Theory of acoustic daylight imaging revisited

Kees Wapenaar, Deyan Draganov, Jan Thorbecke and Jacob Fokkema, Delft University of Technology

Summary

Acoustic daylight imaging (i.e., the principle of using passive noise measurements to derive the reflection response and, subsequently, form an image of the Earth's interior) is based on a relation between reflection and transmission responses derived by Claerbout in 1968. Although this derivation applies to horizontally layered media, Claerbout conjectured that a slightly modified form applies to 3-D inhomogeneous media as well. In this paper we prove this conjecture and we analyze a variant related to interferometric imaging (i.e., inversion of crosscorrelated data) as discussed by Schuster (2001).

Introduction

In 1968 Claerbout showed that the reflection response of a horizontally layered medium can be synthesized from the autocorrelation of its transmission response (Claerbout, 1968). This implies that, when a natural noise source in the Earth's subsurface emits waves to the surface, passive measurements of the noise at the surface suffice to compute the reflection response of the Earth's subsurface. The seismic wavelet in this synthesized reflection response is the autocorrelation of the noise source in the subsurface. The principle of using passive noise measurements to derive the reflection response and, subsequently, form an image of the Earth's interior, was coined "acoustic daylight imaging".

The derivation in the 1968 paper was strictly one-dimensional. Later Claerbout conjectured for the 3-D situation that "by crosscorrelating noise traces recorded at two locations on the surface, we can construct the wave field that would be recorded at one of the locations if there was a source at the other". Rickett and Claerbout (1999) discuss several applications of this principle (including helioseismology and reservoir monitoring) as well as numerical modeling studies to confirm the conjecture. The modeling studies showed that "longer time series, and a white spatial distribution of random noise events would be necessary for the conjecture to work in practice".

Schuster (2001) verified the conjecture for laterally varying media by applying the method of stationary phase to multifold Kirchhoff integrals. Also he analyzed variants of the principle, involving crosscorrelations of different receiver traces in one and the same seismic shot record. He used the term "interferometric imaging" for any algorithm that inverts crosscorrelated data. Schuster's paper clearly shows how the correlation of specific events leads to new events but it does not provide a general proof of Claerbout's conjecture.

In this paper we employ a reciprocity theorem of the correlation type to obtain general relations between reflection and transmission responses, including their coda, for arbitrary inhomogeneous 3-D media. One of these relations proves Claerbout's conjecture for the 3-D situation and gives insight in the requirements with respect to the spatial distribution of random noise events. Another relation validates the interferometric imaging principle for the general 3-D situation and accounts for what Schuster calls "other stuff".

One-way reciprocity theorem

We use acoustic reciprocity as a starting point to derive the relations between reflection and transmission responses. Acoustic reciprocity formulates a relation between two acoustic states in one and the same domain [de Hoop (1988), Fokkema and van den Berg (1993)]. We distinguish between two-way and one-way reciprocity theorems of the convolution type and of the correlation type. In this paper we choose to work with the one-way reciprocity theorem of the correlation type. The choice for 'one-way' stems from the fact that the concepts of reflection and transmission apply to downgoing and upgoing wave fields (rather than to full wave fields). One-way reciprocity theorems are based on a coupled system of one-way wave equations for flux-normalized downgoing and upgoing waves and hold for primary and multiply reflected waves with any propagation angle in 3-D inhomogeneous media (Wapenaar and Grimbergen, 1996). The reason for choosing the correlation type is obvious: both Claerbout's conjecture about acoustic daylight imaging as well as Schuster's proposal for interferometric imaging are based on crosscorrelated data.

We define a plan-parallel domain $\mathcal{D}$ embedded between surfaces $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_m$ at depth levels $x_{3,0}+\epsilon$ and $x_{3,m}-\epsilon$, respectively. The two acoustic states will be denoted by subscripts A and B. We assume that in domain $\mathcal{D}$ the medium parameters in both states are identical, lossless and 3-D inhomogeneous. Furthermore, domain $\mathcal{D}$ is assumed to be source-free. For this situation, the one-way reciprocity theorem of the correlation type reads in the space-frequency $(\mathbf{x},\omega)$ domain

$$\int_{\partial\mathcal{D}_0}\{(P_A^+)^*P_B^+ - (P_A^-)^*P_B^-\}\mathrm{d}^2\mathbf{x} = \int_{\partial\mathcal{D}_m}\{(P_A^+)^*P_B^+ - (P_A^-)^*P_B^-\}\mathrm{d}^2\mathbf{x}, \tag{1}$$

where P^+ and P^- are flux-normalized downgoing and upgoing wave fields and * denotes complex conjugation. In the derivation of this theorem evanescent waves have been neglected; all propagating modes (including multiple reflections) are accounted for. Note that for the special case in which the wave fields in states A and B are identical, equation (1) states that the net power-flux through surface $\partial\mathcal{D}_0$ is equal to the net power-flux through surface $\partial\mathcal{D}_m$. In the following the wave fields in states A and B will be chosen different from each other and equation (1) will be used to derive relations between the reflection and transmission responses of the inhomogeneous medium in domain $\mathcal{D}$.

Proof of Claerbout's conjecture

We consider the configuration of Figure 1. We define a free surface at $x_{3,0}$, i.e., just above $\partial\mathcal{D}_0$, and a homogeneous source-free half-space below $\partial\mathcal{D}_m$. For states A and B we choose sources for downgoing waves at $\mathbf{x}_A$ and $\mathbf{x}_B$ at the free surface, that is, we define $\mathbf{x}_A = (\mathbf{x}_{H,A}, x_{3,0})$ and $\mathbf{x}_B = (\mathbf{x}_{H,B}, x_{3,0})$, where $\mathbf{x}_H$ denotes the horizontal coordinate vector: $\mathbf{x}_H = (x_1, x_2)$. In both states, the total downgoing wave field at $\partial\mathcal{D}_0$ consists of the superposition of a spatial delta function directly below the source and the downward reflected upgoing wave field due to the presence of the free surface. For state A this is illustrated in Figure 1. Hence, for $\mathbf{x}$ at $\partial\mathcal{D}_0$ we have

$$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = \delta(\mathbf{x}_H - \mathbf{x}_{H,A})s_A(\omega) + rP_A^-(\mathbf{x}, \mathbf{x}_A, \omega), \quad (2)$$

$$P_B^+(\mathbf{x}, \mathbf{x}_B, \omega) = \delta(\mathbf{x}_H - \mathbf{x}_{H,B})s_B(\omega) + rP_B^-(\mathbf{x}, \mathbf{x}_B, \omega), \quad (3)$$

$$P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = R(\mathbf{x}, \mathbf{x}_A, \omega)s_A(\omega), \quad (4)$$

$$P_B^-(\mathbf{x}, \mathbf{x}_B, \omega) = R(\mathbf{x}, \mathbf{x}_B, \omega)s_B(\omega), \quad (5)$$

where $s_A(\omega)$ and $s_B(\omega)$ are the source spectra for both states, r is the reflection coefficient of the free surface ($r = -1$) and $R(\mathbf{x}, \mathbf{x}_A, \omega)$ is the reflection response of the inhomogeneous medium in $\mathcal{D}$, including all internal and free surface multiples, for a source at $\mathbf{x}_A$ and a receiver at $\mathbf{x}$. A similar remark applies to $R(\mathbf{x}, \mathbf{x}_B, \omega)$. At $\partial\mathcal{D}_m$ there are only downgoing waves since we assumed that the half-space below $\partial\mathcal{D}_m$ is homogeneous and source-free. Hence, for $\mathbf{x}$ at $\partial\mathcal{D}_m$ we have

$$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = T(\mathbf{x}, \mathbf{x}_A, \omega)s_A(\omega), \quad (6)$$

$$P_B^+(\mathbf{x}, \mathbf{x}_B, \omega) = T(\mathbf{x}, \mathbf{x}_B, \omega)s_B(\omega), \quad (7)$$

$$P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = P_B^-(\mathbf{x}, \mathbf{x}_B, \omega) = 0, \quad (8)$$

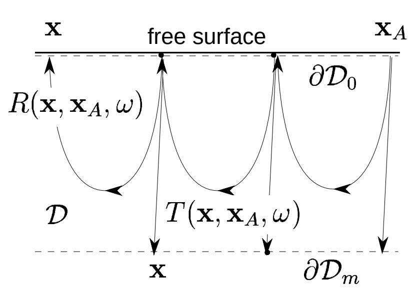

Fig. 1: *Domain $\mathcal{D}$ between surfaces $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_m$. The medium in $\mathcal{D}$ is inhomogeneous in the x_1-, x_2- and x_3-directions. There is a free surface just above $\partial\mathcal{D}_0$; the half-space below $\partial\mathcal{D}_m$ is homogeneous and source-free.*

where $T(\mathbf{x}, \mathbf{x}_A, \omega)$ is the transmission response of the inhomogeneous medium in $\mathcal{D}$, including all internal and free surface multiples. A similar remark applies to $T(\mathbf{x}, \mathbf{x}_B, \omega)$. Substitution of equations (2) through (8) into equation (1), using $r = -1$ and $R(\mathbf{x}_A, \mathbf{x}_B, \omega) = R(\mathbf{x}_B, \mathbf{x}_A, \omega)$, and dividing both sides of the equation by $s_A^*(\omega)s_B(\omega)$ yields

$$\delta(\mathbf{x}_{H,B} - \mathbf{x}_{H,A}) - 2\Re\big[R(\mathbf{x}_A, \mathbf{x}_B, \omega)\big] = \int_{\partial\mathcal{D}_m} T^*(\mathbf{x}, \mathbf{x}_A, \omega)T(\mathbf{x}, \mathbf{x}_B, \omega)\mathrm{d}^2\mathbf{x}, \quad (9)$$

where $\Re$ denotes the real part. Using reciprocity for the transmission responses [$T(\mathbf{x}_A, \mathbf{x}, \omega) = T(\mathbf{x}, \mathbf{x}_A, \omega)$ etc.] and reorganizing the order of the terms we obtain

$$2\Re\big[R(\mathbf{x}_A, \mathbf{x}_B, \omega)\big] = \delta(\mathbf{x}_{H,B} - \mathbf{x}_{H,A}) - \int_{\partial\mathcal{D}_m} T^*(\mathbf{x}_A, \mathbf{x}, \omega)T(\mathbf{x}_B, \mathbf{x}, \omega)\mathrm{d}^2\mathbf{x}. \quad (10)$$

Equation (10) shows that the real part of the reflection response can be obtained from the transmission response. Since the reflection response is causal, the imaginary part can be obtained via the Hilbert transform of the real part. Alternatively, $2\Re\big[R(\mathbf{x}_A, \mathbf{x}_B, \omega)\big]$ can be transformed to the time domain and subsequently be multiplied by the Heaviside step-function, yielding $R(\mathbf{x}_A, \mathbf{x}_B, t)$. The validity of equation (10) is confirmed with a numerical experiment in Figures 2 and 3.

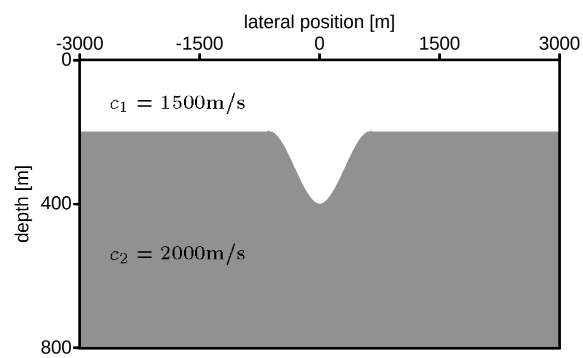

Fig. 2: *Syncline model*

Note that we have nearly achieved the proof of Claerbout's conjecture. The term $T^*(\mathbf{x}_A, \mathbf{x}, \omega)T(\mathbf{x}_B, \mathbf{x}, \omega)$ represents the crosscorrelation of traces recorded at two locations ($\mathbf{x}_A$ and $\mathbf{x}_B$) on the surface for a source at $\mathbf{x}$ in the subsurface; the term $R(\mathbf{x}_A, \mathbf{x}_B, \omega)$ is the wave field that would be recorded at one of the locations ($\mathbf{x}_A$) if there was a source at the other ($\mathbf{x}_B$). The main discrepancy with the conjecture is the integral in equation (10) over all possible source positions $\mathbf{x}$ at surface $\partial\mathcal{D}_m$. It can not be evaluated in practice because the transmission responses are not available for all individual source positions $\mathbf{x}$. To remedy this, we will assume uncorrelated noise sources in the subsurface. The procedure is as follows. We apply an inverse Fourier transform to all terms in equation (10) and discretize the integral, according to

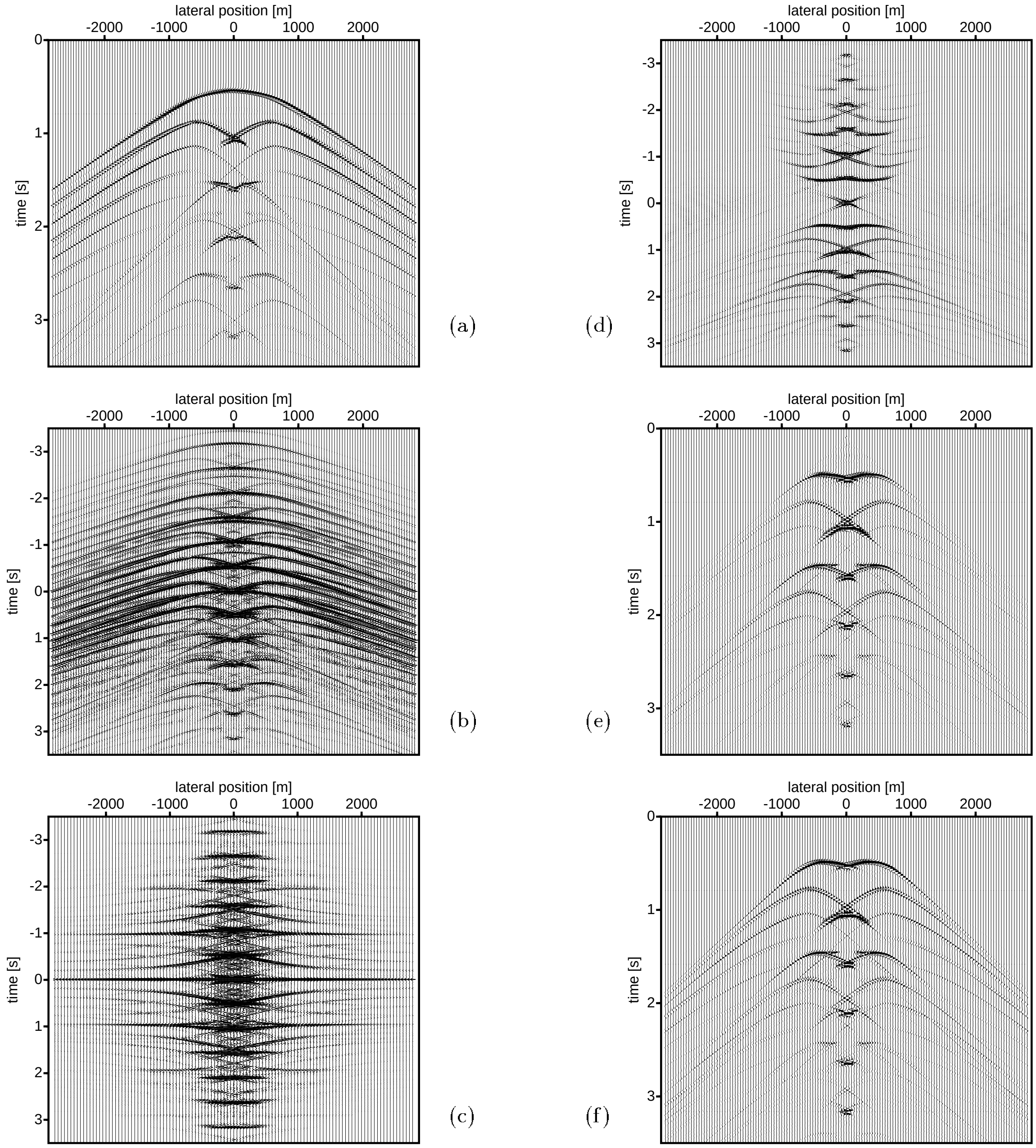

Fig. 3: *Verification of equation (10) for the 2-D inhomogeneous medium in Figure 2. (a) Transmission response* $T(\mathbf{x}_A, \mathbf{x}, \omega)$ *(displayed in the time domain) for a fixed source at* $\mathbf{x} = (0, 800)$ *and a range of receiver positions* $\mathbf{x}_A$ *at the acquisition surface. (b) Correlation of* $T(\mathbf{x}_A, \mathbf{x}, \omega)$ *of figure (a) with 'master trace'* $T(\mathbf{x}_B, \mathbf{x}, \omega)$*, with* $\mathbf{x}_B = (0, 0)$*. (c) As in (b), but now for fixed* $\mathbf{x}_A = \mathbf{x}_B = (0, 0)$ *and all source positions* $\mathbf{x}$ *along the lower boundary. The sum of these traces represents the integral in the right-hand side of equation (10) for* $\mathbf{x}_A = \mathbf{x}_B = (0, 0)$ *and is shown as the central trace in figure (d). (d) Result of the integral in the right-hand side of equation (10) (times -1), for* $\mathbf{x}_B = (0, 0)$ *and all* $\mathbf{x}_A$ *at the acquisition surface. The cross in the centre is a band-limited representation of the delta function in equation (10) (bear in mind that evanescent waves are neglected in equation (1)). (e) Causal part of the data in (d), after muting the bandlimited delta function. This is the reflection response for a source at* $\mathbf{x}_B = (0, 0)$ *and receivers at all* $\mathbf{x}_A$ *at the acquisition surface. (f) For comparison, the directly modeled reflection response.*

$$R(\mathbf{x}_A, \mathbf{x}_B, t) + R(\mathbf{x}_A, \mathbf{x}_B, -t) = \delta(\mathbf{x}_{H,B} - \mathbf{x}_{H,A})\delta(t) - \sum_i \int_{-\infty}^{\infty} T(\mathbf{x}_A, \mathbf{x}_i, t')T(\mathbf{x}_B, \mathbf{x}_i, t + t')\mathrm{d}t', \qquad (11)$$

where the sum is applied over all $\mathbf{x}_i$ at $\partial\mathcal{D}_m$. We now introduce the transmission responses of noise sources at $\mathbf{x}_i$ and $\mathbf{x}_j$, denoted by $T_N(\mathbf{x}_A, \mathbf{x}_i, t) = T(\mathbf{x}_A, \mathbf{x}_i, t) * N_i(t)$ and $T_N(\mathbf{x}_B, \mathbf{x}_j, t) = T(\mathbf{x}_B, \mathbf{x}_j, t) * N_j(t)$. Assuming that $N_i(t)$ and $N_j(t)$ are uncorrelated white noise sources, we have

$$\int_{-\infty}^{\infty} T_N(\mathbf{x}_A, \mathbf{x}_i, t')T_N(\mathbf{x}_B, \mathbf{x}_j, t + t')\mathrm{d}t' = \delta_{ij} \int_{-\infty}^{\infty} T(\mathbf{x}_A, \mathbf{x}_i, t')T(\mathbf{x}_B, \mathbf{x}_j, t + t')\mathrm{d}t'. \qquad (12)$$

We may thus rewrite equation (11) as

$$R(\mathbf{x}_A, \mathbf{x}_B, t) + R(\mathbf{x}_A, \mathbf{x}_B, -t) = \delta(\mathbf{x}_{H,B} - \mathbf{x}_{H,A})\delta(t) - \sum_i \sum_j \int_{-\infty}^{\infty} T_N(\mathbf{x}_A, \mathbf{x}_i, t')T_N(\mathbf{x}_B, \mathbf{x}_j, t + t')\mathrm{d}t', \qquad (13)$$

or

$$R(\mathbf{x}_A, \mathbf{x}_B, t) + R(\mathbf{x}_A, \mathbf{x}_B, -t) = \delta(\mathbf{x}_{H,B} - \mathbf{x}_{H,A})\delta(t) - \int_{-\infty}^{\infty} T_{\mathrm{obs}}(\mathbf{x}_A, t')T_{\mathrm{obs}}(\mathbf{x}_B, t + t')\mathrm{d}t', \qquad (14)$$

with

$$T_{\mathrm{obs}}(\mathbf{x}_A, t) = \sum_i T_N(\mathbf{x}_A, \mathbf{x}_i, t), \qquad (15)$$

$$T_{\mathrm{obs}}(\mathbf{x}_B, t) = \sum_i T_N(\mathbf{x}_B, \mathbf{x}_i, t). \qquad (16)$$

Note that $T_{\mathrm{obs}}(\mathbf{x}_A, t)$ and $T_{\mathrm{obs}}(\mathbf{x}_B, t)$ may be seen as transmission responses, observed at $\mathbf{x}_A$ and $\mathbf{x}_B$ on $\partial\mathcal{D}_0$, due to a distribution of uncorrelated noise sources at a number of positions $\mathbf{x}_i$ on $\partial\mathcal{D}_m$. The integral in the right-hand side of equation (14) describes the crosscorrelation of these observations. This finalizes the proof of Claerbout's conjecture.

Of course in reality the noise sources will not be evenly distributed along a single surface $\partial\mathcal{D}_m$. The actual depth of the sources is almost immaterial, since the extra traveltime between the actual source depth and $\partial\mathcal{D}_m$ drops out in the correlation process. Of course the accuracy degrades in case of an irregular source distribution. For other practical limitations, like crosstalk, non-whiteness etc., the accuracy of equation (14) depends on how well equation (12) is fulfilled. Finally, note that we assumed that the half-space below $\partial\mathcal{D}_m$ is homogeneous. When this assumption is not fulfilled, extra events will appear in the transmission responses which will not be correctly mapped to the reflection response.

Analysis of interferometric imaging

We consider again the configuration of Figure 1, but we remove the free surface and choose both half-spaces above $\partial\mathcal{D}_0$ and below $\partial\mathcal{D}_m$ homogeneous and source-free. We replace state A in equations (2) and (4) for $\mathbf{x}$ at $\partial\mathcal{D}_0$ by

$$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = \delta(\mathbf{x}_H - \mathbf{x}_{H,A})s_A(\omega), \qquad (17)$$

$$P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = R_0(\mathbf{x}, \mathbf{x}_A, \omega)s_A(\omega), \qquad (18)$$

and in equation (6) for $\mathbf{x}$ at $\partial\mathcal{D}_m$ by

$$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = T_0(\mathbf{x}, \mathbf{x}_A, \omega)s_A(\omega), \qquad (19)$$

where $R_0(\mathbf{x}, \mathbf{x}_A, \omega)$ and $T_0(\mathbf{x}, \mathbf{x}_A, \omega)$ are the reflection and transmission responses of the inhomogeneous medium in $\mathcal{D}$, including all internal multiples, but without the free surface multiples. For state B we choose again the wave fields in equations (3), (5) and (7). Substitution in equation (1), using $r = -1$, yields

$$R(\mathbf{x}_A, \mathbf{x}_B, \omega) = \delta(\mathbf{x}_{H,B} - \mathbf{x}_{H,A}) - \int_{\partial\mathcal{D}_0} R_0^*(\mathbf{x}_A, \mathbf{x}, \omega)R(\mathbf{x}_B, \mathbf{x}, \omega)\mathrm{d}^2\mathbf{x} \underbrace{- \int_{\partial\mathcal{D}_m} T_0^*(\mathbf{x}_A, \mathbf{x}, \omega)T(\mathbf{x}_B, \mathbf{x}, \omega)\mathrm{d}^2\mathbf{x}}_{''\text{other stuff}''}. \qquad (20)$$

This equation corresponds with one of the interferometric imaging relations of Schuster (2001). It shows that the reflection response $R(\mathbf{x}_A, \mathbf{x}_B, \omega)$ is obtained by correlating reflections recorded at $\mathbf{x}_A$ and $\mathbf{x}_B$. The last term quantifies what Schuster calls "other stuff". A further discussion is beyond the scope of this paper.

Conclusions

We have used the reciprocity theorem of the correlation type for one-way wave fields to derive different relations between the transmission and reflection responses of an arbitrary 3-D inhomogeneous medium. One of these relations proves Claerbout's conjecture about acoustic daylight imaging and gives insight in the requirements with respect to the spatial distribution of random noise events. Another relation validates the interferometric imaging principle for the arbitrary 3-D situation.

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