with

$$\vec{P}_n = \vec{M}(\vec{\lambda}_n), \qquad (18b)$$

where $\mathbf{F}_n$ is the least-squares inverse of $\mathbf{A}_n$, the elements of $\mathbf{A}_n$ being defined by the first derivatives of simulated measurements $\vec{P}_n$ with respect to parameters $\vec{\lambda}_n$. Knowing that in practical situations measurement vector $\vec{P}$ is always defined in a limited area (see fig. 3) and taking into account that $\vec{\lambda}_n$ is a linear operator, we may conclude that in area I parameter vector $\vec{\lambda}_n$ is not updated by the data. Taking also into account that wave propagation is assumed to occur in a linear and time-invariant earth, the simulated data vector $\vec{P}_n$ is also not updated in area I. This leads to the important result that parameter vector $\vec{\lambda}_n$ is not updated at all in area I for any n.

VI. CONCLUSION

Trend and detail ("area I") cannot be recovered from seismic data and should be included as prior information.

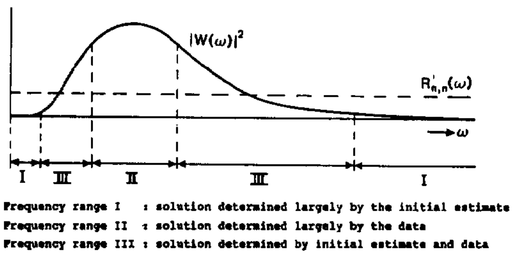

FIG. 1. 3 frequency ranges that contribute differently to the inversion result.

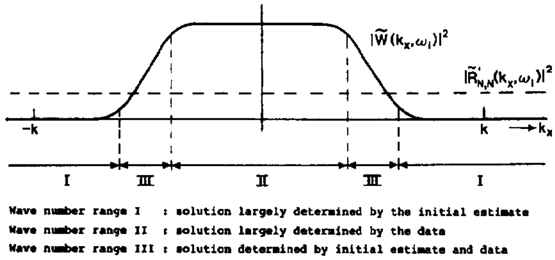

FIG. 2. 3 wavenumber ranges that contribute differently to the inversion result.

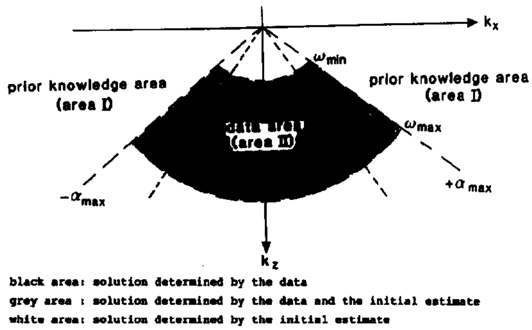

FIG. 3. 3 areas that contribute differently to inversion results.