General representations of electromagnetic interferometry

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***Abstract*— The theoretical and experimental developments in interferometry have advanced spectacularly over the last five years and are still expanding at a rapid pace. With interferometry we mean the retrieval of the Green's function by cross-correlating or cross-convolving two (wave) field recordings. Over the last year several contributions have included interferometric representations for media with losses, in absence of wave phenomena, like in thermal and electromagnetic diffusion or viscous flow, and for moving media. In this paper we use the reciprocity theorems of the time-correlation and time-convolution types to formulate general representations for electromagnetic fields and waves, and give explicit expressions for different possible GPR applications involving controlled or uncontrolled sources, which can be transient or noise sources.**

***Index Terms*— Interferometry, Green's function retrieval, GPR applications, Electromagnetism.**

I. Introduction

SINCE the work of Weaver and Lobkis [1], [2], many others have contributed to our understanding of Green's function retrieval from cross-correlating two recordings in a noise field. From one-dimensional and pulse-echo experiments the subject has evolved to arbitrary three-dimensional media, ranging from having statistical properties to being fully deterministic. Many successful demonstrations of the method on ultrasonic, geophysical and oceanographic data as well as many theoretical developments have been published. Recent developments, in the branch of research that is based on reciprocity, are the extension for situations where time-reversal invariance does not hold (e.g. for electromagnetic waves in conducting media [3], acoustic waves in attenuating media [4], or general scalar diffusion phenomena [5]), as well as for situations where source-receiver reciprocity breaks down (e.g. in moving fluids [6], [7]). Recently we developed a unified representation of Greens functions in terms of cross-correlations that covers all these cases [8], [9]. Another approach that has been developed is based on Green's functions representations in terms of cross-convolutions [10]. In this paper we give an overview of the general electromagnetic formulations and discuss several possible GPR applications of these recent developments.

II. The wave equation in media with losses

The time domain Maxwell equations are most conveniently written down in a matrix-vector equation where the electric and magnetic medium parameters are time-convolutional operators to allow for relaxation mechanisms. Mathematically all relaxation phenomena can be captured in the operator representing electric and magnetic conductivities, yielding

$$\boldsymbol{D}_x\boldsymbol{u}+\boldsymbol{B}\star\boldsymbol{u}+\boldsymbol{A}\partial_t\hat{\boldsymbol{u}}=\hat{\boldsymbol{s}}, \tag{1}$$

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where it assumed that the medium is at rest, or that changes in the medium play a role on time scales much larger than the measurement scale. The vector $\boldsymbol{u}(\boldsymbol{x},t)$ contains the space- and time-dependent electric and magnetic field vectors, the source vector is denoted $\boldsymbol{s}(\boldsymbol{x},t)$, the position dependent matrix $\boldsymbol{A}(\boldsymbol{x})$ contains the instantaneous parts of the electric permittivity and magnetic permeability, the position dependent and time-convolution operator $\boldsymbol{B}(\boldsymbol{x},t)$ takes all electric and magnetic time-relaxation phenomena into account; $\boldsymbol{B}(\boldsymbol{x},t)\star\boldsymbol{u}(\boldsymbol{x},t)=\int_\tau \boldsymbol{B}(\boldsymbol{x},\tau)\boldsymbol{u}(\boldsymbol{x},t-\tau)\mathrm{d}\tau$. The differential operator $\boldsymbol{D}_x$ contains the spatial differential operators $\partial_1,\partial_2,\partial_3$ and ∂_t denotes differentiation with respect to the time coordinate t. The matrix-vector equation can be written out in full using the field and source vectors as

$$\boldsymbol{u}=\left(E_1,E_2,E_3,H_1,H_2,H_3\right)^T, \tag{2}$$
$$\boldsymbol{s}=-\left(J_1^e,J_2^e,J_3^e,J_1^m,J_2^m,J_3^m\right)^T, \tag{3}$$

where the superscript T denotes the transpose of a vector or matrix, while the matrices are given by

$$\boldsymbol{D}_x=\begin{pmatrix}\boldsymbol{O} & \boldsymbol{D}_0^T\\ \boldsymbol{D}_0 & \boldsymbol{O}\end{pmatrix},\boldsymbol{D}_0=\begin{pmatrix}0 & -\partial_3 & \partial_2\\ \partial_3 & 0 & -\partial_1\\ -\partial_2 & \partial_1 & 0\end{pmatrix}, \tag{4}$$
$$\boldsymbol{A}=\begin{pmatrix}\boldsymbol{\varepsilon} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{\mu}\end{pmatrix},\boldsymbol{B}=\begin{pmatrix}\boldsymbol{\sigma}^e & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{\sigma}^m\end{pmatrix}, \tag{5}$$
$$\boldsymbol{\varepsilon}=\begin{pmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}\end{pmatrix},\quad \boldsymbol{\mu}=\begin{pmatrix}\mu_{11} & \mu_{12} & \mu_{13}\\ \mu_{21} & \mu_{22} & \mu_{23}\\ \mu_{31} & \mu_{32} & \mu_{33}\end{pmatrix},$$
$$\boldsymbol{\sigma}^e=\begin{pmatrix}\sigma^e_{11} & \sigma^e_{12} & \sigma^e_{13}\\ \sigma^e_{21} & \sigma^e_{22} & \sigma^e_{23}\\ \sigma^e_{31} & \sigma^e_{32} & \sigma^e_{33}\end{pmatrix},\boldsymbol{\sigma}^m=\begin{pmatrix}\sigma^m_{11} & \sigma^m_{12} & \sigma^m_{13}\\ \sigma^m_{21} & \sigma^m_{22} & \sigma^m_{23}\\ \sigma^m_{31} & \sigma^m_{32} & \sigma^m_{33}\end{pmatrix},$$

where $\varepsilon_{kr},\mu_{jp}$ denote the tensor components of the anisotropic electric permittivity and magnetic permeability and $\sigma^e_{kr},\sigma^m_{jp}$ denote the tensor components of the anisotropic electric and magnetic conductivities, each component of which is a time convolution operator to allow for relaxation mechanisms.

Time convolutions are replaced by products through time-Fourier transformations and time differentiation is transformed to a multiplication $i\omega$, i being the imaginary unit and ω is the radial frequency. We define the time-Fourier transform of a space-time function $\boldsymbol{f}(\boldsymbol{x},t)$ as $\hat{\boldsymbol{f}}(\boldsymbol{x},\omega)=\int_t \boldsymbol{f}(\boldsymbol{x},t)\exp(-\mathrm{i}\omega t)\mathrm{d}t$. Applying the time-Fourier transformation to equation (1) yields

$$\boldsymbol{D}_x\hat{\boldsymbol{u}}+\hat{\boldsymbol{B}}\hat{\boldsymbol{u}}+\mathrm{i}\omega\boldsymbol{A}\hat{\boldsymbol{u}}=\hat{\boldsymbol{s}}. \tag{6}$$

III. Definition of the Green's matrices

To define the electromagnetic Green's matrix $\hat{\boldsymbol{G}}(\boldsymbol{x},\boldsymbol{x}_A,\omega)$ we replace the 6×1 source vector $\hat{\boldsymbol{s}}(\boldsymbol{x},\omega)$ by the 6×6 frequency independent source matrix $\boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}_A)$, $\boldsymbol{I}$ being the identity matrix. The corresponding 6×1 field vector $\hat{\boldsymbol{u}}(\boldsymbol{x}, \omega)$ is replaced by the 6×6 Green's matrix given by

$$\hat{\boldsymbol{G}}(\boldsymbol{x}, \boldsymbol{x}_A, \omega) = \begin{pmatrix} \hat{\boldsymbol{G}}^{Ee} & \hat{\boldsymbol{G}}^{Em} \\ \hat{\boldsymbol{G}}^{He} & \hat{\boldsymbol{G}}^{Hm} \end{pmatrix} (\boldsymbol{x}, \boldsymbol{x}_A, \omega), \tag{7}$$

where the superscripts $\{E, H\}$ denote the observed field type at $\boldsymbol{x}$ and the superscripts $\{e, m\}$ denote the electric or magnetic source type $\{\hat{\boldsymbol{J}}^e, \hat{\boldsymbol{J}}^m\}$ at $\boldsymbol{x}_A$. For each particular field and each source type the Green's sub-matrix is a 3×3 matrix. Each column of $\hat{\boldsymbol{G}}$ represents a field vector $\hat{\boldsymbol{u}}$ at $\boldsymbol{x}$ due one particular point source type and component operative at $\boldsymbol{x}_A$.

IV. General Green's function retrieval

Consider an arbitrary domain $\mathbb{D}$ with boundary $\partial\mathbb{D}$ and outward unit normal vector $\boldsymbol{n} = \{n_1, n_2, n_3\}^T$. We define two points, located at $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ either inside or outside the domain $\mathbb{D}$. Application of electromagnetic reciprocity of the time-correlation type to equation (6) results in a general representation of the Green's matrix retrieval by cross-correlation, see [8] for a unified representation including other field and wave phenomena,

$$\begin{aligned} &\hat{\boldsymbol{G}}(\boldsymbol{x}_B, \boldsymbol{x}_A, \omega)\chi_{\mathbb{D}}(\boldsymbol{x}_A) + \hat{\boldsymbol{G}}^{\dagger}(\boldsymbol{x}_A, \boldsymbol{x}_B, \omega)\chi_{\mathbb{D}}(\boldsymbol{x}_B) \\ &\quad = -\oint_{\partial\mathbb{D}} \hat{\boldsymbol{G}}(\boldsymbol{x}_B, \boldsymbol{x}, \omega)\boldsymbol{N}_x\hat{\boldsymbol{G}}^{\dagger}(\boldsymbol{x}_A, \boldsymbol{x}, \omega)\mathrm{d}^2\boldsymbol{x} \\ &\quad + \int_{\mathbb{D}} \hat{\boldsymbol{G}}(\boldsymbol{x}_B, \boldsymbol{x}, \omega)(\hat{\boldsymbol{B}}^{\dagger} + \hat{\boldsymbol{B}})\hat{\boldsymbol{G}}^{\dagger}(\boldsymbol{x}_A, \boldsymbol{x}, \omega)\mathrm{d}^3\boldsymbol{x}, \end{aligned} \tag{8}$$

where $\hat{\boldsymbol{G}}^{\dagger}$ denotes the complex conjugate transpose of $\hat{\boldsymbol{G}}$ and $\chi_{\mathbb{D}}(\boldsymbol{x}_A) = \{0, 1/2, 1\}$ for $\{\boldsymbol{x}_A \in \mathbb{D}', \boldsymbol{x}_A \in \partial\mathbb{D}, \boldsymbol{x}_A \in \mathbb{D}\}$ denotes the characteristic set of the domain $\mathbb{D}$, while $\mathbb{D}'$ is the complement of $\mathbb{D}$ and $\partial\mathbb{D}$. The matrix $\boldsymbol{N}_x$ is defined similar to $\boldsymbol{D}_x$, but with ∂_i replaced by n_i, $i = 1, 2, 3$. Equation (8) is an exact general representation of the electromagnetic Green's matrix between $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ in terms of cross-correlations of observed electric and magnetic fields observed at $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ due to electric and magnetic sources at $\boldsymbol{x}$ on the boundary $\partial\mathbb{D}$ and inside the domain $\mathbb{D}$. To arrive at this representation for the Green's functions no assumptions have been made on the heterogeneity and relaxation mechanisms inside and outside the domain $\mathbb{D}$.

Similarly applying electromagnetic reciprocity of the time-convolution type to equation (6) leads to a general representation of the Green's matrix retrieval by cross-convolution,

$$\begin{aligned} &\hat{\boldsymbol{G}}(\boldsymbol{x}_B, \boldsymbol{x}_A, \omega)[\chi_{\mathbb{D}}(\boldsymbol{x}_A) - \chi_{\mathbb{D}}(\boldsymbol{x}_B)] \\ &= \oint_{\partial\mathbb{D}} \hat{\boldsymbol{G}}(\boldsymbol{x}_B, \boldsymbol{x}, \omega)\boldsymbol{N}_x\boldsymbol{K}\hat{\boldsymbol{G}}^{T}(\boldsymbol{x}_A, \boldsymbol{x}, \omega)\boldsymbol{K}\mathrm{d}^2\boldsymbol{x}, \end{aligned} \tag{9}$$

where $\boldsymbol{K} = \mathrm{diag}(-1, -1, -1, 1, 1, 1)$. Equation (9) is an exact representation for the electromagnetic Green's function between $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ in terms of cross-convolutions of impulsive field responses observed at the observation points $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ due to electric and magnetic sources on the boundary $\partial\mathbb{D}$ and integrating over all source locations on the closed boundary surface $\partial\mathbb{D}$. From equation (9) it can be observed that the Green's function is only retrieved when either $\boldsymbol{x}_A$ or $\boldsymbol{x}_B$ is inside $\mathbb{D}$. To arrive at this representation for the Green's functions no assumptions have been made on the heterogeneity and relaxation mechanisms inside and outside the domain $\mathbb{D}$. It is worth noting that the medium parameters are not present in equation (9). Possible GPR applications of electromagnetic interferometry using equations (8) and (9) are investigated in the next sections.

V. Application 1. Controlled sources on $\partial\mathbb{D}$

Assuming the loss factors are zero throughout $\mathbb{D}$, the domain integral in equation (8) vanishes and only sources on the boundary are required to construct Green's functions between two points inside $\mathbb{D}$ or between one point inside and one point outside $\mathbb{D}$. When the loss factors are not zero but very small, we neglect the volume integral and a discussion on the corresponding implications can be found in [11]. For GPR the most beneficial application with both receivers inside the domain would be with sources at the surface and receivers in boreholes in an (almost) lossless environment. In case the medium is sufficiently heterogeneous the closed surface can be replaced by an integral over the surface, $\partial\mathbb{D}$, of the Earth [12] and the Green's function between two points in the subsurface is then found as

$$\begin{aligned} &\hat{\boldsymbol{G}}(\boldsymbol{x}_B, \boldsymbol{x}_A, \omega) + \hat{\boldsymbol{G}}^{\dagger}(\boldsymbol{x}_A, \boldsymbol{x}_B, \omega) \\ &\quad = -\int_{\partial\mathbb{D}_0} \hat{\boldsymbol{G}}(\boldsymbol{x}_B, \boldsymbol{x}, \omega)\boldsymbol{N}_x\hat{\boldsymbol{G}}^{\dagger}(\boldsymbol{x}_A, \boldsymbol{x}, \omega)\mathrm{d}^2\boldsymbol{x}, \end{aligned} \tag{10}$$

from which each element of the Green's matrix can be obtained by selecting certain field components in the receiver array and sources at the surface. To obtain the Green's function for a crosshole experiment, as if there was a horizontal electric current source at $\boldsymbol{x}_A$ and a vertical magnetic receiver at $\boldsymbol{x}_B$, from vertical GPR profiles we must choose $\boldsymbol{G}^{He}_{31}(\boldsymbol{x}_B, \boldsymbol{x}_A, t)$ in the left-hand side of equation (10), which is element $(6, 1)$ in the Green's matrix. According to equation (10) it is retrieved by cross-correlating the measured vertical component of the magnetic field [which is the 6th-row of $\boldsymbol{G}(\boldsymbol{x}_B, \boldsymbol{x}, t)$] with the horizontal component of the electric field at $\boldsymbol{x}_A$ [which is the first column of $\boldsymbol{N}_x\boldsymbol{G}^T(\boldsymbol{x}_A, \boldsymbol{x}, t)$], summing over electric and magnetic sources at $\boldsymbol{x}$ (by row-column multiplication) and integrating over all source positions at the surface $\partial\mathbb{D}_0$. This is illustrated in Figure 1. We can write equation (10) in terms of field observations by introducing the diagonal source matrix $\hat{\boldsymbol{S}}(\boldsymbol{x}, \omega) = \mathrm{diag}[\hat{s}_1(\boldsymbol{x}, \omega), \hat{s}_2(\boldsymbol{x}, \omega), \cdots, \hat{s}_6(\boldsymbol{x}, \omega)]^T$. The field observations are then stored in a field matrix,

$$\hat{\boldsymbol{U}}^{\mathrm{obs}}(\boldsymbol{x}_A, \boldsymbol{x}, \omega) = \hat{\boldsymbol{G}}(\boldsymbol{x}_A, \boldsymbol{x}, \omega)\hat{\boldsymbol{S}}(\boldsymbol{x}, \omega), \tag{11}$$

and a similar expression for $\hat{\boldsymbol{U}}(\boldsymbol{x}_B, \boldsymbol{x}, \omega)$. Using these choices in equation (10) , we find

$$\begin{aligned} &\hat{\boldsymbol{U}}^{\mathrm{obs}}(\boldsymbol{x}_B, \boldsymbol{x}_A, \omega) + [\hat{\boldsymbol{U}}^{\mathrm{obs}}(\boldsymbol{x}_A, \boldsymbol{x}_B, \omega)]^{\dagger} = \\ &-\int_{\partial\mathbb{D}_0} \hat{\boldsymbol{U}}^{\mathrm{obs}}(\boldsymbol{x}_B, \boldsymbol{x}, \omega)\hat{\boldsymbol{X}}(\boldsymbol{x}, \omega)[\hat{\boldsymbol{U}}^{\mathrm{obs}}(\boldsymbol{x}_A, \boldsymbol{x}, \omega)]^{\dagger}\mathrm{d}^2\boldsymbol{x}, \end{aligned} \tag{12}$$

where the source compensation matrix is given by

$$\hat{\boldsymbol{X}}(\boldsymbol{x}, \omega) = [\hat{\boldsymbol{S}}(\boldsymbol{x}, \omega)]^{-1}\boldsymbol{N}_x[\hat{\boldsymbol{S}}^{\dagger}(\boldsymbol{x}, \omega)]^{-1}. \tag{13}$$

The fact that the matrix $\hat{\boldsymbol{X}}(\boldsymbol{x}, \omega)$ contains inverses of the source matrix implies that the source spectra should be known to use this method with controlled sources.

Apart from applications with field recordings, these representations can be used advantageously for numerical modeling [13]. In situations where the point-to-point relations of electromagnetic wave propagation need to be computed as is done for single-well or cross-well tomographic studies or

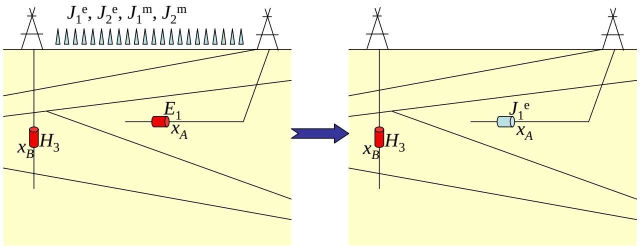

Fig. 1. From left to right panel, interferometry with controlled sources at the surface $\partial\mathbb{D}_0$ as described in Application 1. In this example the vertical magnetic field component is cross-correlated with the horizontal electric electric field component, yielding the vertical magnetic GPR response generated by a horizontal electric source.

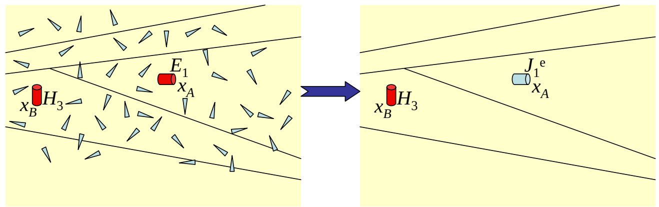

Fig. 2. From left to right panel, interferometry with noise sources distributed throughout $\mathbb{D}$ as described in Application 2. In this example the vertical magnetic field component is cross-correlated with the horizontal electric field component, yielding the vertical magnetic GPR response generated by a horizontal electric source.

for full non-linear electromagnetic inversions methods. Then a considerable reduction in computational cost is achieved because only a sufficient number of sources is required at the Earth surface, while from the stored field values at all grid points data can be created with any point in $\mathbb{D}$ as a source point to any other point in $\mathbb{D}$ as a receiver. This means that in a gridded domain with n^3 nodes, only $\mathcal{O}(n^2)$ sources are needed to obtain data for $\mathcal{O}(n^3)$ source positions.

VI. Application 2. Uncorrelated sources in $\mathbb{D}$

When the loss factors are not negligible, the boundary integral, when $\mathbb{D}$ is taken as infinite domain, has a vanishing contribution, see [4] for an acoustic analogue. Then we consider a volume distribution of noise sources $\hat{\boldsymbol{s}}$ throughout $\mathbb{D}$, see Figure 2, where $\hat{\boldsymbol{s}}(\boldsymbol{x},\omega)$ is a vector with components $\hat{s}_k(\boldsymbol{x},\omega)$. We further assume that any two sources $\hat{s}_k(\boldsymbol{x},\omega), \hat{s}_l(\boldsymbol{x}',\omega)$ are mutually uncorrelated for any $\boldsymbol{x} \neq \boldsymbol{x}'$, i.e. they satisfy the relationship $\langle \hat{\boldsymbol{s}}(\boldsymbol{x},\omega)\hat{\boldsymbol{s}}^\dagger(\boldsymbol{x}',\omega)\rangle = \hat{\boldsymbol{\lambda}}(\boldsymbol{x},\omega)\hat{S}(\omega)\delta(\boldsymbol{x}-\boldsymbol{x}')$, where $\langle\cdot\rangle$ denotes a spatial ensemble, $\hat{S}(\omega)$ denotes the power spectrum of the noise sources and $\hat{\boldsymbol{\lambda}}(\boldsymbol{x},\omega)$ is a matrix containing the source excitation functions. We can then define the field observation vectors at $\boldsymbol{x}_{A;B}$ as,

$$\hat{\boldsymbol{u}}^{\text{obs}}(\boldsymbol{x}_{A;B},\omega) = \int_{\boldsymbol{x}\in\mathbb{D}} \hat{\boldsymbol{G}}(\boldsymbol{x}_{A;B},\boldsymbol{x},\omega)\hat{\boldsymbol{s}}(\boldsymbol{x},\omega)\mathrm{d}^3\boldsymbol{x}. \quad (14)$$

We can evaluate the cross-correlation of the observed fields as

$$\begin{aligned}&\langle \hat{\boldsymbol{u}}^{\text{obs}}(\boldsymbol{x}_B,\omega)\{\hat{\boldsymbol{u}}^{\text{obs}}(\boldsymbol{x}_A,\omega)\}^\dagger\rangle \\ &= \hat{S}(\omega)\int_{\boldsymbol{x}\in\mathbb{D}} \hat{\boldsymbol{G}}(\boldsymbol{x}_B,\boldsymbol{x},\omega)\hat{\boldsymbol{\lambda}}(\boldsymbol{x},\omega)\hat{\boldsymbol{G}}^\dagger(\boldsymbol{x}_A,\boldsymbol{x},\omega)\mathrm{d}^3\boldsymbol{x}. \quad (15)\end{aligned}$$

Comparing this with the right-hand side of equation (8) with vanishing boundary integral we can obtain

$$\begin{aligned}&[\hat{\boldsymbol{G}}(\boldsymbol{x}_B,\boldsymbol{x}_A,\omega) + \hat{\boldsymbol{G}}^\dagger(\boldsymbol{x}_A,\boldsymbol{x}_B,\omega)]\hat{S}(\omega) \\ &\qquad = \langle \hat{\boldsymbol{u}}^{\text{obs}}(\boldsymbol{x}_B,\omega)\{\hat{\boldsymbol{u}}^{\text{obs}}(\boldsymbol{x}_A,\omega)\}^\dagger\rangle, \quad (16)\end{aligned}$$

when $\hat{\boldsymbol{\lambda}}(\boldsymbol{x},\omega) = \hat{\boldsymbol{B}}(\boldsymbol{x},\omega) + \hat{\boldsymbol{B}}^\dagger(\boldsymbol{x},\omega)$. Hence when at least one element of $\hat{\boldsymbol{B}}(\boldsymbol{x},\omega) + \hat{\boldsymbol{B}}^\dagger(\boldsymbol{x},\omega)$ is non-zero, which is the case in most GPR applications, the Green's matrix between the points $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ can be retrieved from cross-correlations of field observations at those points, assuming that a distribution of spatially uncorrelated noise sources are present in $\mathbb{D}$, such that their excitation functions are proportional to the local loss factors. This form of continuous injection of energy is necessary to compensate for the dissipation in $\mathbb{D}$ [9].

VII. Application 3. Uncorrelated sources on $\partial\mathbb{D}$

When $\mathbb{D}$ is taken to be entirely located in the air, and hence the medium in $\mathbb{D}$ is lossless, the volume integral in equation (8) vanishes, see Figure 3. In [10] it was shown that also here we don't need sources on the closed boundary but only in the top part. This is convenient because most natural and man-made electromagnetic noise comes from sources up in the air or from outside our atmosphere. The Earth is a heterogeneous, anisotropic medium with losses and is now completely outside the domain $\mathbb{D}$. Since $\boldsymbol{x}_A \in \mathbb{D}$, while $\boldsymbol{x}_B \notin \mathbb{D}$ we find,

$$\hat{\boldsymbol{G}}(\boldsymbol{x}_B,\boldsymbol{x}_A,\omega) = -\int_{\partial\mathbb{D}_0} \hat{\boldsymbol{G}}(\boldsymbol{x}_B,\boldsymbol{x},\omega)\boldsymbol{N}_x\hat{\boldsymbol{G}}^\dagger(\boldsymbol{x}_A,\boldsymbol{x},\omega)\mathrm{d}^2\boldsymbol{x}. \quad (17)$$

In this configuration the Green's matrix between two points can be obtained only with independent impulsive sources on the boundary $\partial\mathbb{D}_0$. To make equation (17) suitable for uncorrelated noise sources, the matrix $\boldsymbol{N}_x$ must be diagonalized. In general, this diagonalization involves decomposition of the sources on $\partial\mathbb{D}_0$ into sources for inward and outward traveling waves. Since in this particular configuration we can assume that the waves that leave the boundary $\partial\mathbb{D}_0$ in the upward direction never return, and are not recorded, and hence we can diagonalize by changing the tangential magnetic dipole sources, present in equation (17) to equivalent normal component of electric quadrupole sources. Then it is most convenient to write the magnetic field in terms of the electric field using using Maxwell's second equation, and to make a far-field approximation to replace the quadrupole response by a dipole response. We end up with an approximate equation for the electric field Green's matrix due to an electric dipole source [11],

$$\begin{aligned}&\hat{\boldsymbol{G}}^{Ee}(\boldsymbol{x}_B,\boldsymbol{x}_A,\omega) = -\frac{2}{c_0\mu_0} \\ &\times\int_{\partial\mathbb{D}_0} \hat{\boldsymbol{G}}^{Ee}(\boldsymbol{x}_B,\boldsymbol{x},\omega)\{\hat{\boldsymbol{G}}^{Ee}(\boldsymbol{x}_A,\boldsymbol{x},\omega)\}^\dagger\mathrm{d}^2\boldsymbol{x} + \text{"ghost"}, \quad (18)\end{aligned}$$

where the "ghost"-term refers to non-physical events due to the open boundary $\partial\mathbb{D}_0$ and because of the far-field approximation. Both approximations lead to incomplete destructive interference of these non-physical events, which then remain in the data after integrating over all sources on the open boundary $\partial\mathbb{D}_0$. When far-field conditions apply, e.g. when we use sources from outside our atmosphere, and the Earth is sufficiently heterogeneous the "ghost" vanishes. Since in equation (18) we have a direct matrix-matrix product the uncorrelated electric dipole noise sources on the boundary $\partial\mathbb{D}_0$

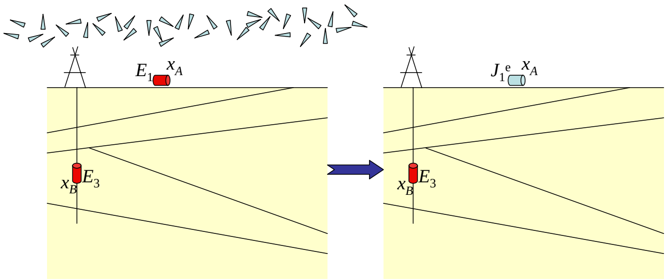

Fig. 3. From left to right panel, interferometry with noise sources in the air on an irregular surface $\partial\mathbb{D}_0$ as described in Application 3. In this example the vertical electric field component is cross-correlated with the horizontal electric electric field component, yielding the vertical electric GPR response generated by a horizontal electric source.

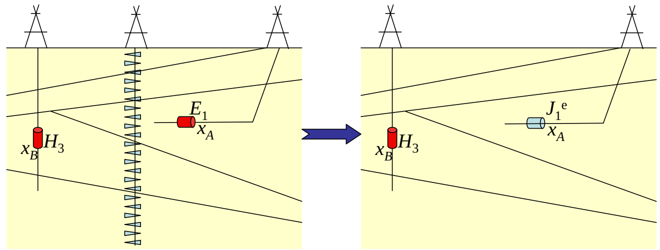

Fig. 4. From left to right panel, interferometry with transient sources in a borehole as described in Application 4. In this example the vertical electric field component is cross-convolved with the horizontal electric electric field component, yielding the vertical electric GPR response generated by a horizontal electric source.

must obey the following relationship $\langle \hat{\boldsymbol{s}}(\boldsymbol{x},\omega)\hat{\boldsymbol{s}}^\dagger(\boldsymbol{x}',\omega)\rangle = 2\hat{S}(\omega)\boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}')/(\mu c)$, $\boldsymbol{I}$ being the identity matrix, and defining the observed electric field vector as $\hat{\boldsymbol{E}}(\boldsymbol{x}_A,\omega) = \int_{\boldsymbol{x}\in\partial\mathbb{D}_0} \hat{\boldsymbol{G}}^{Ee}(\boldsymbol{x}_A,\boldsymbol{x},\omega)\hat{\boldsymbol{s}}(\boldsymbol{x},\omega)\mathrm{d}^2\boldsymbol{x}$, and a similar expression for $\hat{\boldsymbol{E}}(\boldsymbol{x}_B,\omega)$ results in an equation similar to equation (16), but now we retain only the causal Green's matrix and the medium is allowed to be dissipative, while still only sources on the boundary are necessary

$$\hat{\boldsymbol{G}}^{Ee}(\boldsymbol{x}_B,\boldsymbol{x}_A,\omega)\hat{S}(\omega) = -\langle \hat{\boldsymbol{u}}^{\text{obs}}(\boldsymbol{x}_B,\omega)\{\hat{\boldsymbol{u}}^{\text{obs}}(\boldsymbol{x}_A,\omega)\}^\dagger\rangle. \tag{19}$$

In the situation that the earth is almost lossless the configuration as explained in Application 1, with $\mathbb{D}$ containing the Earth, can be used with uncorrelated noise sources above the Earth surface as well, under the same assumption of a sufficiently heterogeneous Earth. In that situation the volume integral in equation (8) can be neglected and the same definitions as used for equation (19) lead to the same relation as given in equation (16).

VIII. Application 4. Transient sources on $\partial\mathbb{D}$

In situations where the boundary of $\mathbb{D}$ can be taken in between two receiver locations both correlation and convolution types of interferometry can be used. The example sketched in Figure 4 yields an approximation to the exact result because sources on a plane are required and not only in a single borehole. This approximate application can still be quite accurate in the kinematics of the cross-hole survey. In situations where the distance between the outer two boreholes is too large for a standard GPR to acquire data, because of too high losses, but the other two combinations can be used, then numerically the data of the configuration indicated in the right panel of Figure 4 can be retrieved. If the subsurface heterogeneities occur mostly in the plane of the three boreholes and far field conditions apply, the result is quite accurate. We define transient sources in a source matrix as introduced in Application 1 and use the same definition for the observed field matrix as in equation (11) to obtain,

$$\hat{\boldsymbol{U}}^{\text{obs}}(\boldsymbol{x}_B,\boldsymbol{x}_A,\omega) = \int_{\partial\mathbb{D}_0} \hat{\boldsymbol{U}}^{\text{obs}}(\boldsymbol{x}_B,\boldsymbol{x},\omega)\hat{\boldsymbol{Y}}(\boldsymbol{x},\omega)[\hat{\boldsymbol{U}}^{\text{obs}}(\boldsymbol{x}_A,\boldsymbol{x},\omega)]^T\boldsymbol{K}\mathrm{d}^2\boldsymbol{x}, \tag{20}$$

where the source compensation matrix is now given by

$$\hat{\boldsymbol{Y}}(\boldsymbol{x},\omega) = [\hat{\boldsymbol{S}}(\boldsymbol{x},\omega)]^{-1}\boldsymbol{N}_x\boldsymbol{K}[\hat{\boldsymbol{S}}(\boldsymbol{x},\omega)]^{-1}. \tag{21}$$

The configuration of Application 1 can also be used in this way when one receiver is in the air above the boundary $\partial\mathbb{D}$ containing the sources.

IX. Conclusion

We have derived general electromagnetic interferometric representations of Green's function in terms of cross-correlations and cross-convolutions of observed radar wave fields. These representations have applications with controlled transient sources, as in cross-borehole GPR tomographic studies, or noise sources for passive surface GPR or vertical GPR profiling.

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