A discussion on stability analysis of wave field depth extrapolation

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Summary

Wave field depth extrapolation operators are typically applied recursively to the data which makes operator stability an issue. In most extrapolation algorithms, the operators are implemented as complex-valued matrices. In this paper, we investigate the stability properties of complex-valued matrices as well as the validity of eigenvalue and singular value analysis to evaluate these properties.

We specifically address the case where the extrapolation operators have been constructed by a modal decomposition of the Helmholtz operator. It is shown that, whether or not applied recursively, these operators are unconditionally stable.

Introduction

In the past two decades much research has been done on the optimization of one-way operators for recursive wave field extrapolation. For 2-D extrapolation we mention Berkhout [1], van der Schoot et al. [12] and Holberg [10]; for 3-D extrapolation some representative references are Blacquiere et al. [2] and Hale [9].

In all the references mentioned above it is assumed that the medium parameters are constant within the lateral extent of the operator. Lateral variations are approximately taken into account by selecting for each gridpoint an operator related to the *local* medium parameters at that gridpoint. In this paper we will call this the 'local explicit method'.

In the frequency domain we can think of the extrapolation process as a matrix-vector product [1], according to

$$\mathbf{P}(z_{n+1}) = \underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)\mathbf{P}(z_n), \tag{1}$$

where $\mathbf{P}(z_n)$ is a vector containing the spatially discretized wave field at depth level z_n and $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$ is the extrapolation matrix. In the local explicit approach, the operators described above, are stored in the rows of $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$. Therefore, unless the medium between z_n and z_{n+1} is laterally invariant, this matrix is not symmetric, i.e., $\underset{\sim}{\mathbf{W}} \neq \underset{\sim}{\mathbf{W}}^T$, where T denotes transposition. This is clearly a shortcoming of the local explicit method, because on basis of reciprocity considerations this matrix should be symmetric [1] [13]. A more rigorous way to derive the extrapolation matrix is based on modal decomposition of the Helmholtz operator ([14], [8]), leading to a symmetric $\underset{\sim}{\mathbf{W}}$. Whatever approach is chosen to accomodate lateral variations, $\underset{\sim}{\mathbf{W}}$ is applied recursively in practice to account for vertical variations. This imposes conditions on the stability of the extrapolation matrix. It has been observed by Etgen [6] that in certain situations the local explicit method leads to unstable results, in spite of the fact that the individual rows of $\underset{\sim}{\mathbf{W}}$ contain stable operators. The instability has been correctly attributed to singular values exceeding unity due to the lateral velocity variations. When the matrix $\underset{\sim}{\mathbf{W}}$ is constructed by means of modal decomposition, it is unconditionally stable: none of the eigenvalues exceed unity ([14], [8]).

Dellinger and Etgen [5] discuss, for real-valued matrix operators, the difference between eigenvalue and singular value decomposition as tools for stability analysis. They show that the mere fact that all eigenvalues are smaller or equal to unity does not guarantee stability of the matrix. In the following we extend the discussion on singular value and eigenvalue analysis in relation to stability. Amongst others, we address complex-valued matrices, since $\underset{\sim}{\mathbf{W}}$ is complex. Furthermore we will show that, in case the extrapolation operators have been constructed by modal decomposition, eigenvalue analysis is decisive for stability as much as singular value analysis.

Some definitions

Throughout this paper we will consider square matrices that may or may not be complex-valued. We will call a matrix *symmetric* when it obeys the property

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{A}}^T, \tag{2}$$

where T denotes transposition. Matrix $\underset{\sim}{\mathbf{A}}$ is called *Hermitian* when

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{A}}^H, \tag{3}$$

where H denotes transposition and complex conjugation. A matrix $\underset{\sim}{\mathbf{U}}$ is *unitary* when it obeys the property

$$\underset{\sim}{\mathbf{U}}^H \underset{\sim}{\mathbf{U}} = \underset{\sim}{\mathbf{I}}, \tag{4}$$

where $\underset{\sim}{\mathbf{I}}$ is the identity matrix. Note that $||\underset{\sim}{\mathbf{U}}\mathbf{f}|| = ||\mathbf{f}||$ for any vector $\mathbf{f}$. A matrix operator is said to be *unconditionally stable* when

$$||\underset{\sim}{\mathbf{A}}\mathbf{f}|| \le ||\mathbf{f}||, \tag{5}$$

for any vector $\mathbf{f}$. Note that

$$||\underset{\sim}{\mathbf{A}}_1\underset{\sim}{\mathbf{A}}_2 \cdots \underset{\sim}{\mathbf{A}}_n \cdots \underset{\sim}{\mathbf{A}}_N\mathbf{f}|| \le ||\mathbf{f}||, \tag{6}$$

for any $\mathbf{f}$ when all matrices $\underset{\sim}{\mathbf{A}}_n$ obey equation (5).

[1] Actually reciprocity for flux-normalized one-way wave fields implies $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n) = \underset{\sim}{\mathbf{W}}^T(z_n, z_{n+1})$. In this paper we assume that the medium parameters are depth-independent between z_n and z_{n+1}, so we also have $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n) = \underset{\sim}{\mathbf{W}}^T(z_{n+1}, z_n)$.

Eigenvalue decomposition and stability analysis

We introduce eigenvalues λ and eigenvectors $\mathbf{e}$ of matrix $\underset{\sim}{\mathbf{A}}$ via

$$\underset{\sim}{\mathbf{A}}\mathbf{e} = \lambda\mathbf{e}. \tag{7}$$

The eigenvalues λ are found by solving

$$\det(\underset{\sim}{\mathbf{A}} - \lambda\underset{\sim}{\mathbf{I}}) = 0. \tag{8}$$

When $\underset{\sim}{\mathbf{A}}$ has a complete set of linearly independent eigenvectors $\mathbf{e}$, then the eigenvalue decomposition of $\underset{\sim}{\mathbf{A}}$ reads [2]

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{\Lambda}}\underset{\sim}{\mathbf{L}}^{-1}, \tag{9}$$

where the columns of matrix $\underset{\sim}{\mathbf{L}}$ are formed by the eigenvectors $\mathbf{e}$ and $\underset{\sim}{\mathbf{\Lambda}}$ is a diagonal matrix, containing the eigenvalues λ on its diagonal. For a matrix $\underset{\sim}{\mathbf{A}}$ that allows the decomposition of equation (9), any power can be written as

$$\underset{\sim}{\mathbf{A}}^N = \underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{\Lambda}}^N\underset{\sim}{\mathbf{L}}^{-1}. \tag{10}$$

Hence, when $|\lambda| \le 1$ for all λ, then $||\underset{\sim}{\mathbf{A}}^N\mathbf{f}||$ remains finite for $N \to \infty$. However, as has been correctly pointed out by Dellinger and Etgen [5], for asymmetric matrices, $||\underset{\sim}{\mathbf{A}}^N\mathbf{f}||$ can become arbitrarily large for finite values of N. Therefore the criterion $|\lambda| \le 1$ alone is not a sufficient condition for stability. Hence, for an extrapolation matrix constructed according to the local explicit method, eigenvalue analysis does not give the final verdict about stability.

A fundamental theorem of matrix algebra states that a Hermitian matrix always has a complete set of orthogonal eigenvectors and real-valued eigenvalues ([7], [3]). Hence, for Hermitian matrices the decomposition defined in equation (9) is always possible and can be rewritten (after a normalization of the eigenvectors) as

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{\Lambda}}\underset{\sim}{\mathbf{L}}^{-1} = \underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{\Lambda}}\underset{\sim}{\mathbf{L}}^H. \tag{11}$$

Since $\underset{\sim}{\mathbf{L}}$ is unitary, in this case the criterion $|\lambda| \le 1$ (for all λ) implies unconditional stability.

Singular value decomposition and stability analysis

The singular values s of an arbitrary matrix $\underset{\sim}{\mathbf{A}}$ are defined as the non-negative square-roots of the eigenvalues of matrix $\underset{\sim}{\mathbf{B}} = \underset{\sim}{\mathbf{A}}\underset{\sim}{\mathbf{A}}^H$. They are found by solving

$$\det(\underset{\sim}{\mathbf{B}} - s^2\underset{\sim}{\mathbf{I}}) = 0. \tag{12}$$

The singular value decomposition of $\underset{\sim}{\mathbf{A}}$ reads [3]

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{U}}\underset{\sim}{\mathbf{S}}\underset{\sim}{\mathbf{V}}^H, \tag{13}$$

where $\underset{\sim}{\mathbf{S}}$ is a diagonal matrix, containing the singular values s on its diagonal in decreasing order. Furthermore, $\underset{\sim}{\mathbf{U}}$ and $\underset{\sim}{\mathbf{V}}$ are unitary matrices. Therefore the condition $s \le 1$ (for all s) implies unconditional stability.

[2] We will see in a later section that even for symmetric matrices [in the sense of equation (2)] the decomposition defined in equation (9) is not always possible.

[3] A discussion on the existence of this decomposition for complex-valued matrices is beyond the scope of this paper.

Some numerical examples

Dellinger and Etgen [5] extensively analyzed the stability properties of asymmetric real-valued 2×2 matrices. Here we show with some examples what can happen to symmetric matrices when they are complex-valued.

Consider the following complex-valued symmetric 2×2 matrix

$$\underset{\sim}{\mathbf{A}} = \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix}. \tag{14}$$

Its eigenvalues are found by solving equation (8), according to

$$(1 + i - \lambda_{1,2})(1 - i - \lambda_{1,2}) - 1 = 0, \tag{15}$$

yielding

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 1. \tag{16}$$

Despite the symmetry of $\underset{\sim}{\mathbf{A}}$ and its unitary eigenvalues, this matrix is unstable, as is easily seen from the following result

$$\underset{\sim}{\mathbf{A}}^{100} = \begin{pmatrix} 1+i & 100 \\ 100 & 1-i \end{pmatrix}. \tag{17}$$

The reason is that the eigenvectors are linearly dependent, so a decomposition in the form of equation (9) is not possible. A square matrix $\underset{\sim}{\mathbf{A}}$ can always be written in the Jordan canonical form

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{C}}\underset{\sim}{\mathbf{\Gamma}}\underset{\sim}{\mathbf{C}}^{-1}, \tag{18}$$

where $\underset{\sim}{\mathbf{\Gamma}}$ is a block-diagonal matrix with 'Jordan blocks' along its diagonal ([11], [7], [4]). For the situation at hand $\underset{\sim}{\mathbf{\Gamma}}$ consists of a single 2×2 Jordan block, with the eigenvalues λ_1 and λ_2 on its diagonal and an extra 'diagonal' containing a single 1:

$$\begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix} = \begin{pmatrix} i & \frac{1}{2} \\ 1 & \frac{i}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ 1 & -i \end{pmatrix}. \tag{19}$$

Analogous to equation (10) we obtain for the N'th power of $\underset{\sim}{\mathbf{A}}$

$$\underset{\sim}{\mathbf{A}}^N = \underset{\sim}{\mathbf{C}}\underset{\sim}{\mathbf{\Gamma}}^N\underset{\sim}{\mathbf{C}}^{-1}. \tag{20}$$

Note that

$$\underset{\sim}{\mathbf{\Gamma}}^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}, \tag{21}$$

which explains the unstable result in equation (17). The situation is even worse than before, since here $||\underset{\sim}{\mathbf{A}}^N\mathbf{f}||$ is unbounded for $N \to \infty$.

The singular values of $\underset{\sim}{\mathbf{A}}$ defined in equation (14) are found by solving equation (12), with

$$\underset{\sim}{\mathbf{B}} = \underset{\sim}{\mathbf{A}}\underset{\sim}{\mathbf{A}}^H = \begin{pmatrix} 3 & 2+2i \\ 2-2i & 3 \end{pmatrix}, \tag{22}$$

yielding

$$s_1 = \sqrt{2} + 1 \quad \text{and} \quad s_2 = \sqrt{2} - 1. \tag{23}$$

Since $s_1 > 1$, this analysis shows immediately that $\underset{\sim}{\mathbf{A}}$ defined in equation (14) is unstable.

Fortunately not all symmetric complex-valued matrices are unstable. Consider the following 2×2 matrix

$$\underset{\sim}{\mathbf{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{24}$$

Note that $\underset{\sim}{\mathbf{A}}$ is symmetric but not Hermitian. Its eigenvalue decomposition reads

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{\Lambda}}\underset{\sim}{\mathbf{L}}^{-1} = \underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{\Lambda}}\underset{\sim}{\mathbf{L}}^{T} = \tag{25}$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1-i}{\sqrt{2}} & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Since $\underset{\sim}{\mathbf{L}}$ is unitary and $|\lambda_1| = |\lambda_2| = 1$, matrix $\underset{\sim}{\mathbf{A}}$ defined in equation (24) is unconditionally stable. This also follows from its singular value decomposition, which yields $s_1 = s_2 = 1$.

Stability analysis of the extrapolation operator, constructed with the modal decomposition method

As we mentioned in the introduction, the exact extrapolation operator $\underset{\sim}{\mathbf{W}}$ should be symmetric on basis of reciprocity considerations. From the discussion and examples above it follows that symmetry alone does not guarantee that an eigenvalue decomposition of $\underset{\sim}{\mathbf{W}}$ exists. In this section we review the construction of $\underset{\sim}{\mathbf{W}}$ according to the modal decomposition approach and discuss its stability properties in terms of its eigenvalues. In [14] and [8] we define the extrapolation matrix $\underset{\sim}{\mathbf{W}}$ as a solution of the one-way wave equation

$$\frac{\partial \underset{\sim}{\mathbf{W}}(z, z_n)}{\partial z} = -j\, \underset{\sim}{\mathbf{H}}_1(z_n)\underset{\sim}{\mathbf{W}}(z, z_n), \tag{26}$$

for $z \geq z_n$, with $\underset{\sim}{\mathbf{W}}(z_n, z_n) = \underset{\sim}{\mathbf{I}}$. The *square-root* operator $\underset{\sim}{\mathbf{H}}_1(z_n)$ is related to the Helmholtz operator $\underset{\sim}{\mathbf{H}}_2(z_n)$, according to

$$\underset{\sim}{\mathbf{H}}_2(z_n) = \underset{\sim}{\mathbf{H}}_1(z_n)\underset{\sim}{\mathbf{H}}_1(z_n). \tag{27}$$

The Helmholtz operator is Hermitian and real-valued, so it can be written as

$$\underset{\sim}{\mathbf{H}}_2(z_n) = \underset{\sim}{\mathbf{L}}(z_n)\underset{\sim}{\mathbf{\Lambda}}(z_n)\underset{\sim}{\mathbf{L}}^{-1}(z_n) = \underset{\sim}{\mathbf{L}}(z_n)\underset{\sim}{\mathbf{\Lambda}}(z_n)\underset{\sim}{\mathbf{L}}^{T}(z_n), \tag{28}$$

with

$$\underset{\sim}{\mathbf{\Lambda}}(z_n) = \mathrm{diag}\Big(\lambda_1(z_n) \cdots \lambda_m(z_n) \cdots \lambda_M(z_n)\Big), \tag{29}$$

where the eigenvalues $\lambda_m(z_n)$ are real-valued and $\underset{\sim}{\mathbf{L}}(z_n)$ is unitary and real-valued. For $\underset{\sim}{\mathbf{H}}_1(z_n)$ we may thus write

$$\underset{\sim}{\mathbf{H}}_1(z_n) = \underset{\sim}{\mathbf{L}}(z_n)\underset{\sim}{\mathbf{\Lambda}}^{1/2}(z_n)\underset{\sim}{\mathbf{L}}^{T}(z_n). \tag{30}$$

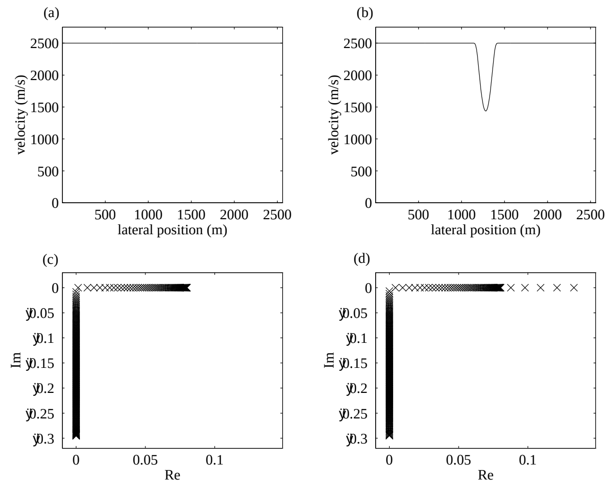

__Fig. 1:__ The location of the eigenvalues $\lambda_m^{1/2}$ of the square-root operator in the complex plane. (a) Homogeneous medium. (b) Laterally variant medium. (c) Eigenvalues for homogeneous medium. (d) Eigenvalues for laterally variant medium.

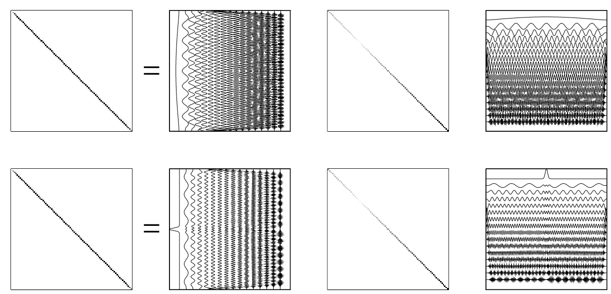

__Fig. 2:__ Schematical representation of the eigenvalue decomposition of the square-root operator for the homogeneous medium (top) and for the laterally variant medium (bottom).

From equation (29) we obtain

$$\underset{\sim}{\mathbf{\Lambda}}^{1/2}(z_n) = \mathrm{diag}\Big(\lambda_1^{1/2}(z_n) \cdots \lambda_m^{1/2}(z_n) \cdots \lambda_M^{1/2}(z_n)\Big). \tag{31}$$

Note that the square-root operator $\underset{\sim}{\mathbf{H}}_1(z_n)$ is symmetric but not Hermitian, since not all eigenvalues $\lambda_m^{1/2}(z_n)$ are real-valued. The sign of $\lambda_m^{1/2}(z_n)$ is chosen such that when $\lambda_m(z_n)$ is positive, then

$$\Re\{\lambda_m^{1/2}(z_n)\} \geq 0, \tag{32}$$

and when $\lambda_m(z_n)$ is negative, then

$$\Im\{\lambda_m^{1/2}(z_n)\} \leq 0, \tag{33}$$

see Figure 1. Equation (30) is visualized in Figure 2.

Using a Taylor series expansion, we obtain from equation (26)

$$\underset{\sim}{\mathbf{W}}(z, z_n) = \sum_{k=0}^{\infty} \frac{(z - z_n)^k}{k!}(-j)^k \underset{\sim}{\mathbf{H}}_1^k(z_n), \tag{34}$$

where, according to equation (30),

$$\underset{\sim}{\mathbf{H}}_1^k(z_n) = \underset{\sim}{\mathbf{L}}(z_n)\underset{\sim}{\mathbf{\Lambda}}^{k/2}(z_n)\underset{\sim}{\mathbf{L}}^T(z_n). \tag{35}$$

Substitution in equation (34) gives

$$\underset{\sim}{\mathbf{W}}(z, z_n) = \underset{\sim}{\mathbf{L}}(z_n)\left[\sum_{k=0}^{\infty} \frac{(z - z_n)^k}{k!}(-j)^k \underset{\sim}{\mathbf{\Lambda}}^{k/2}(z_n)\right]\underset{\sim}{\mathbf{L}}^T(z_n).$$

The term between the brackets is recognized as the Taylor expansion of an exponential function. Hence

$$\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n) = \underset{\sim}{\mathbf{L}}(z_n)\underset{\sim}{\breve{\mathbf{W}}}(z_{n+1}, z_n)\underset{\sim}{\mathbf{L}}^T(z_n), \tag{36}$$

where

$$\underset{\sim}{\breve{\mathbf{W}}}(z_{n+1}, z_n) = \exp\{-j(z_{n+1} - z_n)\underset{\sim}{\mathbf{\Lambda}}^{1/2}(z_n)\}. \tag{37}$$

The latter matrix is a diagonal matrix containing the eigenvalues $\exp\{-j(z_{n+1} - z_n)\lambda_m^{1/2}(z_n)\}$ of $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$. Note that $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$ is symmetric but not Hermitian, since its eigenvalues are complex-valued. Due to the properties of $\lambda_m^{1/2}(z_n)$, as described by equations (32) and (33), the moduli of all eigenvalues of $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$ are equal to or smaller than unity, see Figure 3. Together with the fact that $\underset{\sim}{\mathbf{L}}(z_n)$ is unitary, this implies that $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$, as defined in equation (36), is *unconditionally* stable. In recursive applications, $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$ is usually modified after each recursion step. According to equation (6), this has no negative effect on the stability.

We conclude this section by deriving the singular values of $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$. They are defined as the non-negative square-roots of the eigenvalues of $\underset{\sim}{\mathbf{B}} = \underset{\sim}{\mathbf{W}}\underset{\sim}{\mathbf{W}}^H$ (equation 12). From equation (36) we obtain

$$\underset{\sim}{\mathbf{B}} = \underset{\sim}{\mathbf{W}}\underset{\sim}{\mathbf{W}}^H = \underset{\sim}{\mathbf{L}}(\underset{\sim}{\breve{\mathbf{W}}}\underset{\sim}{\breve{\mathbf{W}}}^H)\underset{\sim}{\mathbf{L}}^T. \tag{38}$$

Apparently the eigenvalues of $\underset{\sim}{\mathbf{B}}$ are equal to the squared moduli of the eigenvalues of $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$. Hence, the singular values of $\underset{\sim}{\mathbf{W}}(z_{n+1}, z_n)$ are equal to the moduli of its eigenvalues.

Conclusions

Extrapolation operators based on the 'local explicit' approach are not symmetric when the medium is laterally variant. As has been pointed out by Dellinger and Etgen [5], the stability analysis of these operators can best be done with a singular value decomposition. When one or more singular values exceed unity, the operator is not unconditionally stable.

Extrapolation operators based on the 'modal decomposition' approach are symmetric, even when the medium is laterally variant. Moreover, since they are constructed from unitary eigenvector matrices and an eigenvalue matrix with eigenvalues not exceeding unity, they are always unconditionally stable ([14], [8]). For these operators, singular value analysis does not give additional information.

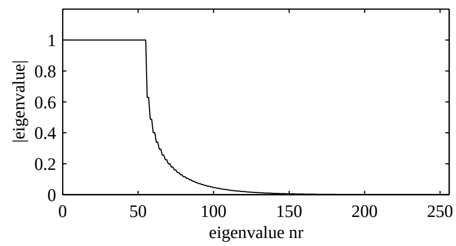

Fig. 3: *The moduli of the eigenvalues of the extrapolation matrix, constructed with the modal decomposition approach, see equation (36). Since the eigenvector matrix is unitary and none of the eigenvalues exceeds unity, the extrapolation matrix defined in equation (36) is unconditionally stable.*

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