On the benefits of auxiliary transmission data for Marchenko-based Green's function retrieval

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Abstract

Green's functions in an unknown medium can be retrieved from single-sided reflection data by solving a multidimensional Marchenko equation. This methodology requires knowledge of the direct wavefield throughout the medium, which should include forward-scattered waveforms. In practice, the direct field is often computed in a smooth background model, where such subtleties are not included. As a result, Marchenko-based Green's function retrieval can be inaccurate, especially in severely complex media. In some cases, auxiliary transmission data may be available. In this extended abstract, we show how these data can be used to modify the Marchenko equation so that forward-scattered waveforms can be retrieved without additional knowledge of the medium.

Introduction

It is well established that a Green's function at an arbitrary location $\boldsymbol{x}$ inside an unknown medium can be retrieved by solving a multidimensional Marchenko equation (Wapenaar et al., 2017). This methodology requires the single-sided reflection response at an acquisition surface and knowledge of the direct wavefield as it has propagated from this surface towards $\boldsymbol{x}$. For accurate results in complex media, forward-scattered waveforms should be included in the direct field (Vargas and Vasconcelos, 2020). In practice, these waveforms are typically not accounted for, resulting in incomplete retrieval of the Green's function (van der Neut et al., 2015). Recently, some novel applications of the Marchenko methodology have emerged in which auxiliary transmission data are available. We mention for instance: Marchenko imaging of joint VSP and reflection data (Liu et al, 2016; Lomas et al., 2020), laboratory experiments (Cui et al., 2018) and transcranial wavefield focusing (Meles et al., 2019). Since transmission data include the desired forward-scattered waveforms, we reason that these data might be used to improve Marchenko-based Green's function retrieval. In this extended abstract, we report on an initial attempt to do so.

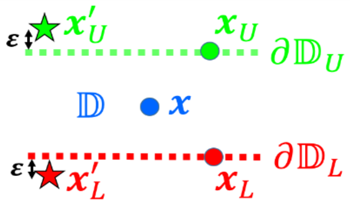

Figure 1 *Configuration in 3D space with $\boldsymbol{x} = (x_1, x_2, x_3)$. An acoustic medium is characterized by the wave velocity $c(\boldsymbol{x})$ and mass density $\rho(\boldsymbol{x})$. A heterogeneous volume $\mathbb{D}$ is enclosed by an (infinite) upper boundary $\partial\mathbb{D}_U$ and an (infinite) lower boundary $\partial\mathbb{D}_L$. Outside $\mathbb{D}$, the medium is non-reflective. Further, $\boldsymbol{x}_U \in \partial\mathbb{D}_U$, $\boldsymbol{x}_L \in \partial\mathbb{D}_L$, $\boldsymbol{x}'_U$ is located at a distance $\varepsilon \to 0$ above $\partial\mathbb{D}_U$ and $\boldsymbol{x}'_L$ is located at a distance $\varepsilon \to 0$ below $\partial\mathbb{D}_L$.*

A Marchenko equation for double-sided reflection data

Over the last year, several novel representations for the Marchenko equation have been presented that do not rely on up/down-decomposition inside the medium (Kiraz et al., 2020; Diekmann and Vasconcelos, 2021; Wapenaar et al, 2021). Here, we modify the derivation of Wapenaar et al. (2021) for a configuration with double-sided illumination, as shown in Figure 1. For any $\boldsymbol{x} \in \mathbb{D}$, the wavefield $p(\boldsymbol{x})$ at frequency ω (which we omitted from the representation for notational convenience) can be expressed as (Wapenaar et al., 2021)

$$p(\boldsymbol{x}) = \int_{\partial\mathbb{D}_U} F_U(\boldsymbol{x}, \boldsymbol{x}_U) p^-(\boldsymbol{x}_U)\, d\boldsymbol{x}_U + \int_{\partial\mathbb{D}_U} F_U^{\star}(\boldsymbol{x}, \boldsymbol{x}_U) p^+(\boldsymbol{x}_U)\, d\boldsymbol{x}_U. \quad (1)$$

In this representation, p^+ and p^- are the down- and upgoing constituents of p at the upper boundary $\partial\mathbb{D}_U$ and superscript $\star$ denotes complex conjugation. Further, F_U is a focusing function, which obeys the focusing condition $F_U(\boldsymbol{x}, \boldsymbol{x}_U)|_{x_3 = x_{3,U}} = \delta(\boldsymbol{x}_H - \boldsymbol{x}_{H,U})$, where δ denotes a 2D Dirac delta function and $\boldsymbol{x}_H = (\mathrm{x}_1, \mathrm{x}_2)$. An equivalent equation can be derived for the lower boundary $\partial\mathbb{D}_L$; that is

$$p(\boldsymbol{x}) = \int_{\partial\mathbb{D}_L} F_L(\boldsymbol{x}, \boldsymbol{x}_L) p^+(\boldsymbol{x}_L)\, d\boldsymbol{x}_L + \int_{\partial\mathbb{D}_L} F_L^{\star}(\boldsymbol{x}, \boldsymbol{x}_L) p^-(\boldsymbol{x}_L)\, d\boldsymbol{x}_L, \quad (2)$$

where F_L is a focusing function that obeys $F_L(\boldsymbol{x}, \boldsymbol{x}_L)|_{x_3 = x_{3,L}} = \delta(\boldsymbol{x}_H - \boldsymbol{x}_{H,L})$. We define a dipole Green's function as $\Gamma_U(\boldsymbol{x}, \boldsymbol{x}'_U) = \frac{-2}{i\omega\rho(\boldsymbol{x}'_U)} \partial'_3 G(\boldsymbol{x}, \boldsymbol{x}'_U)$, where $G(\boldsymbol{x}, \boldsymbol{x}'_U)$ is a monopole Green's function with a source at $\boldsymbol{x}'_U$ and ∂'_3 denotes the vertical partial derivative at $\boldsymbol{x}'_U$. From these definitions, we find $\Gamma_U^+(\boldsymbol{x}, \boldsymbol{x}'_U)|_{x_3 = x_{3,U}} = \delta(\boldsymbol{x}_H - \boldsymbol{x}'_{H,U})$. When we substitute $p = \Gamma_U$ into equation (1), it follows that

$$\Gamma_U(\boldsymbol{x}, \boldsymbol{x}'_U) = \mathcal{R}_{UU} F_U(\boldsymbol{x}, \boldsymbol{x}'_U) + F_U^{\star}(\boldsymbol{x}, \boldsymbol{x}'_U). \quad (3)$$

In this result, we have defined $\mathcal{R}_{UU} F_U(\boldsymbol{x}, \boldsymbol{x}'_U) = \int_{\partial\mathbb{D}_U} F_U(\boldsymbol{x}, \boldsymbol{x}_U) \Gamma_U^-(\boldsymbol{x}_U, \boldsymbol{x}'_U)\, d\boldsymbol{x}_U$, where $\mathcal{R}_{UU}$ can be interpreted as a reflection operator at the upper boundary. We can substitute another dipole Green's function $\Gamma_L(\boldsymbol{x}, \boldsymbol{x}'_L) = \frac{2}{i\omega\rho(\boldsymbol{x}'_L)} \partial'_3 G(\boldsymbol{x}, \boldsymbol{x}'_L)$ into equation (2), yielding

$$\Gamma_L(\boldsymbol{x},\boldsymbol{x}'_L) = \mathcal{R}_{LL}F_L(\boldsymbol{x},\boldsymbol{x}'_L) + F^\star_L(\boldsymbol{x},\boldsymbol{x}'_L), \tag{4}$$

with $\mathcal{R}_{LL}F_L(\boldsymbol{x},\boldsymbol{x}'_L) = \int_{\partial\mathbb{D}_L} F_L(\boldsymbol{x},\boldsymbol{x}_L)\Gamma^+_L(\boldsymbol{x}_L,\boldsymbol{x}'_L)\,d\boldsymbol{x}_L$, where $\mathcal{R}_{LL}$ can be interpreted as a reflection operator at the lower boundary. Equations (3) and (4) can be written in matrix-vector form as

$$\begin{pmatrix}\Gamma_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \Gamma_L(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix} - \begin{pmatrix}\mathrm{F}_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_L(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}^\star = \begin{pmatrix}\mathcal{R}_{UU} & 0\\ 0 & \mathcal{R}_{LL}\end{pmatrix}\begin{pmatrix}\mathrm{F}_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_L(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}. \tag{5}$$

To solve this system of equations for the focusing functions F_U and F_L, we typically design a time window operator (van der Neut et al., 2015) such that $\Theta_U\Gamma_U = 0$ and $\Theta_U F^\star_U = F^\star_{Um}$, where F_{Um} is the coda of F_U; that is $\mathrm{F}_{Um} = \mathrm{F}_U - \mathrm{F}_{Ud}$, with F_{Ud} being the direct field. When we apply Θ_U to the upper row of equation (5) and an equivalent operator Θ_L to the lower row, we obtain the Marchenko equation

$$-\begin{pmatrix}\mathrm{F}_{Um}(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_{Lm}(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}^\star = \begin{pmatrix}\Theta_U & 0\\ 0 & \Theta_L\end{pmatrix}\begin{pmatrix}\mathcal{R}_{UU} & 0\\ 0 & \mathcal{R}_{LL}\end{pmatrix}\left(\begin{pmatrix}\mathrm{F}_{Ud}(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_{Ld}(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix} + \begin{pmatrix}\mathrm{F}_{Um}(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_{Lm}(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}\right). \tag{6}$$

Given the direct fields F_{Ud} and F_{Ld}, equation (6) can be solved for the codas F_{Um} and F_{Lm} by iterative substitution. After convergence, the Green's functions can be computed with equations (3) and (4).

A Marchenko equation for double-sided reflection and transmission data

In this section, we will use transmission data to eliminate the Green's functions on the left-hand side of equation (5). To facilitate this, we substitute $p = \Gamma_U$ into equation (2), yielding

$$\Gamma_U(\boldsymbol{x},\boldsymbol{x}'_U) = \mathcal{T}_{UL}F_L(\boldsymbol{x},\boldsymbol{x}'_L). \tag{7}$$

Here, $\mathcal{T}_{UL}F_L(\boldsymbol{x},\boldsymbol{x}'_L) = \int_{\partial\mathbb{D}_L} F_L(\boldsymbol{x},\boldsymbol{x}_L)\Gamma^+_U(\boldsymbol{x}_L,\boldsymbol{x}'_U)\,d\boldsymbol{x}_L$, where $\mathcal{T}_{UL}$ can be interpreted as a transmission operator from the lower to the upper boundary. Similarly, substituting $p = \Gamma_L$ into equation (1) yields

$$\Gamma_L(\boldsymbol{x},\boldsymbol{x}'_L) = \mathcal{T}_{LU}F_U(\boldsymbol{x},\boldsymbol{x}'_U), \tag{8}$$

with $\mathcal{T}_{LU}F_U(\boldsymbol{x},\boldsymbol{x}'_U) = \int_{\partial\mathbb{D}_U} F_U(\boldsymbol{x},\boldsymbol{x}_U)\Gamma^-_L(\boldsymbol{x}_U,\boldsymbol{x}'_L)\,d\boldsymbol{x}_U$, where $\mathcal{T}_{LU}$ is a transmission operator from the upper to the lower boundary. Equations (7) and (8) can be written in matrix-vector form as

$$\begin{pmatrix}\Gamma_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \Gamma_L(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix} = \begin{pmatrix}0 & \mathcal{T}_{UL}\\ \mathcal{T}_{LU} & 0\end{pmatrix}\begin{pmatrix}\mathrm{F}_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_L(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}, \tag{9}$$

To eliminate the Green's functions on the left-hand side, we subtract equation (9) from (5), yielding

$$-\begin{pmatrix}\mathrm{F}_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_L(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}^\star = \begin{pmatrix}\mathcal{R}_{UU} & -\mathcal{T}_{UL}\\ -\mathcal{T}_{LU} & \mathcal{R}_{LL}\end{pmatrix}\begin{pmatrix}\mathrm{F}_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_L(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}. \tag{10}$$

This result holds for any $\boldsymbol{x} \in \mathbb{D}$. Hence, equation (10) cannot be solved without additional constraints specifying this location. Although there are probably better solutions to deal with this issue (which we are currently investigating), we present an initial attempt here by combining the upper row of equation (10) with the lower row of equation (6); that is

$$-\begin{pmatrix}\mathrm{F}_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_{Lm}(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}^\star = \begin{pmatrix}1 & 0\\ 0 & \Theta_L\end{pmatrix}\begin{pmatrix}\mathcal{R}_{UU} & -\mathcal{T}_{UL}\\ 0 & \mathcal{R}_{LL}\end{pmatrix}\left(\begin{pmatrix}0\\ \mathrm{F}_{Ld}(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix} + \begin{pmatrix}\mathrm{F}_U(\boldsymbol{x},\boldsymbol{x}'_U)\\ \mathrm{F}_{Lm}(\boldsymbol{x},\boldsymbol{x}'_L)\end{pmatrix}\right). \tag{11}$$

Given F_{Ld}, equation (11) can be solved by iterative substitution. Unlike the conventional Marchenko equation in (6), no time window is applied to the upper row. Consequently, forward-scattered components of F_U can be retrieved by this procedure, as we will demonstrate in the following section.

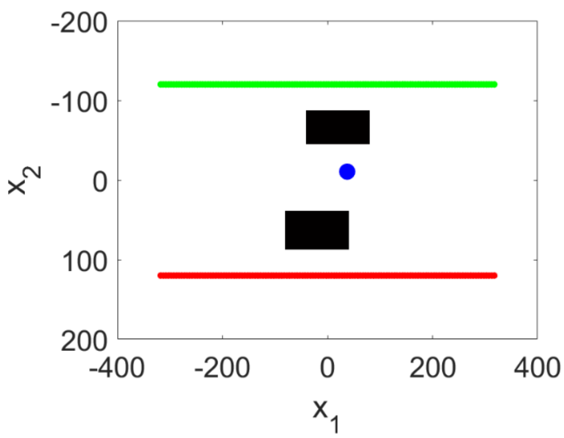

Figure 2 *Configuration with a constant velocity $c = 1500\, m\, s^{-1}$. The black boxes denote two density anomalies with $\rho = 2000\, kg\, m^{-3}$. Outside these boxes, we have a constant density of $\rho = 1000\, kg\, m^{-3}$. In green and red, we denote the upper and lower array. Focusing functions and Green's functions are evaluated at the blue dot.*

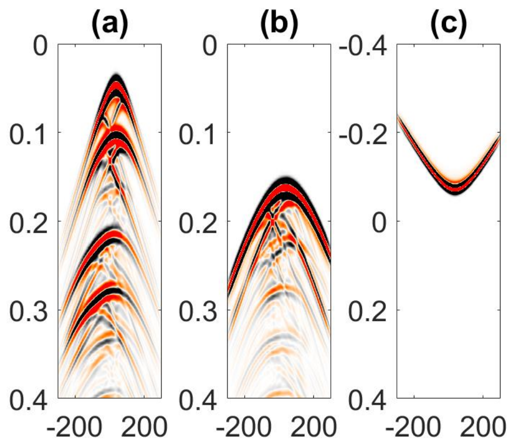

Figure 3 *(a) Reflection response at the upper array for a source at the lateral position of the blue dot in Figure 2; (b) Transmission response of the same source, recorded at the lower array; (c) Initial focusing function at the upper array. All responses are clipped at 10% of their maximum value.*

Numerical example

In this section, we support our theory with a numerical experiment. The configuration is shown in Figure 2. The upper and lower acquisition arrays contain 128 source / receiver pairs each, with a 5m source / receiver spacing (we applied a cosine taper to the first 12.5% and the last 12.5% of the traces). Synthetic data are computed by solving an interface contrast source integral equation (van den Berg, 2021). We use the second derivative of a Gaussian with a peak frequency of 50Hz as a wavelet and analytic time-reversed dipole Green's functions in a homogeneous background medium as initial focusing functions. In Figure 3, we show some of the data. We compute the focusing functions F_U and F_L by 20 iterative substitutions of equation (6) or (11), and the Green's function Γ_U and Γ_L with help of equations (3) and (4). For reference, we also construct a ground truth Green's function Γ_U by direct modeling, which is shown in Figure 4(a). A ground truth focusing function F_U is computed by direct inversion of equation (3) (using the ground truth Γ_U), see Figure 5(a). In Figure 4(b), we show the focusing function that is retrieved from equation (6). While most major events are retrieved correctly, the forward-scattered contributions are clearly underestimated, as pointed out by the green arrow in Figure 4(b). In Figure 4(c), we show the equivalent results as obtained from equation (11). This time, the forward-scattered contributions have been recovered correctly, as pointed out by the green arrow in Figure 4(c). The retrieved Green's functions are shown in Figures 5(b) and 5(c). Despite this preliminary success, we can also observe a drawback of our current approach. The orange arrow in Figure 4(c) points at an artifact that cannot be found in Figure 4(b). An intuitive explanation is that errors in F_{Ld} and F_{Lm} are transported to F_U by the action of $-\mathcal{T}_{UL}$ in equation (11). Hence, the methodology still suffers from incorrect forward-scattered waveforms in the second row of equation (11). As a consequence, some of the events in the Green's function are retrieved with erroneous amplitudes, as pointed out by the purple arrow in Figure 5(c). To overcome this issue, we reason that equation (10) rather than (11) should be inverted. We are currently investigating if this can indeed be done by least-squares inversion with additional constraints that specify the focal location $\boldsymbol{x}$.

Discussion and conclusion

In the conventional Marchenko equation, time windows are fundamentally required to separate the Green's function and focusing on the left-hand side of the underlying representation. In this abstract, we have shown that the Green's function can be eliminated from the representations by using auxiliary

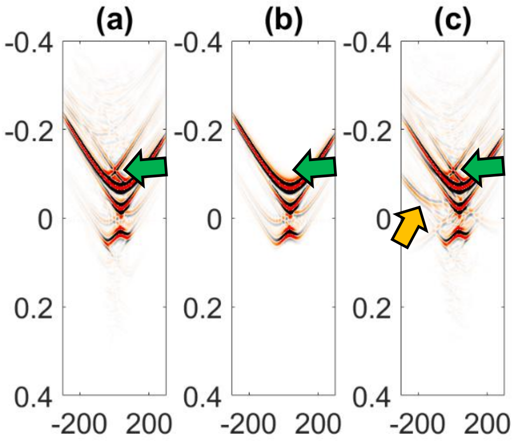

Figure 4 *Focusing function F_U: (a) Ground truth, (b) retrieved from equation (6) and (c) retrieved from equation (11). All responses are clipped at 10% of their maximum value.*

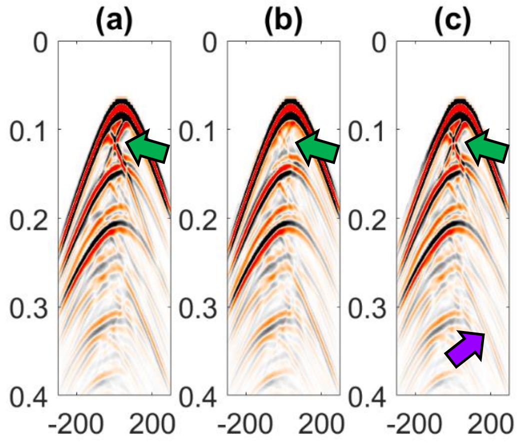

Figure 5 *Green's function Γ_U as retrieved from equation (3): (a) Ground truth and by using focusing functions from (b) equation (6) and (c) equation (11). All responses are clipped at 10% of their maximum value.*

transmission data. Hence, the conditions for time windowing can be relaxed, allowing the retrieval of waveforms beyond the scope of the conventional Marchenko equation. Although more research is required to exploit this advantage to its full potential, we have demonstrated some of the prospected gains in this abstract. In this process, we have focused our attention to diffraction-type events in the data. However, the methodology might also help us to cope with short-period multiples, which are difficult to retrieve with the conventional Marchenko methodology due to the bandlimited nature of our data, especially in case of severe lateral variations (Elison et al., 2020).

Acknowledgements
The research of Kees Wapenaar has received funding from the European Research Council (grant no. 742703).

References
- Cui, T., Becker, T. S., van Manen, D. -J., Rickett, J. E. and Vasconcelos, I. [2018] Marchenko redatuming in a dissipative medium: Numerical and experimental implementation. *Phys. Rev. Appl.*, **19**, 044022.
- Diekmann, L. and Vasconcelos, I. [2021] Focusing and Green’s function retrieval in three-dimensional inverse scattering revisited: A single-sided Marchenko integral for the full wave field. *Phys. Rev. Res.*, **3**, 013206.
- Elison, P., Dukalski, M. S., de Vos, K., van Manen, D. -J. and Robertsson, J. O. A. [2020] Data-driven control over short-period internal multiples in media with a horizontally layered overburden. *Geoph. J. Int.*, 221, 769-787.
- Kiraz, M. S. R., Snieder, R. and Wapenaar, K. [2020] Marchenko focusing without up/down decomposition. *90th Annual SEG Meeting, 3593-3597.*
- Liu, Y., van der Neut, J., Arntsen, B. and Wapenaar, K. [2016] Combination of surface and borehole seismic data for robust target-oriented imaging. *Geoph. J. Int.*, **205**, 758-775.
- Lomas, A., Singh, S. and Curtis, A. [2020] Imaging vertical structures using Marchenko methods with vertical seismic-profile data. *Geophysics*, **85**, S103-S113.
- Meles, G. A., van der Neut, J., van Dongen, K. W. A. and Wapenaar, K. [2019] Wavefield finite time focusing with reduced spatial exposure. *J. Acoust. Soc. Am.*, **145**, 3521-3530.
- van den Berg, P. M., [2021] *Forward and inverse scattering algorithms based on contrast source integral equations*. John Wiley and Sons, Inc., Hoboken, NJ.
- van der Neut, J., Vasconcelos, I. and Wapenaar, K. [2015] On Green's function retrieval by iterative substitution of the coupled Marchenko equations. *Geoph. J. Int.*, **203**, 792-813.
- Vargas, D. and Vasconcelos, I. [2020] Rayleigh-Marchenko redatuming using scattered fields in highly complex media. In: *EAGE Extended Abstracts*.
- Wapenaar, K., Thorbecke, J., van der Neut, J., Slob, E., and Snieder, R. [2017] Review paper: Virtual sources and their responses, Part II: Data-driven single-sided focusing. *Geophysical Prospecting*, **65**, 1430-1451.
- Wapenaar, K., Snieder, R., de Ridder, S., and Slob, E. [2021] Green’s function representations for Marchenko imaging without up/down decomposition. *arXiv:2103.07734* [physics.geo-ph].