# Extracting the Green's function for static problems from dynamic fields

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## Summary

The theory of Green's function extraction from field fluctuations has originally been derived in geoscience applications for wave propagation problems. Although current application of this technique is not restricted to wave propagation, there was no theory for Green's function extraction of static fields. We present the theory of Green's function extraction of static fields and illustrate the theory with a numerical example. The theory presented here is applicable to potential fields and to DC resistivity problems. The ability to extract static fields from field fluctuations makes it possible, in principle, to extract static fields from passive measurements of field fluctuations. This can be particularly relevant for continuous monitoring of the subsurface with electrical fields.

## Introduction

Extraction the Green's function from field fluctuations is a rapidly growing field in science and engineering (Curtis et al., 2006; Larose et al., 2006; Wapenaar et al., 2008). In seismological applications this technique is usually referred to as *seismic interferometry*. The principle of Green's function extraction has up to this point been applied to time-dependent problems. This may have been caused by the fact the underlying theory was originally related to time-reversal, e.g. (Derode et al., 2003). The current theory has mostly been applied to problems involving wave propagation, such problems are inherently time-dependent.

In this work we show that the principle of Green's function extraction can be applied to static fields. The theory presented here is a simplified version of the more general formulation of *Lagrangian Green's function extraction* (Snieder et al., 2010), and we restrict ourselves to the Green's function extraction for a potential field problem. It may appear to be a paradox that one can extract the Green's function of static fields from dynamic field fluctuations. The underlying principle is, however, not complicated. As an example, we show numerical simulations where random dipoles are present in an electrostatic system. At every moment in time, different dipoles generate the field, which at any moment in time depends only on the instantaneous dipole distribution. (This is actually the definition of the quasi-static response.) We show theoretically and numerically that by averaging over all dipole distributions one can extract the electrostatic response. Since the response is quasi-static, it does not matter whether the fields are generated by time-dependent electrical dipoles, or by an ensemble of dipoles; in the end this gives the same response. In the following section we derive the theory, which we illustrate with a numerical example in the subsequent section.

## Green's function extraction in electrostatics

In linear dielectric media, the electric displacement $\mathbf{D}$ is related to the electric field $\mathbf{E}$ by the relation $\mathbf{D} = \varepsilon\mathbf{E}$, with $\varepsilon(\mathbf{r})$ the electrical permittivity (Jackson, 1975). Using the field equation $(\nabla \cdot \mathbf{D}) = q$, with $q(\mathbf{r})$ the charge density, and the relation $\mathbf{E} = -\nabla u$, with $u(\mathbf{r})$ the electric potential, the following field equation results

$$0 = \nabla \cdot (\varepsilon(\mathbf{r})\nabla u(\mathbf{r})) + q(\mathbf{r}) \ . \tag{1}$$

If instead of the electrostatic problem we consider direct currents in a conducting medium, then the charge density $q(\mathbf{r})$ is replaced by a volume density of charge injection or extraction rate $-\dot{q}(\mathbf{r})$ (the minus sign comes from the historical convention that the loss of charge from the source region constitutes a positive electric current), and the electric permittivity $\varepsilon(\mathbf{r})$ is replaced by the conductivity $\sigma(\mathbf{r})$. This leaves equation (1) intact with different symbols (Stratton, 1941). This results in a formulation for the extraction of the resistivity tensor in conducting media from field fluctuations (Slob et al., 2010).

The derivation of the Green's function extraction is based on representation theorems, but is actually simpler than earlier derivation for Green's function extraction for acoustic waves (Derode et al., 2003; Wapenaar et al., 2005) because the static fields considered here are real. We consider two states, labeled $A$ and $B$, and take the field equation (1) for state $A$, multiply it with the field of state $B$ and integrate over a volume $V$ to give

$$\int_V u_B(\mathbf{r})q_A(\mathbf{r})dV = -\int_V u_B(\mathbf{r})\nabla \cdot (\varepsilon(\mathbf{r})\nabla u_A(\mathbf{r}))\, dV \ . \tag{2}$$

Using the identity $u_B\nabla\cdot(\varepsilon\nabla u_A) = \nabla\cdot(\varepsilon u_B\nabla u_A) - \varepsilon(\nabla u_A\cdot\nabla u_B)$ and applying Gauss' theorem gives

$$\begin{aligned}\int_V u_B(\mathbf{r})q_A(\mathbf{r})dV = \quad & -\oint_{\partial V} \varepsilon(\mathbf{r})u_B(\mathbf{r})\frac{\partial u_A(\mathbf{r})}{\partial n}dS \\ & + \int \varepsilon(\mathbf{r})\left(\nabla u_A(\mathbf{r})\cdot\nabla u_B(\mathbf{r})\right) dV \ ,\end{aligned} \tag{3}$$

where $\partial V$ denotes the boundary of the volume $V$. When the potential $u_B$ or the normal component of the electric field $\partial u_A/\partial n$ vanishes on the boundary, the first integral on the right hand side vanishes, and

$$\int_V u_B(\mathbf{r})q_A(\mathbf{r})dV = \int \varepsilon(\mathbf{r})\left(\nabla u_A(\mathbf{r})\cdot\nabla u_B(\mathbf{r})\right) dV \ . \tag{4}$$

The fields decay sufficiently rapid with distance that the surface integral also vanishes when the surface is take at infinity. We next take use a point source for the two states ($q_A(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_A)$ and $q_B(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_B)$), the corresponding fields are, by definition, given by the Green's functions $G(\mathbf{r}, \mathbf{r}_A)$ and $G(\mathbf{r}, \mathbf{r}_B)$, respectively. Inserting this in expression (4) gives

$$G(\mathbf{r}_A, \mathbf{r}_B) = \int_V \varepsilon(\mathbf{r}) \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G(\mathbf{r}, \mathbf{r}_B)\right) dV \ . \tag{5}$$

In order to establish the connection of this equation with the Green's function extraction from field fluctuations we use the field generated by an electric dipole distribution $\mathbf{p}(\mathbf{r})$ (Jackson, 1975)

$$u(\mathbf{r}_0) = \int_V \left(\nabla G(\mathbf{r}_0, \mathbf{r})\right) \cdot \mathbf{p}(\mathbf{r}) dV \ . \tag{6}$$

We next consider random dipole sources that are spatially and directionally uncorrelated and satisfy

$$\langle p_i(\mathbf{r}_1) p_j(\mathbf{r}_2)\rangle = |S|^2 \varepsilon(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta_{ij} \ , \tag{7}$$

where $|S|^2$ measures the strength of the dipole sources. For the moment we consider an ensemble of identical electrostatic systems, each with their own excitation by dipoles, and $\langle \cdots \rangle$ denotes an ensemble average. Multiplying expression (5) with $|S|^2$, using the fact that for this problem $G$ is real, and using the summation convention, gives

$$\begin{aligned}
G(\mathbf{r}_A, \mathbf{r}_B)|S|^2 &= |S|^2 \int_V \varepsilon(\mathbf{r}) \partial_i G(\mathbf{r}, \mathbf{r}_A) \partial_i G^*(\mathbf{r}, \mathbf{r}_B) dV \\
&= \int_V \int_V |S|^2 \varepsilon(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2) \\
&\qquad \times \delta_{ij} \partial_i G(\mathbf{r}_1, \mathbf{r}_A) \partial_j G^*(\mathbf{r}_2, \mathbf{r}_B) dV_1 dV_2 \\
&= \langle \int_V \partial_i G(\mathbf{r}_1, \mathbf{r}_A) p_i(\mathbf{r}_1) dV_1 \\
&\qquad \times \int_V \partial_j G^*(\mathbf{r}_2, \mathbf{r}_A) p_j(\mathbf{r}_2) dV_2 \rangle \\
&= \langle u(\mathbf{r}_A) u^*(\mathbf{r}_B)\rangle \ ,
\end{aligned} \tag{8}$$

where the identity $\int f_i(\mathbf{r}) g_i(\mathbf{r}) dV = \int\int f_i(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta_{ij} g_j(\mathbf{r}_2) dV_1 dV_2$ has been used in the second identity, expression (7) in the third equality, and expression (6) in the last identity. This means that the electrostatic Green's function $G(\mathbf{r}_A, \mathbf{r}_B)$ follows from the ensemble average of the correlation of field fluctuations recorded at $\mathbf{r}_A$ and $\mathbf{r}_B$ that are excited by uncorrelated dipole sources.

In reality one may not have an ensemble of identical electrostatic systems, but one may have a system where random sources fluctuate with time. When the characteristic time of the temporal variations in these dipole sources is large compared to the time it takes for light to propagate through the system, the response of the system is quasi-static. In that case the ensemble average can be replaced by a temporal average over the field fluctuations. In fact, the approach to replace an ensemble average by an average over time is common in seismology where averaging over multiple non-overlapping time windows is used to extract the dynamic Green's function (Larose et al., 2006; Sabra et al., 2005; Shapiro et al., 2005). By applying the same principle to quasi-static field fluctuations one can extract the electrostatic Green's function from temporal field fluctuations.

## Numerical simulation of Green's function retrieval in electrostatics

In this example we illustrate the theory for the Green's function extraction for the electrostatic potential by cross-correlating the fields generated by random electric dipoles within a conducting spherical shell with radius $R$ at which the potential vanishes. Using the method of images, one can show that the potential generated by a dipole $\mathbf{p}$ at location $\mathbf{r}$ inside the shell vanishes at the shell $r = R$ when one adds the fields generated by a monopole $q' = (R/r^2)(\mathbf{p} \cdot \hat{\mathbf{r}})$ and a dipole $\mathbf{p}' = -(R/r)^3 \left(\mathbf{p} - 2(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\right)$, both at location $\mathbf{r}' = (R/r)^2 \mathbf{r}$ outside the shell (Snieder et al., 2010). We use a system of units scaled in such a way that $4\pi\varepsilon_0$ and $R$ are both equal to 1.

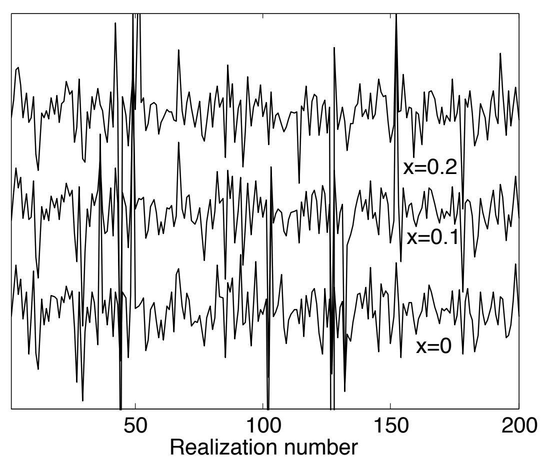


Fig. 1: Two hundred realizations of the potential at the $x$-axis at $x = 0$, $x = 0.1$, and $x = 0.2$, respectively.

The Green's function extraction is applied to equation (8). In each realization the field $u(\mathbf{r})$ is generated by ten random dipoles at locations that are drawn from a uniform distribution within the sphere. Each component from each dipole is drawn from a uniform distribution between -1 and +1. In equation (8) we use $\mathbf{r}_B = (0.3, 0.2, 0)$ and choose $\mathbf{r}_A$ along the $x$-axis of a coordinate system that has its origin at the center of the sphere. Figure 1 shows the potential in 200 realizations at three points along the $x$-axis. The potential at each point has the character of white noise, which is not surprising because the dipoles in every realization are uncorrelated and have zero mean. The potential at the different locations is, however, correlated. It is these correlations in the fields generated by random sources that contains the information that ultimately leads to the extraction of the Green's function. Note that it does not matter whether one considers figure 1 to show different realizations, or whether it shows a time series of the quasi-static electric response of a system that exhibits quasi-random electric dipoles as a function of time.

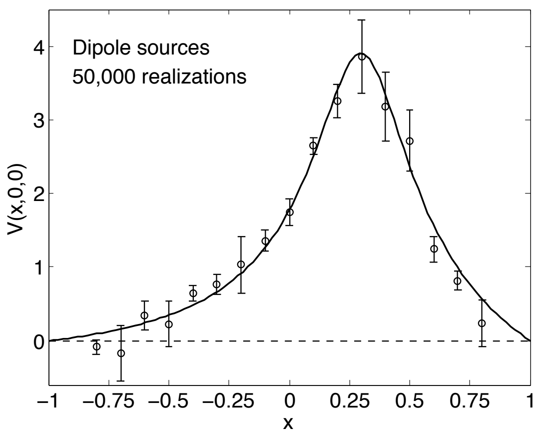


Fig. 2: Open symbols: the potential and its standard deviation reconstructed from 50,000 realizations of random dipoles. Solid line: the true potential for a monopole at $(0.3, 0.2, 0)$.

Figure 2 shows the cross correlation of 50,000 realizations of the field fluctuations recorded at location $\mathbf{r}_B = (0.3, 0.2, 0)$ and various locations along the $x$-axis. In all examples shown, the realizations are divided in 10 bins that each contains 5,000 realizations that are drawn from the same statistical distribution. The cross-correlation of field fluctuations in the different bins is used to compute the mean of the cross-correlation of field fluctuations and the variance in this mean (Hogg and Craig, 1978). The variance of the mean is shown with the error bars on figure 2. The mean and error thus computed do not depend much on the number of bins chosen, which means that the averaging over the bins has the same statistical effect as the averaging over realizations within each bin. The solid line in figure 2 shows the potential due to a unit point charge at location $\mathbf{r}_B$, which is the desired Green's function. The cross-correlations of field fluctuations, and their errors, are multiplied with a common scale factor that minimizes the difference of the cross-correlation and the Green's function. This scale factor accounts for the term $|S|^2$ in equation (8). Note that the potential estimated from the cross-correlation of field fluctuations generated by random dipole sources agrees with the true Green's function within the shown standard deviations. This confirms that the Green's function for this static example can indeed by extracted from the cross-correlation of field fluctuations.

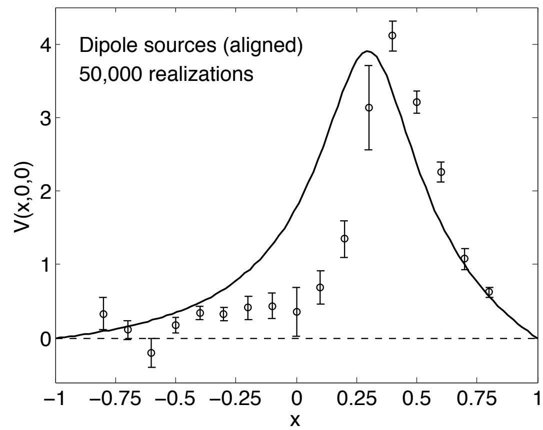


Fig. 3: Open symbols: the potential and its standard deviation reconstructed from 50,000 realizations of aligned dipoles with dipole moment $\mathbf{p} = (1, 1, 1)$. Solid line: the true potential for a monopole at $(0.3, 0.2, 0)$.

The Green's function in the example of figure 2 was extracted from the cross-correlation of field fluctuations excited by random dipoles. In each realization, the field was generated by the simultaneous action of 10 dipoles. In order to investigate what happens when the dipoles do not have a random orientation, we repeated the numerical experiment of figure 2, but now used the fixed dipole $\mathbf{p} = (1, 1, 1)$ in every realization. Ten of these dipoles were placed randomly within the spherical shell in every realization. The estimate of the Green's function estimated from the cross-correlations of the associated field fluctuations is shown in figure 3. In this case cross-correlation of field fluctuations does not lead to an acceptable estimate of the Green's function. The reason for this discrepancy is that according to equation (7) the orientation of different dipole vectors must be uncorrelated. This assumption is violated when a constant dipole vector is used for every dipole.

One might think that it does not really matter whether field fluctuations are generated by dipoles or by monopoles (point charges). We show in figure 4 the field extracted from cross-correlation of 10,000 realizations of field fluctuations generated by monopoles. In each realization ten point charges are placed at random positions within the shell. Each point charge is drawn from a uniform distribution between -1 and 1. Note that the standard deviation in figure 4 is much smaller than that in figure 2, despite the fact that five time less realizations are used. There are two reasons for this. First, for random dipole orientations, one carries out an implicit averaging over the direction of the dipole vectors, such averaging is not needed for monopole sources. Second, the dipole fields vary more rapidly with space than the monopole field do, hence the dipole fields must be sampled more finely by point charges to mimic the volume integrals in expression (8). Note that the cross-correlation of field fluctuations caused by random monopoles does not lead

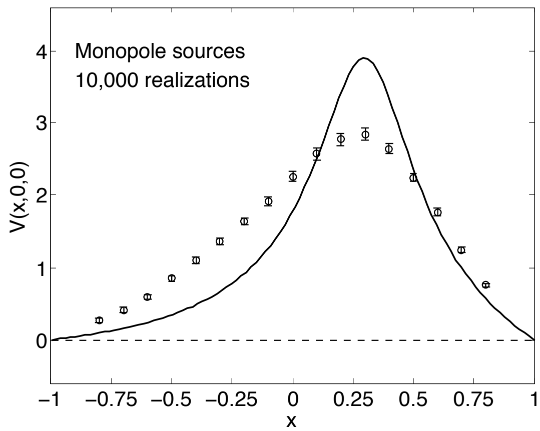


Fig. 4: Open symbols: the potential and its standard deviation reconstructed from 10,000 realizations of random monopoles. Solid line: the true potential for a monopole at $(0.3, 0.2, 0)$.

to an acceptable extraction of the Green's function.

The examples of the figures 2 and 4 illustrate the paradox that one needs field fluctuations excited by random dipoles to extract the monopole field. This is fortunate; natural field fluctuations cannot be caused by monopoles because the random occurrence of monopoles is not consistent with charge conservation. Charges are neither created nor destroyed in a source-free region. This means that only dipoles or higher order multipoles can excite field fluctuations. For the expected localized fluctuations, where the local charge separation is orders of magnitude smaller than that of the measurement scale, the contribution of the dipole moments dominates the potential field, and the electric potential is given by equation (6). The occurrence of dipole moments in a material can have four basic causes: electronic, ionic, dipolar and space charge polarization (Khesin et al., 1996). Relaxation times of the first three processes are usually smaller than 1 $\mu$s, while for the last process the relaxation time can be as large as 1 s. These time scales are extremely long compared to the propagation time of light through system of the size of a laboratory experiment or geophysical field experiments. Field fluctuations can be generated, for example in natural rocks by electromagnetic radiation in fracturing rocks or in stressed rocks before fracturing. During fracturing, bonds are broken. The larger the number of cut bonds, the larger is the number of excited atoms, and hence the greater becomes the electromagnetic radiation amplitude. These electromagnetic oscillations behave like surface vibrational optical waves, where positive charges move together in a diametrically opposite phase to the negative ones and decay exponentially into the material like Rayleigh waves. The resulting oscillating electric dipole is the source of the electromagnetic radiation. The pulse amplitude decays due to an interaction with bulk phonons and the life time of the measurable electric field varies between several to 100 $\mu$s (Bahat et al., 2005). Random field fluctuations caused by charge separation have been observed in water-saturated porous media that were drained (Haas and Revil, 2009). Each charge separation in that system is thought to be caused by the burst of a meniscus in the pore space, the so-called *Haines jump*. The application to direct current resistivity problems and the connection with the fluctuation-dissipation theorem is discussed elsewhere in more detail (Slob et al., 2010).

## Conclusion

The theory and numerical examples presented here show that the principle of Green's function extraction from field fluctuations can be applied to static problems. We illustrate this with an example from electrostatics, but we have shown elsewhere that the electrical conductivity tensor can be extracted from field fluctuations in a DC current problem (Slob et al., 2010). Extending the theory for Green's function extraction from dynamic problems creates the possibility to extract the Green's function for potential fields and DC currents from field fluctuations. This makes it possible, in principle, to interrogate the properties of static fields in the subsurface without active sources. This is useful when active sources can not de deployed. Furthermore, extracting static fields from field fluctuations could make it possible to monitor the static fields in the subsurface in a continuous fashion. This can be a valuable tool for continuous monitoring. The spectacular examples of continuous monitoring from seismic noise (Wegler and Sens-Schönfelder, 2007; Brenguier et al., 2008) offer hope that the principle of Green's function extraction can also be applied for continuous monitoring of quasi-static fields.

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