

# Elastodynamic reciprocity theorems for time-lapse seismic methods

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## Summary

In this paper we present elastodynamic reciprocity theorems for the full and one-way wave equations and we discuss their application in time-lapse seismic methods.

## Introduction

Reciprocity theorems play an important role in formulating true amplitude operations on seismic wave fields, such as multiple elimination, migration and characterization. In general, a reciprocity theorem interrelates the quantities that characterize two admissible physical states that could occur in one and the same domain (de Hoop, 1988). One state is identified with an actual measurement, while the other state can either be a computational state (e.g. migration operators), a desired state (e.g. multiple-free data) or another measurement (characterizing time-lapse differences in the reservoir). In previous work we discussed the application of acoustic reciprocity theorems for time-lapse seismic methods. In particular, in Fokkema et al. (1999) we discussed applications based on the full and one-way wave equations. Here we extend this to the elastodynamic situation.

## Reciprocity theorem for the full wave field

In the space-frequency  $(\mathbf{x}, \omega)$  domain, the equations that govern elastodynamic wave motion read

$$j\omega \rho V_i - \partial_j T_{ij} = F_i \quad (1)$$

$$j\omega s_{ijkl} T_{kl} - \frac{1}{2}(\partial_i V_j + \partial_j V_i) = -H_{ij}, \quad (2)$$

where  $T_{ij}$  is the stress,  $V_i$  is the particle velocity,  $\rho$  is the volume density of mass,  $s_{ijkl}$  is the compliance,  $F_i$  is the volume source density of volume force and  $H_{ij}$  is the volume source density of deformation rate. The Latin subscripts take on the values 1 to 3 and the summation convention applies to repeated subscripts. The stress obeys the symmetry relation  $T_{ij} = T_{ji}$ . The compliance obeys  $s_{ijkl} = s_{jikl} = s_{jilk}$  and, assuming that the wave motion occurs adiabatically,  $s_{ijkl} = s_{klij}$ . We introduce two elastodynamic states (i.e., wave fields, medium parameters and sources), that will be distinguished by the subscripts  $A$  and  $B$ . For these two states we consider the interaction quantity  $\partial_j \{T_{ij,A} V_{i,B} - V_{i,A} T_{ij,B}\}$ . Applying the product rule for differentiation, substituting equations (1) and (2) for states  $A$  and  $B$ , integrating the result over a volume  $\mathcal{V}$  with boundary  $\partial\mathcal{V}$  and outward pointing normal vector  $\mathbf{n} = (n_1, n_2, n_3)$  (see Figure 1) and applying the theorem of Gauss yields

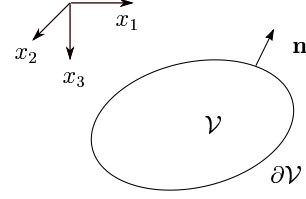


Fig. 1: Configuration for Betti-Rayleigh reciprocity theorem.

$$\int_{\mathbf{x} \in \partial\mathcal{V}} \{T_{ij,A} V_{i,B} - V_{i,A} T_{ij,B}\} n_j dA = \quad (3)$$

$$j\omega \int_{\mathbf{x} \in \mathcal{V}} \{T_{ij,A} \Delta s_{ijkl} T_{kl,B} - V_{i,A} \Delta \rho V_{i,B}\} dV +$$

$$\int_{\mathbf{x} \in \mathcal{V}} \{T_{ij,A} H_{ij,B} + V_{i,A} F_{i,B} - F_{i,A} V_{i,B} - H_{ij,A} T_{ij,B}\} dV,$$

with  $\Delta s_{ijkl} = s_{ijkl,B} - s_{ijkl,A}$  and  $\Delta \rho = \rho_B - \rho_A$ . Equation (3) is the Betti-Rayleigh reciprocity theorem.

## Reciprocity theorem for one-way wave fields

We introduce a system of coupled equations for the one-way wave fields  $\mathbf{P}^+$  and  $\mathbf{P}^-$ , propagating in the positive and negative depth direction, respectively, according to

$$\partial_3 \mathbf{P} = \hat{\mathbf{B}} \mathbf{P} + \mathbf{S}, \quad (4)$$

(the hat denotes a pseudo-differential operator), with

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}^+ \\ \mathbf{P}^- \end{pmatrix}, \quad \mathbf{P}^\pm = \begin{pmatrix} \Phi^\pm \\ \Psi^\pm \\ \Upsilon^\pm \end{pmatrix} \quad (5)$$

and

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}^+ \\ \mathbf{S}^- \end{pmatrix}, \quad \mathbf{S}^\pm = \begin{pmatrix} S_\phi^\pm \\ S_\psi^\pm \\ S_\nu^\pm \end{pmatrix}, \quad (6)$$

where  $\Phi^\pm$ ,  $\Psi^\pm$  and  $\Upsilon^\pm$  represent the (flux-normalized) down- and upgoing quasi-P, quasi-S1 and quasi-S2 waves, respectively.  $\mathbf{S}^+$  and  $\mathbf{S}^-$  are source vectors for these one-way wave fields. The one-way operator matrix  $\hat{\mathbf{B}}$  is defined as

$$\hat{\mathbf{B}} = \begin{pmatrix} -j\omega \hat{\mathbf{A}}^+ & \mathbf{O} \\ \mathbf{O} & j\omega \hat{\mathbf{A}}^- \end{pmatrix} + \begin{pmatrix} \hat{\mathbf{T}}^+ & \hat{\mathbf{R}}^- \\ -\hat{\mathbf{R}}^+ & \hat{\mathbf{T}}^- \end{pmatrix}, \quad (7)$$

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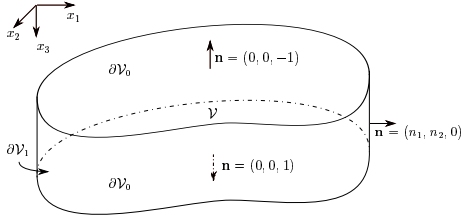


Fig. 2: Modified configuration for the one-way reciprocity theorem. The combination of the two planar surfaces is denoted by  $\partial\mathcal{V}_0$ ; the cylindrical surface is denoted by  $\partial\mathcal{V}_1$ .

where  $\hat{\mathbf{A}}^\pm$  is the vertical slowness operator, and  $\hat{\mathbf{R}}^\pm$  and  $\hat{\mathbf{T}}^\pm$  are the reflection and transmission operators, respectively. We introduce two different states that will be distinguished by the subscripts  $A$  and  $B$ . For these two states we consider the interaction quantity  $\partial_3\{\mathbf{P}_A^t \mathbf{N} \mathbf{P}_B\}$ , with  $\mathbf{N} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{pmatrix}$  or, written alternatively,  $\partial_3\{(\mathbf{P}_A^+)^t \mathbf{P}_B^- - (\mathbf{P}_A^-)^t \mathbf{P}_B^+\}$ . The superscript  $t$  denotes transposition. Applying the product rule for differentiation, substituting the one-way wave equation (4) for states  $A$  and  $B$ , integrating the result over a cylindrical volume  $\mathcal{V}$  with boundary  $\partial\mathcal{V}_0 \cup \partial\mathcal{V}_1$  (see Figure 2), applying the theorem of Gauss and using the symplectic relation  $\hat{\mathbf{B}}^t \mathbf{N} = -\mathbf{N} \hat{\mathbf{B}}$ , yields the following one-way reciprocity theorem

$$\begin{aligned} \int_{\mathbf{x} \in \partial\mathcal{V}_0} \mathbf{P}_A^t \mathbf{N} \mathbf{P}_B n_3 dA &= \int_{\mathbf{x} \in \mathcal{V}} \mathbf{P}_A^t \mathbf{N} \{ \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A \} \mathbf{P}_B dV \\ &+ \int_{\mathbf{x} \in \mathcal{V}} \{ \mathbf{P}_A^t \mathbf{N} \mathbf{S}_B + \mathbf{S}_A^t \mathbf{N} \mathbf{P}_B \} dV. \end{aligned} \quad (8)$$

For elastodynamic one-way wave fields this reciprocity theorem has been strictly proven only for laterally invariant media; for acoustic one-way wave fields it has been proven for laterally varying media as well (Wapenaar and Grimbergen, 1996). In the following we will use it without further proof for elastodynamic one-way wave fields in laterally varying media.

### Reciprocity theorems for time-lapse seismic

Since in a reciprocity theorem two states interact, it is optimally fitted to formulate the relation between two measurements in a time-lapse seismic experiment. State  $A$  is associated with the reference wave field at, say,  $t = t_1$ , while state  $B$  is associated with the monitoring wave field at, say,  $t = t_2 > t_1$ . It is noted that  $t_2 - t_1$  is much longer than the seismic experiment time. In our analysis  $\mathbb{R}^3$  is divided in three domains (Figure 3):  $\mathcal{V}_o$  is the domain where there are no differences between the material parameters in the two states, mostly associated with the domain above the reservoir (i.e.,  $x_3 \leq x_3^1$ ); the domain  $\mathcal{V}_c$ , for example associated with the reservoir ( $x_3^1 < x_3 \leq x_3^2$ ), where there is a difference between the material parameters in the two states mostly due

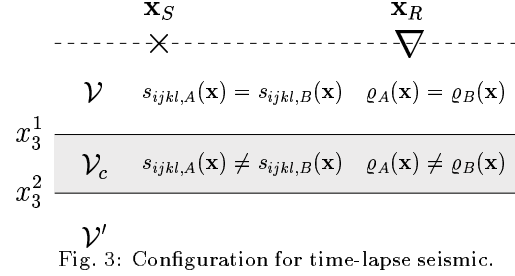


Fig. 3: Configuration for time-lapse seismic.

to the reservoir production history; and  $\mathcal{V}'$  denotes the complement of  $\mathcal{V} = \mathcal{V}_o \cup \mathcal{V}_c$  (i.e.,  $x_3 > x_3^2$ ); the material parameters in this domain may or may not be different.

#### Full wave equation

In order to simplify the analysis we only consider point sources of the volume force type. The source of state  $A$  is taken at  $\mathbf{x} = \mathbf{x}_S$  in the  $x_m$ -direction, while the source of state  $B$  is taken at  $\mathbf{x} = \mathbf{x}_R$  in the  $x_n$ -direction, according to

$$F_{i,A}(\mathbf{x}, \omega) = f_A(\omega) \delta(\mathbf{x} - \mathbf{x}_S) \delta_{im}, \quad (9)$$

$$F_{i,B}(\mathbf{x}, \omega) = f_B(\omega) \delta(\mathbf{x} - \mathbf{x}_R) \delta_{in}. \quad (10)$$

Application of reciprocity theorem (3) to domain  $\mathcal{V} = \mathcal{V}_o \cup \mathcal{V}_c$  yields

$$\begin{aligned} f_B(\omega) V_{n,A}(\mathbf{x}_R | \mathbf{x}_S) - f_A(\omega) V_{m,B}(\mathbf{x}_S | \mathbf{x}_R) \\ = -j\omega \int_{\mathbf{x} \in \mathcal{V}} \{ T_{ij,A} \Delta s_{ijkl} T_{kl,B} - V_{i,A} \Delta \varrho V_{i,B} \} dV \\ + \int_{x_3=x_3^2} \{ T_{i3,A}(\mathbf{x} | \mathbf{x}_S) V_{i,B}(\mathbf{x} | \mathbf{x}_R) \\ - V_{i,A}(\mathbf{x} | \mathbf{x}_S) T_{i3,B}(\mathbf{x} | \mathbf{x}_R) \} dA. \end{aligned} \quad (11)$$

The surface integral on the right-hand side of equation (11) takes into account a possible difference of the material parameters in  $\mathcal{V}'$ , below the reservoir; it vanishes when there is no difference between the two states in  $\mathcal{V}'$ .

#### One-way wave equation

In the one-way analysis we consider point-sources for downgoing waves in both states:

$$\mathbf{S}_A(\mathbf{x}, \omega) = \{ \mathbf{s}_A^+(\omega) \}^t 0 \delta(\mathbf{x} - \mathbf{x}_S), \quad (12)$$

$$\mathbf{S}_B(\mathbf{x}, \omega) = \{ \mathbf{s}_B^+(\omega) \}^t 0 \delta(\mathbf{x} - \mathbf{x}_R). \quad (13)$$

Application of reciprocity theorem (8) to domain  $\mathcal{V} = \mathcal{V}_o \cup \mathcal{V}_c$  yields

$$\begin{aligned} \{ \mathbf{s}_B^+(\omega) \}^t \mathbf{P}_A^-(\mathbf{x}_R | \mathbf{x}_S) - \{ \mathbf{s}_A^+(\omega) \}^t \mathbf{P}_B^-(\mathbf{x}_S | \mathbf{x}_R) \\ = \int_{\mathbf{x} \in \mathcal{V}_c} \mathbf{P}_A^t(\mathbf{x} | \mathbf{x}_S) \mathbf{N} \{ \hat{\mathbf{B}}_B(\mathbf{x}) - \hat{\mathbf{B}}_A(\mathbf{x}) \} \mathbf{P}_B(\mathbf{x} | \mathbf{x}_R) dV \\ + \int_{x_3=x_3^2} [ \{ \mathbf{P}_A^-(\mathbf{x} | \mathbf{x}_S) \}^t \mathbf{P}_B^+(\mathbf{x} | \mathbf{x}_R) \\ - \{ \mathbf{P}_A^+(\mathbf{x} | \mathbf{x}_S) \}^t \mathbf{P}_B^-(\mathbf{x} | \mathbf{x}_R) ] dA. \end{aligned} \quad (14)$$

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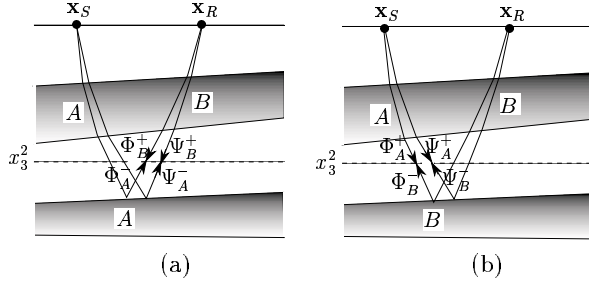


Fig. 4: Simplified representation of the two terms in the boundary integral in equation (14). Both terms accomplish a forward extrapolation of upgoing waves from  $x_3^2$  to the surface.

As in the previous case, the surface integral on the right-hand side of equation (14) vanishes when there is no difference between the two states in  $\mathcal{V}'$  (i.e., below  $x_3 = x_3^2$ ). Let us analyze this boundary integral, however, for the situation in which there are changes below  $x_3 = x_3^2$ . Figure 4 shows a configuration with two regions in which changes occur (the grey areas). Figure 4a shows some wavepaths in the first term of the boundary integral in equation (14), which can be written as

$$\int_{x_3=x_3^2} \{P_A^-(\mathbf{x}|\mathbf{x}_S)\}^t P_B^+(\mathbf{x}|\mathbf{x}_R) dA = \quad (15)$$

$$\int_{x_3=x_3^2} \{\Phi_A^-(\mathbf{x}|\mathbf{x}_S)\Phi_B^+(\mathbf{x}|\mathbf{x}_R) + \Psi_A^-(\mathbf{x}|\mathbf{x}_S)\Psi_B^+(\mathbf{x}|\mathbf{x}_R)\} dA$$

(note that we ignore the contribution  $\Upsilon_A^-\Upsilon_B^+$ ). If  $\Phi_B^+$  and  $\Psi_B^+$  are interpreted as Green's functions for state  $B$  (multiplied by a source function), then it is understood that this integral performs an upward extrapolation of  $\Phi_A^-$  and  $\Psi_A^-$  in state  $A$  from the depth level  $x_3^2$  to  $\mathbf{x}_R$  at the acquisition surface. This results in a virtual experiment in which the downgoing waves propagate from  $\mathbf{x}_S$  through the medium in state  $A$  (before the changes took place), reflection at the second reservoir occurs in state  $A$ , and the upgoing waves propagate through the medium in state  $B$  (after the changes took place) to  $\mathbf{x}_R$ . The second term in the boundary integral in equation (14) (see Figure 4b) represents a similar virtual experiment with the same propagation paths, except with reflection taking place at the second reservoir in state  $B$ . Hence, since the traveltimes in these virtual experiments are the same, the difference of these terms (as expressed by the boundary integral in equation (14)) is proportional to the time-lapse changes of the elastodynamic reflectivity of the second reservoir.

### Example

Figure 5 shows a subsurface model, including a reservoir layer in which changes take place. The propagation velocity in the reservoir in state  $A$  (before the changes took place) is given by  $c_A = 2500$  m/s; in state  $B$  (after the

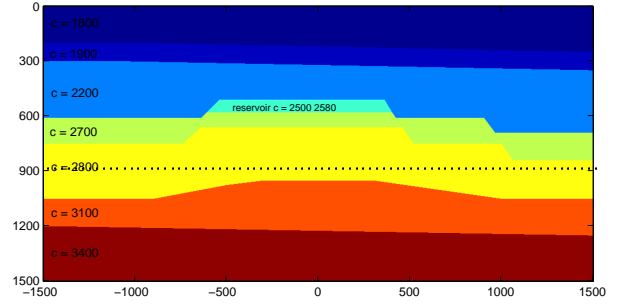


Fig. 5: Subsurface model, including a reservoir in which changes take place:  $c_A = 2500$  m/s,  $c_B = 2580$  m/s.

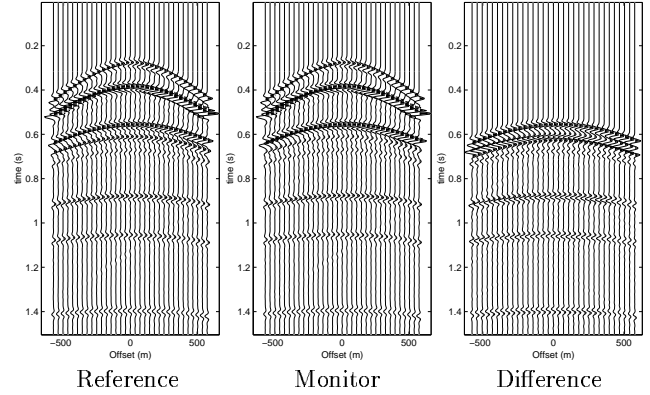


Fig. 6: Shot gather for the model of Figure 5 in state  $A$  (the 'reference' shot gather) and in state  $B$  (the 'monitor' shot gather) and the difference of these two shot gathers.

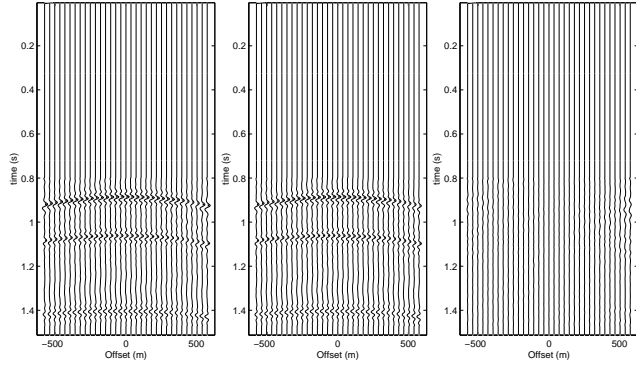
changes took place) it is given by  $c_B = 2580$  m/s. Figure 6 shows a shot record in state  $A$  (the 'reference' shot gather) and in state  $B$  (the 'monitor' shot gather) as well as the difference of these two shot gathers. Note that the first event in the difference shot gather is representative for the time-lapse changes of the reflectivity of the top of the reservoir. The other events in this difference shot gather, however, are spurious events: they are caused by the traveltime differences of the waves propagating through the reservoir and bear no relation with the time-lapse differences with the reflectors below the reservoir (as a matter of fact there are no time-lapse differences below the reservoir, so these events should actually be zero).

For the acoustic situation the boundary integral in equation (14) simplifies to

$$\int_{x_3=x_3^2} [P_A^-(\mathbf{x}|\mathbf{x}_S)P_B^+(\mathbf{x}|\mathbf{x}_R) - P_A^+(\mathbf{x}|\mathbf{x}_S)P_B^-(\mathbf{x}|\mathbf{x}_R)] dA. \quad (16)$$

Let  $x_3 = x_3^2$  denote the dotted line in Figure 5 below the reservoir. Following the explanation in the previous section, the two terms in the integral in equation (16) should cancel, because they can be seen as virtual experiments with the same propagation paths and with the same re-

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$$\int_{x_3=x_3^2} P_A^-(\mathbf{x}|\mathbf{x}_S)P_B^+(\mathbf{x}|\mathbf{x}_R)dA - \int_{x_3=x_3^2} P_A^+(\mathbf{x}|\mathbf{x}_S)P_B^-(\mathbf{x}|\mathbf{x}_R)dA \quad \text{Difference}$$

Fig. 7: Evaluation of the two terms of the integral in equation (16) and their difference.

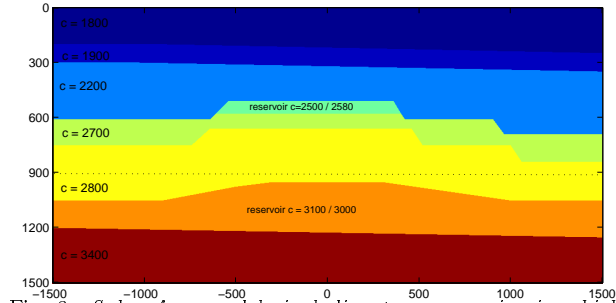


Fig. 8: Subsurface model, including two reservoirs in which changes take place. Reservoir 1:  $c_A = 2500$  m/s,  $c_B = 2580$  m/s. Reservoir 2:  $c_A = 3100$  m/s,  $c_B = 3000$  m/s.

flectors below the dotted line in Figure 5. Figure 7 shows the two contributions of this integral (anti-causal events have been muted) as well as their difference (Scherpenhuijsen, 2000). Unlike the difference section in Figure 6, the difference section in Figure 7 is indeed zero.

Figure 8 shows a subsurface model, including two reservoirs in which changes take place. In reservoir 1 the velocity changes from  $c_A = 2500$  m/s to  $c_B = 2580$  m/s; in reservoir 2 from  $c_A = 3100$  m/s to  $c_B = 3000$  m/s. Figure 9 shows the difference of two shot gathers, contaminated by traveltme differences (left panel), as well as the result of evaluating the boundary integral (16) (right panel, after muting non-causal events). The latter result contains the true time-lapse changes of the reflectivity of the second reservoir.

### Conclusions

We have formulated elastodynamic reciprocity theorems for time-lapse seismic methods, based on the full and the one-way wave equations. The latter form allows a straightforward physical interpretation of the various contributing terms. Its implementation requires wave field decomposition (Schalkwijk et al., 1998) and one-way wave field extrapolation of down- and upgoing  $P$  and

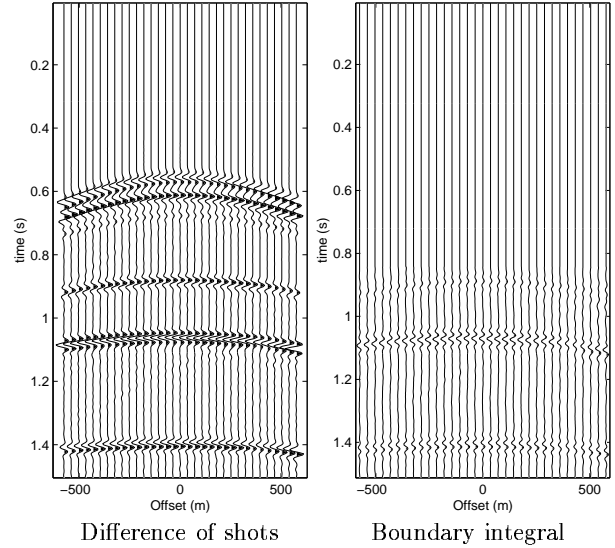


Fig. 9: Results for the stacked reservoir model of Figure 8. Left panel: difference of two shot gathers, contaminated by traveltme differences. Right panel: evaluation of the boundary integral (16). This result contains the true time-lapse changes of the reflectivity of the second reservoir.

$S$  waves. We have illustrated the evaluation of the boundary integral in the one-way reciprocity theorem for the acoustic situation. Unlike difference data taken at the acquisition surface, the boundary integral represents the true time-lapse changes of the reflectivity of a reservoir below the boundary at which the integral is evaluated.

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