Three-dimensional Marchenko equation for Green's function retrieval "beyond seismic interferometry"

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SUMMARY

In recent work we showed with heuristic arguments that the Green's response to a virtual source in the subsurface can be obtained from reflection data at the surface. This method is called "Green's function retrieval beyond seismic interferometry", because, unlike in seismic interferometry, no receiver is needed at the position of the virtual source. Here we present a formal derivation of Green's function retrieval beyond seismic interferometry, based on a 3-D extension of the Marchenko equation. We illustrate the theory with a numerical example and indicate the potential applications in seismic imaging and AVA analysis.

INTRODUCTION

With seismic interferometry, virtual sources are created in an unknown medium at positions where there are only receivers. Recently we have introduced a method "beyond seismic interferometry", which does not need receivers at the positions of the virtual sources (Broggini and Snieder, 2012; Broggini et al., 2012; Wapenaar et al., 2012a). We showed with heuristic arguments that the Green's response to a virtual source in the subsurface can be obtained from reflection data at the surface. In addition to the reflection data, the method requires an estimate of the direct arrivals between the virtual source position and the surface. However, a detailed model of the subsurface is not needed. Nevertheless, the Green's function retrieved with this new method properly contains all internal multiple reflections. This method provides the basis for employing internal multiples in seismic imaging, leading to a true-amplitude image, free of ghost images related to internal multiples (Wapenaar et al., 2012b). Recently we presented a first mathematical derivation for Green's function retrieval "beyond interferometry" (Wapenaar et al., 2013). The derivation makes use of a 3-D Marchenko equation. Here we present a more concise derivation, using a shortcut proposed by Slob et al. (2013) for the 1D situation, and we illustrate the theory with a 2D numerical example.

REVIEW OF RECIPROCITY THEOREMS FOR ONE-WAY WAVE FIELDS

An acoustic wave field in the space-time domain is represented by the acoustic pressure $p(\mathbf{x},t)$, where $\mathbf{x}$ is the Cartesian spatial coordinate vector and t denotes time. The temporal Fourier transform is defined as $p(\mathbf{x},\omega)=\int p(\mathbf{x},t)\exp(i\omega t)\mathrm{d}t$, where ω is the angular frequency and i the imaginary unit ($i=\sqrt{-1}$). To keep the notation simple, the same symbol is used for time- and frequency-domain functions (here p). The downward and upward propagating, mutually coupled, constituents of the wave field are denoted by p^+ and p^-, respectively. Throughout this paper we assume that these one-way wave fields are normalized with respect to acoustic power flux. Consider an inhomogeneous lossless medium below an acoustically transparent boundary $\partial\mathbb{D}_0$. The upper half-space above $\partial\mathbb{D}_0$ is homogeneous. The propagation velocity and mass density of the inhomogeneous medium are defined as $c(\mathbf{x})$ and $\rho(\mathbf{x})$, respectively. From here onward, the spatial coordinate vector $\mathbf{x}$ is defined as $\mathbf{x}=(\mathbf{x}_\mathrm{H},x_3)$, in which $\mathbf{x}_\mathrm{H}=(x_1,x_2)$ is the horizontal coordinate vector and x_3 the vertical coordinate; the positive x_3-axis is pointing downward. The boundary $\partial\mathbb{D}_0$ is chosen at depth level $x_3=x_{3,0}$. Coordinates at $\partial\mathbb{D}_0$ are denoted as $\mathbf{x}_0=(\mathbf{x}_\mathrm{H},x_{3,0})$. A second boundary $\partial\mathbb{D}_i$ is defined at an arbitrary depth level $x_3=x_{3,i}$, with $x_{3,i}>x_{3,0}$. Coordinates at $\partial\mathbb{D}_i$ are denoted as $\mathbf{x}_i=(\mathbf{x}_\mathrm{H},x_{3,i})$. The domain enclosed by boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_i$ is denoted as $\mathbb{D}$.

Acoustic reciprocity theorems formulate general relations between two acoustic states in one and the same domain (de Hoop, 1988; Fokkema and van den Berg, 1993). Flux-normalized one-way wave fields obey, in the space-frequency domain, the following reciprocity relation (Wapenaar et al., 2004)

$$\int_{\partial\mathbb{D}_0}\{p_A^+p_B^- - p_A^-p_B^+\}\mathrm{d}\mathbf{x}_0=\int_{\partial\mathbb{D}_i}\{p_A^+p_B^- - p_A^-p_B^+\}\mathrm{d}\mathbf{x}_i, \quad (1)$$

where subscripts A and B refer to two independent states. The underlying assumptions are that there are no sources in the domain $\mathbb{D}$ between $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_i$, and that the medium parameters in $\mathbb{D}$ are the same in states A and B. According to the definitions of $\mathbf{x}_0$ and $\mathbf{x}_i$, the integrations in equation 1 take place at $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_i$, respectively, along the horizontal coordinate vector $\mathbf{x}_\mathrm{H}$. A second reciprocity theorem for one-way wave fields is given by

$$\int_{\partial\mathbb{D}_0}\{(p_A^+)^*p_B^+ - (p_A^-)^*p_B^-\}\mathrm{d}\mathbf{x}_0=\int_{\partial\mathbb{D}_i}\{(p_A^+)^*p_B^+ - (p_A^-)^*p_B^-\}\mathrm{d}\mathbf{x}_i, \quad (2)$$

where the asterisk ($*$) denotes complex conjugation. In addition to the assumptions mentioned above, evanescent waves are neglected in equation 2 at the boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_i$. In the following two sections, the reciprocity theorems of equations 1 and 2 are used to derive relations between fundamental solutions of the wave equation and the Green's function of the inhomogeneous medium below $\partial\mathbb{D}_0$.

FUNDAMENTAL SOLUTIONS

In the derivation of the Marchenko equations for 1-D scattering problems, so-called "fundamental solutions" of the source-free Schrödinger equation play an essential role (Lamb, 1980). For the situation of a localized potential $u(x)$, the fundamental solutions $f_1(x,t)$ and $f_2(x,t)$ of the Schrödinger equation are those solutions that reduce to outgoing waves for $x\to\infty$ and $x\to-\infty$, respectively. Any other solution can be expressed as a linear combination of the two fundamental solutions. The concept of fundamental solutions has been extended for the 3-D situation (Wapenaar et al., 2013). Here a slightly different approach is followed by defining the fundamental solutions in a reference medium instead of in the actual medium, similar as proposed by Slob et al. (2013) for

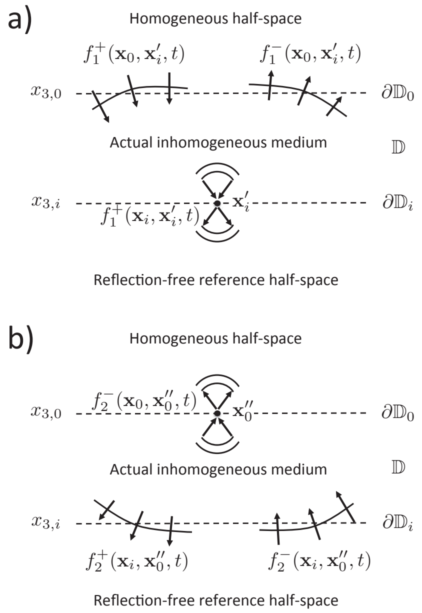

Fig. 1: *Fundamental solutions of the 3-D wave equation in a reference medium, which is equal to the actual medium above $\partial\mathbb{D}_i$ and reflection-free below $\partial\mathbb{D}_i$.*

the 1-D situation. The reference medium is chosen such that it is equal to the actual medium discussed in the previous section above boundary $\partial\mathbb{D}_i$, whereas it is taken reflection-free below this boundary. The first fundamental solution $f_1(\mathbf{x},t)$, with $\mathbf{x}\in\mathbb{R}^3$ and $t\in\mathbb{R}$, consists in the homogeneous upper half-space $x_3\le x_{3,0}$ of a downgoing field $f_1^+(\mathbf{x},t)$ and an upgoing field $f_1^-(\mathbf{x},t)$, with $f_1^+(\mathbf{x},t)$ shaped such that $f_1(\mathbf{x},t)$ focuses at $\mathbf{x}_i'=(\mathbf{x}_{\rm H}',x_{3,i})$ at $t=0$, and continues as a diverging downgoing field $f_1^+(\mathbf{x},t)$ into the reflection-free reference half-space $x_3\ge x_{3,i}$. The focusing condition is formally defined as $f_1(\mathbf{x}_{\rm H},x_3=x_{3,i},t)=\delta(\mathbf{x}_{\rm H}-\mathbf{x}_{\rm H}')\delta(t)$, where $\delta(\mathbf{x}_{\rm H}-\mathbf{x}_{\rm H}')$ and $\delta(t)$ are 2-dimensional and 1-dimensional Dirac delta functions, respectively. For the following derivations it is necessary to consider the lateral position $\mathbf{x}_{\rm H}'$ of the focal point as a variable and therefore it is useful to include the focal point $\mathbf{x}_i'$ in the argument list of the wave fields. Hence, $f_1^{\pm}(\mathbf{x},t)$ is from here onward denoted as $f_1^{\pm}(\mathbf{x},\mathbf{x}_i',t)$ etc., see Figure 1(a). In the frequency domain, the first fundamental solution is written as $f_1(\mathbf{x},\mathbf{x}_i',\omega)=f_1^+(\mathbf{x},\mathbf{x}_i',\omega)+f_1^-(\mathbf{x},\mathbf{x}_i',\omega)$, with $f_1^-(\mathbf{x},\mathbf{x}_i',\omega)=0$ for $x_3\ge x_{3,i}$. The focusing condition reads in the frequency domain

$$f_1(\mathbf{x}_i,\mathbf{x}_i',\omega)=f_1^+(\mathbf{x}_i,\mathbf{x}_i',\omega)=\delta(\mathbf{x}_{\rm H}-\mathbf{x}_{\rm H}'). \tag{3}$$

Note that $f_1(\mathbf{x},\mathbf{x}_i',\omega)$ can only collapse into a delta function at $x_3=x_{3,i}$ when the entire evanescent field is included. Since the evanescent field decays exponentially during downward propagation and should remain finite at $x_3=x_{3,i}$, the illuminating field $f_1^+(\mathbf{x},\mathbf{x}_i',\omega)$ in the upper-half-space $x_3\le x_{3,0}$ should have very high amplitudes, which would make this fundamental solution unstable. To avoid this unstable behaviour, evanescent waves are excluded at the focusing depth level $x_3=x_{3,i}$ (i.e., at boundary $\partial\mathbb{D}_i$). This means that the delta function in equation 3 and in subsequent equations should be interpreted as a spatially band-limited delta function.

The second fundamental solution $f_2(\mathbf{x},t)$ consists in the reflection-free reference half-space $x_3\ge x_{3,i}$ of an upgoing field $f_2^-(\mathbf{x},t)$ and a downgoing field $f_2^+(\mathbf{x},t)$, with $f_2^-(\mathbf{x},t)$ shaped such that $f_2(\mathbf{x},t)$ focuses at $\mathbf{x}_0''=(\mathbf{x}_{\rm H}'',x_{3,0})$ at $t=0$, and continues as a diverging upgoing field $f_2^-(\mathbf{x},t)$ into the homogeneous upper half-space $x_3\le x_{3,0}$. The focusing condition is formally defined as $f_2(\mathbf{x}_{\rm H},x_3=x_{3,0},t)=\delta(\mathbf{x}_{\rm H}-\mathbf{x}_{\rm H}'')\delta(t)$. From here onward the focal point $\mathbf{x}_0''$ is included in the argument list, hence $f_2^{\pm}(\mathbf{x},t)$ is denoted as $f_2^{\pm}(\mathbf{x},\mathbf{x}_0'',t)$, etc., see Figure 1(b). This fundamental solution reads in the frequency domain $f_2(\mathbf{x},\mathbf{x}_0'',\omega)=f_2^+(\mathbf{x},\mathbf{x}_0'',\omega)+f_2^-(\mathbf{x},\mathbf{x}_0'',\omega)$, with $f_2^+(\mathbf{x},\mathbf{x}_0'',\omega)=0$ for $x_3\le x_{3,0}$. The focusing condition becomes in the frequency domain

$$f_2(\mathbf{x}_0,\mathbf{x}_0'',\omega)=f_2^-(\mathbf{x}_0,\mathbf{x}_0'',\omega)=\delta(\mathbf{x}_{\rm H}-\mathbf{x}_{\rm H}''). \tag{4}$$

Reciprocity theorems 1 and 2 can be used to find relations between the fundamental solutions at the boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_i$. Substituting $p_A^{\pm}(\mathbf{x},\omega)=f_1^{\pm}(\mathbf{x},\mathbf{x}_i',\omega)$ and $p_B^{\pm}(\mathbf{x},\omega)=f_2^{\pm}(\mathbf{x},\mathbf{x}_0'',\omega)$, using equations 3 and 4, yields

$$f_1^+(\mathbf{x}_0'',\mathbf{x}_i',\omega)=f_2^-(\mathbf{x}_i',\mathbf{x}_0'',\omega) \tag{5}$$

and

$$-\{f_1^-(\mathbf{x}_0'',\mathbf{x}_i',\omega)\}^*=f_2^+(\mathbf{x}_i',\mathbf{x}_0'',\omega), \tag{6}$$

respectively.

GREEN'S FUNCTION REPRESENTATION

The flux-normalized one-way Green's function of the actual inhomogeneous medium is introduced, with its source at $\mathbf{x}_0''$ just above $\partial\mathbb{D}_0$, see Figure 2. Directly below the source, at $\partial\mathbb{D}_0$, the downward propagating constituent of the Green's function is defined as

$$G^+(\mathbf{x}_0,\mathbf{x}_0'',t)=\delta(\mathbf{x}_{\rm H}-\mathbf{x}_{\rm H}'')\delta(t). \tag{7}$$

The upward propagating constituent of the Green's function at $\partial\mathbb{D}_0$ is defined as

$$G^-(\mathbf{x}_0,\mathbf{x}_0'',t)=R(\mathbf{x}_0,\mathbf{x}_0'',t). \tag{8}$$

This Green's function is by definition the reflection response of the inhomogeneous medium, represented by $R(\mathbf{x}_0,\mathbf{x}_0'',t)$ in the right-hand side. Figure 2 also shows the downward and upward propagating constituents $G^+(\mathbf{x}_i,\mathbf{x}_0'',t)$ and $G^-(\mathbf{x}_i,\mathbf{x}_0'',t)$, respectively, at $\partial\mathbb{D}_i$. In the frequency domain, the Green's function is written as $G(\mathbf{x},\mathbf{x}_0'',\omega)=G^+(\mathbf{x},\mathbf{x}_0'',\omega)+G^-(\mathbf{x},\mathbf{x}_0'',\omega)$. Equations 7 and 8 become

$$G^+(\mathbf{x}_0,\mathbf{x}_0'',\omega)=\delta(\mathbf{x}_{\rm H}-\mathbf{x}_{\rm H}'') \tag{9}$$

and

$$G^-(\mathbf{x}_0,\mathbf{x}_0'',\omega)=R(\mathbf{x}_0,\mathbf{x}_0'',\omega), \tag{10}$$

respectively. Relations between the Green's functions $G^{\pm}$ and the fundamental solutions $f_1^{\pm}$ can be found by applying the reciprocity theorems 1 and 2 to the domain $\mathbb{D}$ enclosed by $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_i$. Substituting $p_A^{\pm}(\mathbf{x},\omega)=f_1^{\pm}(\mathbf{x},\mathbf{x}_i',\omega)$ and $p_B^{\pm}(\mathbf{x},\omega)=G^{\pm}(\mathbf{x},\mathbf{x}_0'',\omega)$ into equations 1 and 2, using equations 3, 9 and 10, yields

$$\int_{\partial\mathbb{D}_0} f_1^+(\mathbf{x}_0,\mathbf{x}_i',\omega)R(\mathbf{x}_0,\mathbf{x}_0'',\omega){\rm d}\mathbf{x}_0 -f_1^-(\mathbf{x}_0'',\mathbf{x}_i',\omega)=G^-(\mathbf{x}_i',\mathbf{x}_0'',\omega) \tag{11}$$

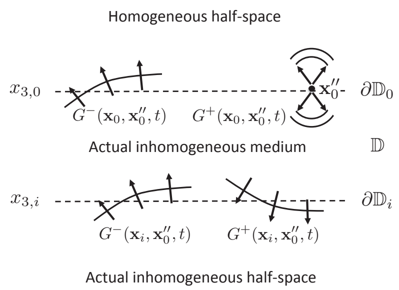

Fig. 2: *Flux-normalized one-way Green's functions of the 3-D wave equation in the actual inhomogeneous medium.*

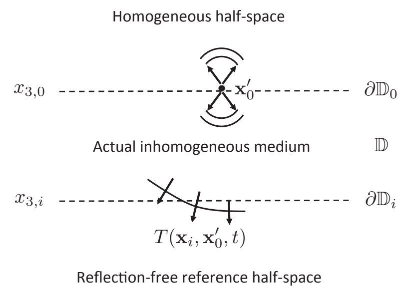

Fig. 3: *Transmission response of the reference medium.*

and

$$-\int_{\partial\mathbb{D}_0}\{f_1^-(\mathbf{x}_0,\mathbf{x}_i',\omega)\}^*R(\mathbf{x}_0,\mathbf{x}_0'',\omega)\mathrm{d}\mathbf{x}_0 + \{f_1^+(\mathbf{x}_0'',\mathbf{x}_i',\omega)\}^* = G^+(\mathbf{x}_i',\mathbf{x}_0'',\omega), \quad (12)$$

respectively. Slob and Wapenaar (2013) derive similar expressions for EM waves (for TE and TM modes). Adding these expressions, using equations 5 and 6, gives

$$G(\mathbf{x}_i',\mathbf{x}_0'',\omega) = \int_{\partial\mathbb{D}_0} f_2(\mathbf{x}_i',\mathbf{x}_0,\omega)R(\mathbf{x}_0,\mathbf{x}_0'',\omega)\mathrm{d}\mathbf{x}_0 + \{f_2(\mathbf{x}_i',\mathbf{x}_0'',\omega)\}^*. \quad (13)$$

This equation shows that the Green's function of the actual inhomogeneous medium, propagating from a source just above the surface $\partial\mathbb{D}_0$ (Figure 2) to a receiver at an arbitrary depth level $\partial\mathbb{D}_i$ (or vice versa), can be obtained from the reflection response of the actual medium, observed at the surface $\partial\mathbb{D}_0$, and the fundamental solution f_2 of the reference medium (Figure 1(b)). In the next section the right-hand side of equation 13 is used to derive the 3-D Marchenko equation, which is an integral equation for the fundamental solution f_2.

3-D MARCHENKO EQUATION

Transforming equation 13 to the time domain gives

$$G(\mathbf{x}_i',\mathbf{x}_0'',t) = \int_{\partial\mathbb{D}_0}\mathrm{d}\mathbf{x}_0\int_{-\infty}^{\infty} f_2(\mathbf{x}_i',\mathbf{x}_0,t')R(\mathbf{x}_0,\mathbf{x}_0'',t-t')\mathrm{d}t' + f_2(\mathbf{x}_i',\mathbf{x}_0'',-t). \quad (14)$$

Let $t_\mathrm{d}(\mathbf{x}_i',\mathbf{x}_0'')$ denote the traveltime of the direct arrival between $\mathbf{x}_0''$, located just above $\partial\mathbb{D}_0$, and $\mathbf{x}_i'$ at $\partial\mathbb{D}_i$. In the following, equation 14 is evaluated only for $t < t_\mathrm{d}(\mathbf{x}_i',\mathbf{x}_0'')$, hence

$$0 = \int_{\partial\mathbb{D}_0}\mathrm{d}\mathbf{x}_0\int_{-\infty}^{\infty} f_2(\mathbf{x}_i',\mathbf{x}_0,t')R(\mathbf{x}_0,\mathbf{x}_0'',t-t')\mathrm{d}t' + f_2(\mathbf{x}_i',\mathbf{x}_0'',-t), \quad t < t_\mathrm{d}(\mathbf{x}_i',\mathbf{x}_0''). \quad (15)$$

Assuming $R(\mathbf{x}_0,\mathbf{x}_0'',t)$ is known (obtained from reflection measurements at the surface), the aim is to determine the fundamental solution $f_2(\mathbf{x}_i',\mathbf{x}_0'',t)$. An ansatz needs to be made for the form of this function. To this end, first an auxiliary property of $f_2^-(\mathbf{x}_i,\mathbf{x}_0'',t)$ is derived. Figure 3 shows the transmission response $T(\mathbf{x}_i,\mathbf{x}_0',t)$ of the reference medium to a source at $\mathbf{x}_0'$ just above $\partial\mathbb{D}_0$. Substituting $p_A^+(\mathbf{x}_0,\omega) = \delta(\mathbf{x}_H-\mathbf{x}_H')$, $p_A^+(\mathbf{x}_i,\omega) = T(\mathbf{x}_i,\mathbf{x}_0',\omega)$, $p_A^-(\mathbf{x}_i,\omega) = 0$ and $p_B^\pm(\mathbf{x},\omega) = f_2^\pm(\mathbf{x},\mathbf{x}_0'',\omega)$ into equation 1, using equation 4, yields

$$\delta(\mathbf{x}_H'-\mathbf{x}_H'') = \int_{\partial\mathbb{D}_i} T(\mathbf{x}_i,\mathbf{x}_0',\omega)f_2^-(\mathbf{x}_i,\mathbf{x}_0'',\omega)\mathrm{d}\mathbf{x}_i. \quad (16)$$

Hence, $f_2^-(\mathbf{x}_i,\mathbf{x}_0'',\omega)$ is the inverse of the transmission response $T(\mathbf{x}_i,\mathbf{x}_0'',\omega)$ in the sense of equation 16. In the time domain this is formulated as

$$f_2^-(\mathbf{x}_i,\mathbf{x}_0'',t) = \{T(\mathbf{x}_i,\mathbf{x}_0'',t)\}^{\mathrm{inv}}. \quad (17)$$

The ansatz for the form of $f_2(\mathbf{x}_i,\mathbf{x}_0'',t)$ is inspired by the 1-D case. In the 1-D derivation (Lamb, 1980), $f_2(x,t)$ is defined as a delta pulse traveling in the negative x-direction, followed by a scattering coda caused by the potential $u(x)$. Moreover, the incident field is defined such that the scattering coda vanishes beyond the scattering region, leaving only a delta pulse beyond this region. Since $f_2(\mathbf{x}_i,\mathbf{x}_0'',t)$ focuses at $\mathbf{x}_0''$ at $t=0$ (see Figure 1(b)), it is reasonable to assume that the first event arriving at $\mathbf{x}_0''$ is the direct arrival of the upgoing field $f_2^-(\mathbf{x}_i,\mathbf{x}_0'',t)$. According to equation 17, $f_2^-(\mathbf{x}_i,\mathbf{x}_0'',t)$ is the inverse of the transmission response, hence, the aforementioned assumption is equivalent with assuming that the first event of $f_2(\mathbf{x}_i,\mathbf{x}_0'',t)$ is the direct arrival of the inverse transmission response, which is denoted as $\{T_\mathrm{d}(\mathbf{x}_i,\mathbf{x}_0'',t)\}^{\mathrm{inv}}$. The traveltime of this direct arrival is $-t_\mathrm{d}(\mathbf{x}_i,\mathbf{x}_0'')$. Analogous to the 1-D situation, the ansatz for $f_2(\mathbf{x}_i,\mathbf{x}_0'',t)$ is a superposition of the direct wave and a scattering coda, according to

$$f_2(\mathbf{x}_i,\mathbf{x}_0'',t) = \{T_\mathrm{d}(\mathbf{x}_i,\mathbf{x}_0'',t)\}^{\mathrm{inv}} + \theta(t+t_\mathrm{d}(\mathbf{x}_i,\mathbf{x}_0''))M(\mathbf{x}_i,\mathbf{x}_0'',t). \quad (18)$$

Here $M(\mathbf{x}_i,\mathbf{x}_0'',t)$ is the coda following the direct arrival, which is assumed to be causal, i.e.,

$$M(\mathbf{x}_i,\mathbf{x}_0'',t) = 0 \quad \text{for} \quad t < -t_\mathrm{d}(\mathbf{x}_i,\mathbf{x}_0''). \quad (19)$$

This causality is emphasized by the multiplication with the Heaviside function $\theta(t+t_\mathrm{d}(\mathbf{x}_i,\mathbf{x}_0''))$ in equation 18.

Note that the ansatz (equation 18) limits the validity of the following derivation to configurations for which the ansatz holds true. For example, it holds in layered media with moderately curved interfaces as long as $|\mathbf{x}_\mathrm{H}-\mathbf{x}_\mathrm{H}''|$ is not too large (to avoid occurrence of turning waves, headwaves etc.). The conditions underlying the ansatz need further investigation, which is beyond the scope of this paper.

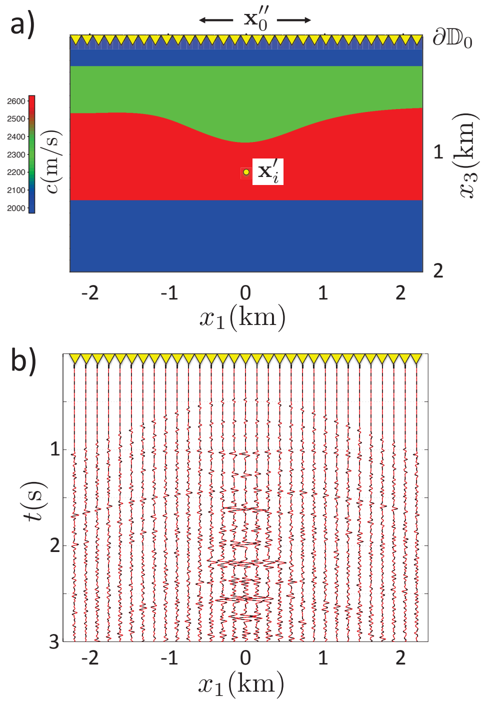

Fig. 4: *(a) Subsurface model. (b) Retrieved Green's function* $G(\mathbf{x}_0'', \mathbf{x}_i', t)$ *(black) and reference solution (red).*

Substituting equation 18 into 15 yields

$$\begin{aligned} 0 \;&= \int_{\partial\mathbb{D}_0} \mathrm{d}\mathbf{x}_0 \int_{-\infty}^{\infty} \{T_\mathrm{d}(\mathbf{x}_i', \mathbf{x}_0, t')\}^\mathrm{inv} R(\mathbf{x}_0, \mathbf{x}_0'', t-t')\mathrm{d}t' \\ &+ \int_{\partial\mathbb{D}_0} \mathrm{d}\mathbf{x}_0 \int_{-t_\mathrm{d}(\mathbf{x}_i', \mathbf{x}_0)}^{\infty} M(\mathbf{x}_i', \mathbf{x}_0, t') R(\mathbf{x}_0, \mathbf{x}_0'', t-t')\mathrm{d}t' \\ &+ M(\mathbf{x}_i', \mathbf{x}_0'', -t), \quad t < t_\mathrm{d}(\mathbf{x}_i', \mathbf{x}_0''). \end{aligned} \tag{20}$$

This is the 3-D Marchenko equation. It is an integral equation for the coda $M(\mathbf{x}_i', \mathbf{x}_0'', t)$ of the fundamental solution $f_2(\mathbf{x}_i', \mathbf{x}_0'', t)$.

GREEN'S FUNCTION RETRIEVAL

To solve the Marchenko equation for $M(\mathbf{x}_i', \mathbf{x}_0'', t)$, we rewrite equation 20 into the following iterative scheme

$$\begin{aligned} &M_k(\mathbf{x}_i', \mathbf{x}_0'', -t) = M_0(\mathbf{x}_i', \mathbf{x}_0'', -t) \\ &- \int_{\partial\mathbb{D}_0} \mathrm{d}\mathbf{x}_0 \int_{-t_\mathrm{d}(\mathbf{x}_i', \mathbf{x}_0)}^{\infty} M_{k-1}(\mathbf{x}_i', \mathbf{x}_0, t') R(\mathbf{x}_0, \mathbf{x}_0'', t-t')\mathrm{d}t', \end{aligned} \tag{21}$$

with

$$\begin{aligned} &M_0(\mathbf{x}_i', \mathbf{x}_0'', -t) = \\ &- \int_{\partial\mathbb{D}_0} \mathrm{d}\mathbf{x}_0 \int_{-\infty}^{\infty} \{T_\mathrm{d}(\mathbf{x}_i', \mathbf{x}_0, t')\}^\mathrm{inv} R(\mathbf{x}_0, \mathbf{x}_0'', t-t')\mathrm{d}t', \end{aligned} \tag{22}$$

for $t < t_\mathrm{d}(\mathbf{x}_i', \mathbf{x}_0'')$. Evaluation of equation 22 requires that, apart from the reflection response, information about the direct arrivals of the transmission response is available. These direct arrivals can for example be modeled in an estimated background model. Once $M_0(\mathbf{x}_i', \mathbf{x}_0'', -t)$ has been evaluated, it is used in the iterative scheme of equation 21, which starts for $k = 1$ and continues until convergence. The scheme is expected to converge to $M(\mathbf{x}_i', \mathbf{x}_0'', t)$ as long as the reflection response does not return all emitted energy back to the surface $\partial\mathbb{D}_0$. Substituting the end result $M(\mathbf{x}_i', \mathbf{x}_0'', t)$ into equation 18 gives the fundamental solution $f_2(\mathbf{x}_i', \mathbf{x}_0'', t)$. Finally, the Green's function $G(\mathbf{x}_i', \mathbf{x}_0'', t) = G(\mathbf{x}_0'', \mathbf{x}_i', t)$ is obtained from equation 14. Unlike in other Green's function retrieval methods (Weaver and Lobkis, 2001; Campillo and Paul, 2003; Wapenaar et al., 2004; Schuster, 2009), no physical receiver is needed at the position of the virtual source. This is illustrated with a 2D numerical example in Figure 4. Figure 4(a) shows an inhomogeneous medium with a synclinal interface (the colors represent the propagation velocity c). The yellow dot represents the virtual source position $\mathbf{x}_i' = (0, 1150)$ and the yellow triangles represent 31 receiver positions $\mathbf{x}_0''$ at the surface. Figure 4(b) shows the Green's function $G(\mathbf{x}_0'', \mathbf{x}_i', t)$ obtained with the iterative scheme (black dashed traces), overlain on the directly modeled Green's function (red traces). All traces have been multiplied by $t^{2.5}$ to emphasize the scattering coda. Note that this coda, including the triplications related to the synclinal structure, is very well recovered. Thorbecke et al. (2013) show that this result is robust with respect to errors in the direct arrival.

APPLICATION IN SEISMIC IMAGING

Because no physical receiver is needed at the position of the virtual source, the virtual source can be chosen anywhere in the medium, yielding $G(\mathbf{x}_i', \mathbf{x}_0'', t) = G(\mathbf{x}_0'', \mathbf{x}_i', t)$ for any $\mathbf{x}_i'$. Instead of using equation 13 as the starting point for deriving the Marchenko equation, the system of equations 11 and 12 can be used to derive a system of Marchenko equations, which finally yield $G^+(\mathbf{x}_i', \mathbf{x}_0'', t)$ and $G^-(\mathbf{x}_i', \mathbf{x}_0'', t)$ for any $\mathbf{x}_i'$ (Slob et al., 2013). These functions are related to each other via

$$G^-(\mathbf{x}_i', \mathbf{x}_0'', t) = \int_{\partial\mathbb{D}_i} \mathrm{d}\mathbf{x}_i \int_{-\infty}^{\infty} R(\mathbf{x}_i', \mathbf{x}_i, t') G^+(\mathbf{x}_i, \mathbf{x}_0'', t-t')\mathrm{d}t', \tag{23}$$

where $R(\mathbf{x}_i', \mathbf{x}_i, t)$ is the reflection response of the medium below $\partial\mathbb{D}_i$. This reflection response can be resolved from equation 23 by multidimensional deconvolution. Van der Neut et al. (2013) show that a more stable result can be obtained by first recasting equation 23 into a Fredholm integral equation of the second kind. By repeated application of Green's function retrieval and resolving the reflection response, a true-amplitude ghost-free image of the subsurface is obtained by extracting $R(\mathbf{x}_i, \mathbf{x}_i, t = 0)$ for all $\mathbf{x}_i$ (Wapenaar et al., 2012b; Broggini et al., 2013). True-amplitude angle-dependent reflection information can be obtained from $R(\mathbf{x}_i', \mathbf{x}_i, t)$ via local $\tau - p$ transforms (de Bruin et al., 1990).

CONCLUSIONS

We derived a 3-D extension of the Marchenko equation and showed that this is the basis for retrieving the Green's function from seismic reflection data at the surface. Unlike in interferometric Green's function retrieval methods, no physical receiver is required at the position of the virtual source; only an estimate of the first arrivals is needed. With a numerical example we showed that the retrieved Green's function contains the correct scattering coda of the inhomogeneous medium. We indicated the potential applications in ghost-free true-amplitude imaging and AVA analysis. These and other applications of Green's function retrieval "beyond seismic interferometry" are currently subject of intensive research.

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