The propagator and transfer matrix for a 3D inhomogeneous dissipative acoustic medium, expressed in Marchenko focusing functions

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Summary

Standard Marchenko redatuming and imaging schemes neglect evanescent waves and are based on the assumption that decomposition into downgoing and upgoing waves is possible in the subsurface. Recently we have shown that propagator matrices, which circumvent these assumptions, can be expressed in terms of Marchenko focusing functions. In this paper we generalize the relation between the propagator matrix and the Marchenko focusing functions for a 3D inhomogeneous dissipative medium. Moreover, for the same type of medium we discuss a relation between the transfer matrix and the Marchenko focusing functions.

Introduction

Marchenko redatuming and imaging schemes suppress internal multiples in a data-driven way. The nucleus of these schemes is formed by the Marchenko focusing function, comprised of a time-reversed event (similar as the standard primary focusing operator), and a specific multiple coda. Whereas the time-reversed event is defined by a macro subsurface model, the scattering coda is retrieved from the reflection data measured at the surface. The main underlying assumption of most Marchenko schemes is that evanescent waves can be neglected and that decomposition into downgoing and upgoing waves is possible in the subsurface. Recently, several authors suggested ways of circumventing one or more of these assumptions (Kiraz et al., 2021; Diekmann and Vasconcelos, 2021; Wapenaar et al., 2021). Moreover, it has been shown that the Marchenko focusing function is closely related to the propagator matrix concept (Becker et al., 2016; Wapenaar et al., 2017; Elison et al., 2021; Wapenaar and de Ridder, 2022). A propagator matrix (Gilbert and Backus, 1966; Kennett, 1972; Woodhouse, 1974) "propagates" a full wave field (pressure and particle velocity) from one depth level to another through a multilayered medium. It implicitly accounts for downgoing and upgoing, propagating and evanescent waves.

This work is structured as follows. We start by reviewing the propagator matrix and its relation with the Marchenko focusing function. However, we generalize this relation for dissipative media and for all propagating and evanescent waves (in our previous derivation we assumed the medium is lossless and we ignored evanescent waves in the upper half-space; evanescent waves in high velocity layers in the subsurface were already included (Wapenaar and de Ridder, 2022)).

Recently, Dukalski et al. (2022a) discussed a similar relation between the transfer matrix and the Marchenko focusing function. Following their definition, a transfer matrix "transfers" a decomposed wave field (downgoing and upgoing constituents) from one depth level to another through a multilayered medium (Born and Wolf, 1965; Katsidis and Siapkas, 2002). The relation is extended for elastodynamic waves in horizontally layered media in a companion paper (Dukalski et al., 2022b). In the current paper we also address the transfer matrix, but we follow a different approach. First, we express the transfer matrix in terms of the propagator matrix for an arbitrary inhomogeneous dissipative acoustic medium. Next, we use the relation between the propagator matrix and the Marchenko focusing function to obtain a relation between the transfer matrix and the focusing function. We discuss how this relates to the results of Dukalski et al. (2022b).

The propagator matrix for a dissipative medium

Our starting point is the following matrix-vector wave equation in the space-frequency $(\mathbf{x},\omega)$ domain

$$\partial_3\mathbf{q} = \mathbf{A}\,\mathbf{q} + \mathbf{d}, \tag{1}$$

where $\mathbf{q}(\mathbf{x},\omega)$ is a wave-field vector, $\mathbf{A}(\mathbf{x},\omega)$ an operator matrix and $\mathbf{d}(\mathbf{x},\omega)$ a source vector. They are defined as

$$\mathbf{q} = \begin{pmatrix} p \\ v_3 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 & i\omega\rho \\ i\omega\kappa - \frac{1}{i\omega}\partial_\alpha \frac{1}{\rho}\partial_\alpha & 0 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} \hat{f}_3 \\ q \end{pmatrix} \tag{2}$$

(Corones, 1975; Kosloff and Baysal, 1983; Fishman and McCoy, 1984; Wapenaar and Berkhout, 1986). Here $p(\mathbf{x},\omega)$ is the acoustic pressure, $v_3(\mathbf{x},\omega)$ the vertical component of the particle velocity, $\kappa(\mathbf{x},\omega)$ and $\rho(\mathbf{x},\omega)$ are the compressibility and mass density of the medium, $q(\mathbf{x},\omega)$ is the volume injection-rate density and $\hat{f}_3(\mathbf{x},\omega)$ is the vertical component of external force density. Since we consider a dissipative medium, the compressibilty and mass density are complex-valued and frequency-dependent, with $\Im\kappa(\mathbf{x},\omega) \geq 0$ and $\Im\rho(\mathbf{x},\omega) \geq 0$, where $\Im$ denotes the imaginary part. We introduce the propagator matrix $\mathbf{W}(\mathbf{x},\mathbf{x}_R,\omega)$ via

$$\mathbf{q}(\mathbf{x},\omega) = \int_{\partial\mathbb{D}_R} \mathbf{W}(\mathbf{x},\mathbf{x}_R,\omega)\mathbf{q}(\mathbf{x}_R,\omega)\mathrm{d}\mathbf{x}_R, \tag{3}$$

where $\partial\mathbb{D}_R$ is a horizontal boundary at depth level $x_{3,R}$ (Gilbert and Backus, 1966; Kennett, 1972; Woodhouse, 1974). Substitution of equation 3 into equation 1, assuming the source vector $\mathbf{d}$ is zero between $\partial\mathbb{D}_R$ and depth level x_3, yields

$$\partial_3\mathbf{W} = \mathbf{A}\mathbf{W}. \tag{4}$$

Choosing $x_3 = x_{3,R}$ in equation 3, we obtain the boundary condition

$$\mathbf{W}(\mathbf{x},\mathbf{x}_R,\omega)|_{x_3=x_{3,R}} = \mathbf{I}\delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},R}), \tag{5}$$

with horizontal coordinate vectors $\mathbf{x}_\mathrm{H} = (x_1,x_2)$ and $\mathbf{x}_{\mathrm{H},R} = (x_{1,R},x_{2,R})$. $\mathbf{W}$ is a 2×2-matrix, which can be written as follows

$$\mathbf{W}(\mathbf{x},\mathbf{x}_R,\omega) = \begin{pmatrix} W^{p,p} & W^{p,v} \\ W^{v,p} & W^{v,v} \end{pmatrix}(\mathbf{x},\mathbf{x}_R,\omega). \tag{6}$$

We introduce an adjoint medium, with parameters $\kappa^*(\mathbf{x},\omega)$ and $\rho^*(\mathbf{x},\omega)$, where the asterisk denotes complex conjugation.

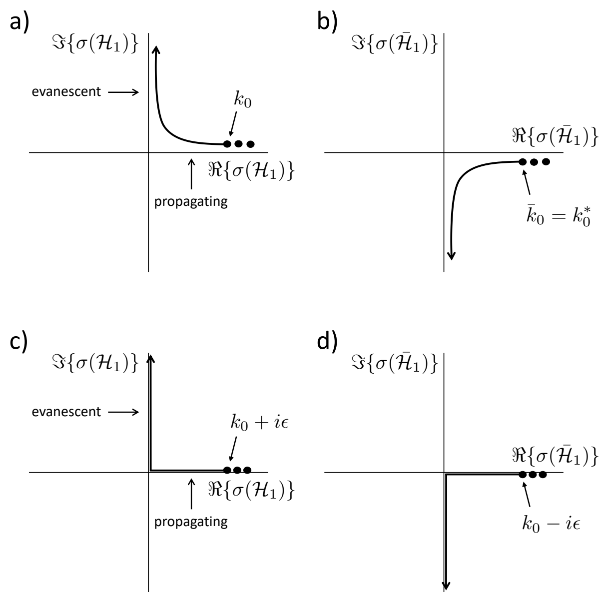

Figure 1: Eigenvalue spectra of operators $\mathscr{H}_1$ and $\bar{\mathscr{H}}_1$ in the complex plane for a dissipative medium (a) and its adjoint (b) and for a lossless medium (c,d).

Waves propagating through the adjoint medium gain energy. We define an operator matrix $\bar{\mathbf{A}}$ for the adjoint medium, which is defined like $\mathbf{A}$ in equation 2, but with $\kappa(\mathbf{x},\omega)$ and $\rho(\mathbf{x},\omega)$ replaced by $\bar{\kappa}(\mathbf{x},\omega) = \kappa^*(\mathbf{x},\omega)$ and $\bar{\rho}(\mathbf{x},\omega) = \rho^*(\mathbf{x},\omega)$, respectively. Similarly, we denote solutions of the wave equation for the adjoint medium with a bar above their symbols, hence, $\bar{\mathbf{W}}(\mathbf{x},\mathbf{x}_R,\omega)$ is the propagator matrix for the adjoint medium. This matrix is related to the propagator matrix of the original (dissipative) medium as follows (Wapenaar, 2022)

$$\begin{pmatrix} \bar{W}^{p,p} & \bar{W}^{p,v} \\ \bar{W}^{v,p} & \bar{W}^{v,v} \end{pmatrix}(\mathbf{x},\mathbf{x}_R,\omega) = \begin{pmatrix} W^{p,p} & -W^{p,v} \\ -W^{v,p} & W^{v,v} \end{pmatrix}^*(\mathbf{x},\mathbf{x}_R,\omega). \tag{7}$$

The propagator matrix expressed in focusing functions

From here onward we assume that $\partial\mathbb{D}_R$ is the acquisition boundary. The medium above $\partial\mathbb{D}_R$ is assumed to be homogeneous and dissipative; the medium below $\partial\mathbb{D}_R$ is arbitrary inhomogeneous, dissipative and source-free. At the boundary $\partial\mathbb{D}_R$ we define $\mathbf{q} = \mathbf{L}\mathbf{p}$, with $\mathbf{q}$ defined in equation 2 and

$$\mathbf{L} = \begin{pmatrix} 1 & 1 \\ \frac{1}{\omega\rho_0}\mathscr{H}_1 & -\frac{1}{\omega\rho_0}\mathscr{H}_1 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} p^+ \\ p^- \end{pmatrix}. \tag{8}$$

Here $\mathscr{H}_1$ is the square-root of the Helmholtz operator $\mathscr{H}_2 = k_0^2 + \partial_\alpha\partial_\alpha$, with $k_0^2 = \omega^2\kappa_0\rho_0$, where $\kappa_0(\omega)$ and $\rho_0(\omega)$ are the complex-valued parameters of the upper half-space. Furthermore, $p^+(\mathbf{x},\omega)$ and $p^-(\mathbf{x},\omega)$ are pressure-normalized downgoing and upgoing waves, respectively. Figure 1(a) shows the eigenvalue spectrum $\sigma(\mathscr{H}_1)$ for a dissipative medium. Figure 1(b) shows the eigenvalue spectrum $\sigma(\bar{\mathscr{H}}_1)$ for the adjoint medium. Note that $\sigma(\bar{\mathscr{H}}_1) = \{\sigma(\mathscr{H}_1)\}^*$. Similar as its spectrum, the operator $\bar{\mathscr{H}}_1$ is related to $\mathscr{H}_1$ via

$$\bar{\mathscr{H}}_1 = \mathscr{H}_1^*. \tag{9}$$

Figures 1(c) and 1(d) show the spectra for the limiting case of a lossless medium. Hence, for this situation the bar only denotes that the amplitudes of the "evanescent" wave modes increase exponentially.

Substituting $\mathbf{q}(\mathbf{x}_R,\omega) = \mathbf{L}(x_{3,R},\omega)\mathbf{p}(\mathbf{x}_R,\omega)$ into equation 3 gives

$$\mathbf{q}(\mathbf{x},\omega) = \int_{\partial\mathbb{D}_R} \mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega)\mathbf{p}(\mathbf{x}_R,\omega)\mathrm{d}\mathbf{x}_R, \tag{10}$$

for $x_3 \geq x_{3,R}$, with $\mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega) = \mathbf{W}(\mathbf{x},\mathbf{x}_R,\omega)\mathbf{L}(x_{3,R},\omega)$, or

$$\mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega) = \begin{pmatrix} W^{p,p} & W^{p,v} \\ W^{v,p} & W^{v,v} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \overleftarrow{\mathscr{H}}_1\frac{1}{\omega\rho_0} & -\overleftarrow{\mathscr{H}}_1\frac{1}{\omega\rho_0} \end{pmatrix}. \tag{11}$$

Here $\overleftarrow{\mathscr{H}}_1(x_{3,R},\omega)$ acts on the quantity left of it (this follows from the fact that $\mathscr{H}_1$ is a symmetric operator). From equation 11 it follows that for the elements of $\mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega)$ we may write

$$Y_{11}(\mathbf{x},\mathbf{x}_R,\omega) = W^{p,p} + \frac{1}{\omega\rho_0}\mathscr{H}_1 W^{p,v}, \tag{12}$$

$$Y_{12}(\mathbf{x},\mathbf{x}_R,\omega) = W^{p,p} - \frac{1}{\omega\rho_0}\mathscr{H}_1 W^{p,v}, \tag{13}$$

$$Y_{21}(\mathbf{x},\mathbf{x}_R,\omega) = W^{v,p} + \frac{1}{\omega\rho_0}\mathscr{H}_1 W^{v,v}, \tag{14}$$

$$Y_{22}(\mathbf{x},\mathbf{x}_R,\omega) = W^{v,p} - \frac{1}{\omega\rho_0}\mathscr{H}_1 W^{v,v}. \tag{15}$$

From equations 5, 6 and 13 we find

$$Y_{12}(\mathbf{x},\mathbf{x}_R,\omega)|_{x_3=x_{3,R}} = \delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},R}), \tag{16}$$

hence, $Y_{12}(\mathbf{x},\mathbf{x}_R,\omega)$ behaves as a focusing function, with its focal point $\mathbf{x}_R$ at $\partial\mathbb{D}_R$. We define

$$Y_{12}(\mathbf{x},\mathbf{x}_R,\omega) = F^p(\mathbf{x},\mathbf{x}_R,\omega), \tag{17}$$

where F^p is the focusing function in terms of acoustic pressure (it is similar to the focusing function f_2 defined in our previous papers in the sense that it focuses at the acquisition boundary, but it is normalised differently, it holds for dissipative media, and it accounts for evanescent waves). Furthermore, we define

$$Y_{22}(\mathbf{x},\mathbf{x}_R,\omega) = F^v(\mathbf{x},\mathbf{x}_R,\omega), \tag{18}$$

where F^v is the particle velocity associated to the focusing function F^p (note that F^v is defined differently from that in Wapenaar (2022)).

Using equations 7 and 9, we find $\bar{Y}_{12}(\mathbf{x},\mathbf{x}_R,\omega) = Y_{11}^*(\mathbf{x},\mathbf{x}_R,\omega)$ and $\bar{Y}_{22}(\mathbf{x},\mathbf{x}_R,\omega) = -Y_{21}^*(\mathbf{x},\mathbf{x}_R,\omega)$, or

$$Y_{11}(\mathbf{x},\mathbf{x}_R,\omega) = \bar{F}^{p*}(\mathbf{x},\mathbf{x}_R,\omega), \tag{19}$$

$$Y_{21}(\mathbf{x},\mathbf{x}_R,\omega) = -\bar{F}^{v*}(\mathbf{x},\mathbf{x}_R,\omega). \tag{20}$$

Hence, for matrix $\mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega)$ we obtain

$$\mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega) = \begin{pmatrix} \bar{F}^{p*} & F^p \\ -\bar{F}^{v*} & F^v \end{pmatrix}(\mathbf{x},\mathbf{x}_R,\omega). \tag{21}$$

Using this in equation 10, with $\mathbf{q}$ and $\mathbf{p}$ defined in equations 2 and 8, we obtain for the first element of $\mathbf{q}$

$$\begin{aligned} p(\mathbf{x},\omega) &= \int_{\partial\mathbb{D}_R} \bar{F}^{p*}(\mathbf{x},\mathbf{x}_R,\omega)p^+(\mathbf{x}_R,\omega)\mathrm{d}\mathbf{x}_R \\ &+ \int_{\partial\mathbb{D}_R} F^p(\mathbf{x},\mathbf{x}_R,\omega)p^-(\mathbf{x}_R,\omega)\mathrm{d}\mathbf{x}_R. \qquad (22) \end{aligned}$$

This expression is almost identical to equation 17 of Wapenaar and de Ridder (2022), except for the bar on F^p in the first integral, which denotes that the focusing function is defined in the adjoint medium. Equation 22 is exact and holds for dissipative media.

If we take a point source of vertical force at $\mathbf{x}_S$ just above $\partial\mathbb{D}_R$, then at $\partial\mathbb{D}_R$ we have $p^+(\mathbf{x}_R,\omega)=\frac{1}{2}\delta(\mathbf{x}_{\mathrm{H},R}-\mathbf{x}_{\mathrm{H},S})$ and $p^-(\mathbf{x}_R,\omega)=\frac{1}{2}R(\mathbf{x}_R,\mathbf{x}_S,\omega)$ (where R is the reflection response), and for $\mathbf{x}$ in the subsurface we have $p(\mathbf{x},\omega)=G^{p,f}(\mathbf{x},\mathbf{x}_S,\omega)$, which is the Green's function in terms of pressure at $\mathbf{x}$ in response to a vertical force source at $\mathbf{x}_S$. Substitution into equation 22 gives

$$\begin{aligned} 2G^{p,f}(\mathbf{x},\mathbf{x}_S,\omega) &= \int_{\partial\mathbb{D}_R} F^p(\mathbf{x},\mathbf{x}_R,\omega)R(\mathbf{x}_R,\mathbf{x}_S,\omega)\mathrm{d}\mathbf{x}_R \\ &+ \bar{F}^{p*}(\mathbf{x},\mathbf{x}_S,\omega). \qquad (23) \end{aligned}$$

This representation is consistent with Slob (2016), but has been derived here without decomposition at $\mathbf{x}$ in the subsurface.

In the limiting case of a lossless medium the bar only affects the behavior of the evanescent waves (as discussed below equation 9). When evanescent waves at $\partial\mathbb{D}_R$ are neglected (and the medium is lossless), the bar can be omitted and equations 22 and 23 reduce to equations 17 and 27 of Wapenaar and de Ridder (2022).

Finally, we express the elements of the propagator matrix in terms of the focusing functions $F(\mathbf{x},\mathbf{x}_R,\omega)$ and $\bar{F}(\mathbf{x},\mathbf{x}_R,\omega)$. Using $\mathbf{W}(\mathbf{x},\mathbf{x}_R,\omega)=\mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega)\mathbf{L}^{-1}(x_{3,R},\omega)$, we obtain

$$W^{p,p}(\mathbf{x},\mathbf{x}_R,\omega)=\frac{1}{2}\big(\bar{F}^{p*}+F^p\big)(\mathbf{x},\mathbf{x}_R,\omega), \qquad (24)$$

$$W^{p,v}(\mathbf{x},\mathbf{x}_R,\omega)=\frac{\omega\rho_0}{2}\mathscr{H}_1^{-1}\big(\bar{F}^{p*}-F^p\big)(\mathbf{x},\mathbf{x}_R,\omega), \qquad (25)$$

$$W^{v,p}(\mathbf{x},\mathbf{x}_R,\omega)=\frac{1}{2}\big(-\bar{F}^{v*}+F^v\big)(\mathbf{x},\mathbf{x}_R,\omega), \qquad (26)$$

$$W^{v,v}(\mathbf{x},\mathbf{x}_R,\omega)=-\frac{\omega\rho_0}{2}\mathscr{H}_1^{-1}\big(\bar{F}^{v*}+F^v\big)(\mathbf{x},\mathbf{x}_R,\omega). \qquad (27)$$

We illustrate equation 24 for the horizontally layered lossless medium of Figure 2(a). We define the spatial Fourier transform of $u(\mathbf{x},\omega)$ along the horizontal coordinate $\mathbf{x}_\mathrm{H}$ as

$$\tilde{u}(\mathbf{s},x_3,\omega)=\int_{\mathbb{R}^2}\exp\{-i\omega\mathbf{s}\cdot\mathbf{x}_\mathrm{H}\}u(\mathbf{x}_\mathrm{H},x_3,\omega)\mathrm{d}^2\mathbf{x}_\mathrm{H}, \qquad (28)$$

where $\mathbf{s}=(s_1,s_2)$ is the horizontal slowness vector. Next, the inverse temporal transform is defined as

$$u(\mathbf{s},x_3,\tau)=\frac{1}{\pi}\Re\int_0^\infty \tilde{u}(\mathbf{s},x_3,\omega)\exp\{-i\omega\tau\}\mathrm{d}\omega, \qquad (29)$$

where $\Re$ denotes the real part (Stoffa, 1989). We apply these transforms to $F^p(\mathbf{x},\mathbf{x}_R,\omega)$, choosing the focal point at $\mathbf{x}_R =$

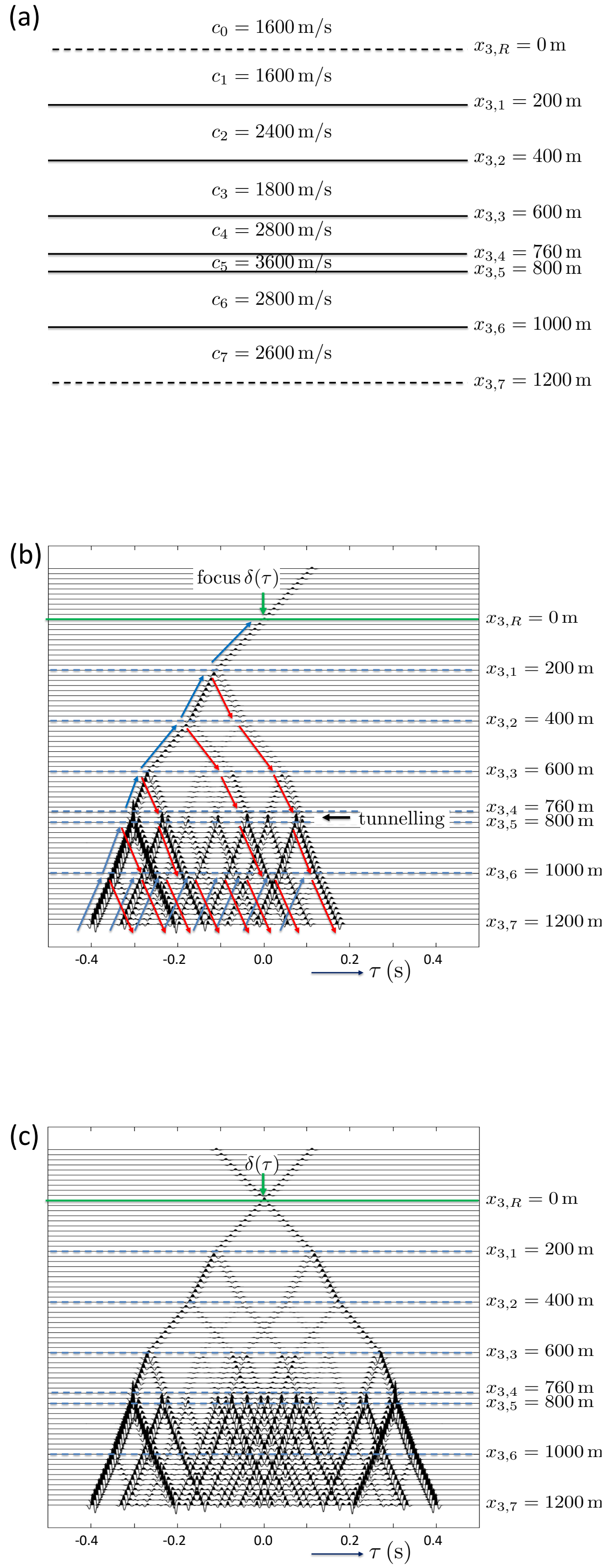

Figure 2: (a) Horizontally layered medium. (b) Focusing function $F^p(s_1,x_3,x_{3,R},\tau)$ (for fixed s_1). (c) Propagator matrix element $W^{p,p}(s_1,x_3,x_{3,R},\tau)$ (for fixed s_1).

$(0,0,x_{3,R})$ and setting $s_2=0$. This yields the transformed focusing function $F^p(s_1,x_3,x_{3,R},\tau)$, which is shown in Figure 2(b) for fixed $s_1=1/3500$ s/m and convolved with a Ricker wavelet with a central frequency of 50 Hz. Starting at the bottom, we observe upgoing waves (blue arrows), which are tuned such that a single wave propagates upward through the upper layer and focuses at depth level $x_{3,R}$ at $\tau=0$. After that, it continues as an upgoing wave into the upper half-space. Note that the focusing function is evanescent in the high velocity layer between 760 and 800 m, through which it tunnels. Equation 24 transforms to

$$W^{p,p}(s_1,x_3,x_{3,R},\tau)=\frac{1}{2}\Big(F^p(s_1,x_3,x_{3,R},-\tau)+F^p(s_1,x_3,x_{3,R},\tau)\Big). \quad (30)$$

This element of the propagator matrix is shown in Figure 2(c).

The transfer matrix for a dissipative medium

Analogous to equation 3, we introduce the transfer matrix $\mathbf{T}(\mathbf{x},\mathbf{x}_R,\omega)$ via

$$\mathbf{p}(\mathbf{x},\omega)=\int_{\partial\mathbb{D}_R}\mathbf{T}(\mathbf{x},\mathbf{x}_R,\omega)\mathbf{p}(\mathbf{x}_R,\omega)\mathrm{d}\mathbf{x}_R. \quad (31)$$

Hence, the transfer matrix "transfers" the downgoing and upgoing waves p^+ and p^- from acquisition boundary $\partial\mathbb{D}_R$ to an arbitrary depth level in the subsurface. Usually, this matrix is analysed for horizontally layered media and it is built up recursively from interface to interface. This is also the approach followed by Dukalski et al. (2022a,b), who in addition express the transfer matrix in terms of Marchenko focusing functions. Here we follow a different approach. Substituting $\mathbf{q}=\mathbf{Lp}$ into equation 3 we obtain equation 31, with

$$\mathbf{T}(\mathbf{x},\mathbf{x}_R,\omega)=\mathbf{L}^{-1}(\mathbf{x},\omega)\mathbf{W}(\mathbf{x},\mathbf{x}_R,\omega)\mathbf{L}(x_{3,R},\omega), \quad (32)$$

where

$$\mathbf{L}^{-1}(\mathbf{x},\omega)=\frac{1}{2}\begin{pmatrix}1 & \omega\mathscr{H}_1^{-1}(\mathbf{x},\omega)\rho(\mathbf{x},\omega)\\ 1 & -\omega\mathscr{H}_1^{-1}(\mathbf{x},\omega)\rho(\mathbf{x},\omega)\end{pmatrix}. \quad (33)$$

Equation 32 is valid in an arbitrary inhomogeneous dissipative acoustic medium, assuming the operator $\mathbf{L}(\mathbf{x},\omega)$ and its inverse exists inside the medium. This is a more serious assumption than we made in the previous section, where we only assumed that $\mathbf{L}(x_{3,R},\omega)$ and its inverse exists at the boundary $\partial\mathbb{D}_R$, where the medium is homogeneous. Despite this additional restriction, it is worthwhile to analyse the transfer matrix further and relate it to Marchenko focusing functions.

The transfer matrix expressed in focusing functions

Using $\mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega)=\mathbf{W}(\mathbf{x},\mathbf{x}_R,\omega)\mathbf{L}(x_{3,R},\omega)$, we rewrite equation 32 as

$$\mathbf{T}(\mathbf{x},\mathbf{x}_R,\omega)=\mathbf{L}^{-1}(\mathbf{x},\omega)\mathbf{Y}(\mathbf{x},\mathbf{x}_R,\omega). \quad (34)$$

First we analyze the right columns of the matrices at both sides of this equation. Using the definition of $\mathbf{L}^{-1}$ in equation 33 and the right column of $\mathbf{Y}$ in equation 21, we obtain, analogous to $\mathbf{L}^{-1}\mathbf{q}=\mathbf{p}$,

$$\frac{1}{2}\begin{pmatrix}1 & \omega\mathscr{H}_1^{-1}\rho\\ 1 & -\omega\mathscr{H}_1^{-1}\rho\end{pmatrix}\begin{pmatrix}F^p\\ F^v\end{pmatrix}=\begin{pmatrix}F^{p+}(\mathbf{x},\mathbf{x}_R,\omega)\\ F^{p-}(\mathbf{x},\mathbf{x}_R,\omega)\end{pmatrix}. \quad (35)$$

Here $F^{p+}(\mathbf{x},\mathbf{x}_R,\omega)$ and $F^{p-}(\mathbf{x},\mathbf{x}_R,\omega)$ are the downgoing and upgoing parts at $\mathbf{x}$ (pressure-normalized) of the focusing function $F^p(\mathbf{x},\mathbf{x}_R,\omega)$. For the left columns of the matrices at both sides of equation 34 we obtain, using equation 9 and the left column of $\mathbf{Y}$ in equation 21,

$$\frac{1}{2}\begin{pmatrix}1 & \omega\mathscr{H}_1^{-1}\rho\\ 1 & -\omega\mathscr{H}_1^{-1}\rho\end{pmatrix}\begin{pmatrix}\bar{F}^{p*}\\ -\bar{F}^{v*}\end{pmatrix}=$$
$$\frac{1}{2}\begin{pmatrix}1 & -\omega\bar{\mathscr{H}}_1^{-1}\bar{\rho}\\ 1 & \omega\bar{\mathscr{H}}_1^{-1}\bar{\rho}\end{pmatrix}^*\begin{pmatrix}\bar{F}^{p}\\ \bar{F}^{v}\end{pmatrix}^*=\begin{pmatrix}\bar{F}^{p-}(\mathbf{x},\mathbf{x}_R,\omega)\\ \bar{F}^{p+}(\mathbf{x},\mathbf{x}_R,\omega)\end{pmatrix}^*. \quad (36)$$

Combining the obtained columns in one matrix, we obtain

$$\mathbf{T}(\mathbf{x},\mathbf{x}_R,\omega)=\begin{pmatrix}\bar{F}^{p-*}(\mathbf{x},\mathbf{x}_R,\omega) & F^{p+}(\mathbf{x},\mathbf{x}_R,\omega)\\ \bar{F}^{p+*}(\mathbf{x},\mathbf{x}_R,\omega) & F^{p-}(\mathbf{x},\mathbf{x}_R,\omega)\end{pmatrix}. \quad (37)$$

This equation expresses the transfer matrix for an arbitrary inhomogeneous dissipative acoustic medium in terms of decomposed focusing functions of the medium and its adjoint.

For the special case of a horizontally layered medium we apply the spatial Fourier transform of equation 28, choosing the focal point at $\mathbf{x}_R=(0,0,x_{3,R})$. This yields

$$\tilde{\mathbf{T}}(\mathbf{s},x_3,x_{3,R},\omega)=\begin{pmatrix}\tilde{\bar{F}}^{p-*}(-\mathbf{s},x_3,x_{3,R},\omega) & \tilde{F}^{p+}(\mathbf{s},x_3,x_{3,R},\omega)\\ \tilde{\bar{F}}^{p+*}(-\mathbf{s},x_3,x_{3,R},\omega) & \tilde{F}^{p-}(\mathbf{s},x_3,x_{3,R},\omega)\end{pmatrix}.$$

This expression is consistent with Dukalski et al. (2022a,b), when taking into account that their path-reversal operator $\mathscr{P}$ implies (1) taking the adjoint medium (or, in the limiting case of a lossless medium, taking exponentially increasing "evanescent" wave modes), (2) taking the complex conjugate, and (3) changing the sign of the horizontal slowness (for the acoustic case, however, this sign change can be omitted).

Conclusions

We have derived mutual relations between the propagator matrix, the transfer matrix and the Marchenko focusing functions, for an arbitrary inhomogeneous dissipative acoustic medium. The relations are exact, meaning that they hold for all propagating and evanescent wave modes. The relations involving the transfer matrix are restricted by the assumption that up/down decomposition is possible inside the inhomogeneous medium; the relation between the propagator matrix and the focusing function does not rely on this assumption.

The relations support further research on redatuming and imaging methods accounting for internal multiples. On the one hand, when the medium is accurately known, the propagator matrix can be numerically modelled in that medium and its adjoint, from which the focusing functions and transfer matrix can be derived. On the other hand, under specific circumstances the focusing functions can be derived from the single-sided reflection response (for lossless media) or from the double-sided reflection and transmission response (for dissipative media), from which the propagator matrix and transfer matrix can be derived.

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