# Sparse inversion for solving the coupled Marchenko equations including free-surface multiples

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## Summary

## Introduction

In recent years, much progress has been made in Marchenko redatuming. Starting with the work of Broggini et al. (2012), Wapenaar et al. (2014) extended the method to 2D and 3D, which was recently followed by the inclusion of free-surface multiples (Singh et al. (2015)). The coupled Marchenko equations are conventionally solved by iterative substitution. However, this is not guaranteed to produce the correct solution when free-surface multiples are included. When comparing the interferometric interpretation of the Marchenko scheme without free-surface multiples to the interferometric interpretation of the scheme including free-surface multiples, a difference in dynamics becomes evident. This intuitive idea is mathematically confirmed, proving that iterative substitution might not be the most logical choice for solving these equations. We suggest sparse inversion as a more suitable method, that does not only retrieve the correct wavefields where the iterative scheme fails, but that might also provide a more flexible and robust algorithm.

## Theory

The coupled Marchenko equations are displayed in equations 1 and 2. On the left side are the equations without free-surface multiples, on the right side the equations that also include free-surface multiples. Both are written in a discretized notation, as introduced by van der Neut et al. (2015). Here $R_0$ denotes the reflection response without free-surface multiples and $R$ is the reflection response including free-surface multiples, which have been obtained by sources and receivers at the acquisition level. Both responses contain internal multiples. The star denotes complex conjugation. $r$ denotes the reflection coefficient of the free-surface, which we assumed to be equal to $-1$.

Marchenko redatuming uses one-way focusing functions to focus both primary reflections and multiples at the desired depth level. The upgoing focusing function is represented by $f_1^-$. The downgoing focusing function has a direct part $f_{1d}^+$ and a coda $f_{1m}^+$. The direct part $f_{1d}^+$ is obtained by time-reversing the transmission response from the acquisition level to the focal level, modeled in a macro velocity model. The estimate of this direct wavefield is needed to initiate the scheme and to determine the truncation times of the filter $\Theta$. This filter is a time-symmetric windowing function that uses a difference in causality properties to separate the Green's functions from the focusing functions.

$$f_1^- = \Theta R_0 f_{1d}^+ + \Theta R_0 f_{1m}^+ \Rightarrow f_1^- = \Theta R f_{1d}^+ + \Theta R f_{1m}^+ - r\Theta R f_1^- \tag{1}$$

$$f_{1m}^+ = \Theta R_0^* f_1^- \Rightarrow f_{1m}^+ = \Theta R^* f_1^- - rR^* f_{1m}^+ \tag{2}$$

These equations are conventionally solved by iterative substitution (Slob et al. (2014)). The iterative scheme would ideally converge within a few iterations. At each iteration, the one-way Green's functions that contain primaries and all orders of multiples can be obtained from the one-way focusing functions. Figure 1 shows the convergence of the Green's function for 3 different models, both for solving the Marchenko equations without free-surface multiples and including free-surface multiples. The error in the convergence plots has been calculated by taking the $l_2$-norm of the absolute difference between iterations. Clearly, the scheme without free-surface multiples converges in a fast and straightforward manner. However, a completely different behaviour is observed for the scheme without free-surface multiples: the Green's function is not always correctly retrieved. For the first model, convergence takes place within a few iterations, but it occurs less fast and direct compared to the scheme without free-surface multiples. A behaviour of converging and diverging is observed for the second model, and complete divergence is found for the third model. This suggests that iterative substitution is perhaps not the most straightforward method for solving for the coupled Marchenko equations including free-surface multiples.

The underlying mechanism for iteratively solving the equations with free-surface multiples is different from the equations without free-surface multiples (Staring et al. (2016)). When no free-surface multiples are present, the dynamics are straightforward. Internal multiples can generate artefacts in the focusing functions in the first iteration. A form of multiple prediction is used to immediately generate counter-events from these artefacts in the next iteration. The following iterations contain only amplitude updates of these counterevents, until the removal of artefacts is complete. No new artefacts are being created

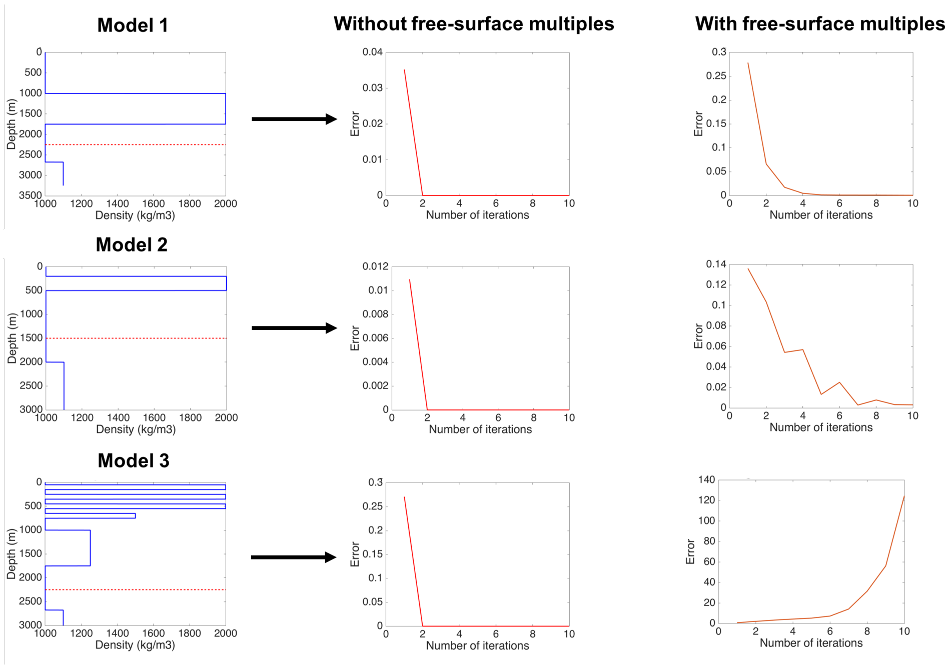


***Figure 1*** *Three models and their convergence, for the scheme without free-surface multiples and the scheme including free-surface multiples. The redatuming level is indicated by the dotted red line.*

after the first iteration. This can be seen when looking at the convergence plots in the middle of figure 1.

However, a completely different mechanism is found when studying the scheme including free-surface multiples. Free-surface multiples can also create artefacts in the upgoing focusing function during the first iteration, but their removal is not straight-forward. Although all necessary counter-events are being generated in the upgoing focusing function during the second iteration, they are being sabotaged by reproductions of the artefacts themselves, and new artefacts are being added. These new artefacts will not receive counter-events in next iterations, but will only be removed once the original artefacts are being eliminated. This is delicate, since the original artefacts are boosting and reproducing themselves while the scheme tries to remove them. Depending on the play of these dynamics, the scheme can produce convergence behaviour as seen in the three plots on the right in figure 1.

Since this is only an intuitive explanation found by studying the different updates of the one-way wave-fields, a more exact and mathematical explanation is needed to confirm these findings. We start by writing equations 1 and 2 in matrix form:

$$\underbrace{\begin{pmatrix} -r\Theta R & \Theta R \\ \Theta R^* & -r\Theta R^* \end{pmatrix} \begin{pmatrix} 0 \\ f_{1d}^+ \end{pmatrix}}_{\text{b}} = \underbrace{\left[ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} -r\Theta R & \Theta R \\ \Theta R^* & -r\Theta R^* \end{pmatrix} \right]}_{\text{A=I-M}} \underbrace{\begin{pmatrix} f_1^- \\ f_{1m}^+ \end{pmatrix}}_{\text{x}}. \tag{3}$$

This is a Fredholm equation of the second kind, which can be expanded as a Neumann series:

$$\begin{pmatrix} f_1^- \\ f_{1m}^+ \end{pmatrix} = \sum_{k=1}^{\infty} \underbrace{\begin{pmatrix} -r\Theta R & \Theta R \\ \Theta R^* & -r\Theta R^* \end{pmatrix}}_{\text{M}}^{k} \begin{pmatrix} 0 \\ f_{1d}^+ \end{pmatrix}. \tag{4}$$

Thus, our iterative solution can be interpreted as a Neumann series, where $k$ indicates the iteration number. This series is guaranteed to converge if $\left\| \begin{pmatrix} -r\Theta R & \Theta R \\ \Theta R^* & -r\Theta R^* \end{pmatrix}^k \begin{pmatrix} 0 \\ f_{1d}^+ \end{pmatrix} \right\|_2 \rightarrow 0$ as $k \rightarrow \infty$ (Fokkema and van den Berg (2013)), where the 2 indicates the $l_2$-norm. In order for this to hold, $||M||_2 < 1$ has to be

***Table 1*** *$l_2$ norms of the matrix M*

| | Model 1 | Model 2 | Model 3 |
|---|---|---|---|
| Without free-surface multiples | 0.58 | 0.82 | 0.98 |
| With free-surface multiples | 1.25 | 3.74 | 4.55 |

satisfied, such that $\|MP\|_2 \leq \|P\|_2$ for any wavefield P. Note that when the norm is above 1, convergence is no longer guaranteed, but may still occur. This fits with our intuitive explanation and the observations from figure 1. The value of this norm can only be found by numerical testing. Table 1 displays the norms of matrix M of the three models displayed in figure 1. Without exception, the norms for the scheme without free-surface multiples are below 1 and the iterative scheme is guaranteed to converge. However, norms are above 1 when including free-surface multiples, meaning that convergence is no longer guaranteed. From these observations, we can conclude that the iterative scheme might not be the most suitable method for solving the coupled Marchenko equations including free-surface multiples. Therefore, we decided to look at an alternative approach.

**Sparse inversion: an alternative approach**

In order to find a more suitable method, we need to specifically look for something that is capable of tackling the convergence issue. We propose a sparse inversion for solving the linear system of equations $Ax = b$, as given in equation 3. Here the vector $x$, containing the coda of the one-way focusing functions, is the unknown that we wish to solve for. However, we have to be careful not to encounter the exact same problem, since the Neumann lemma states that if $\|M\|_2 < 1$ then the matrix $A = I - M$ is invertible. Therefore, we propose to solve this system of equations by sparse inversion. Such an inversion is aimed at solving under-determined systems of equations by minimizing a norm, thereby making the matrix $A$ invertible. In addition, it allows us to enforce sparsity on the focusing functions, these should be free of artefacts and should thus satisfy a minimum energy criterion. Based on these requirements, we have chosen the SPG$l_1$ solver (Van Den Berg and Friedlander (2008)). This solver seems to be particularly suitable for this type of problem, since it attempts to minimize $\|x\|_1$, where the subscript 1 indicates the $l_1$-norm. In addition, solving the coupled Marchenko equations by inversion gives us the possibility to add additional constraints, for example in the form of smoothing and damping. When the data is incomplete or contaminated with noise, an inversion with these extra constraints might still lead to an acceptable result. Also, simultaneous inversion for the source wavelet, that is usually not accurately known, might be possible.

In order to satisfy the causality requirements that form the base of the Marchenko scheme, a restriction matrix containing the filter $\Theta$ on its diagonal should be applied to the matrix $x$ to ensure that we obtain a solution inside the domain of the focusing functions:

$$\underbrace{\begin{pmatrix} -r\Theta R & \Theta R \\ \Theta R^* & -r\Theta R^* \end{pmatrix} \begin{pmatrix} 0 \\ f_{1d}^+ \end{pmatrix}}_{\text{b}} = \underbrace{\left[ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} -r\Theta R & \Theta R \\ \Theta R^* & -r\Theta R^* \end{pmatrix} \right]}_{\text{A}} \underbrace{\begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix} \begin{pmatrix} f_1^- \\ f_{1m}^+ \end{pmatrix}}_{\text{x}}. \tag{5}$$

**Results**

In this section, we will compare the downgoing Green's functions retrieved by iterative substitution and sparse inversion with SPG$l_1$. Data including free-surface multiples is used, such that we can see whether the inversion is capable of overcoming the convergence issue from which the iterative scheme suffers. Figure 2 shows the results, where the red line represents the correct downgoing Green's function that was obtained by modeling and the blue line represents the results obtained by solving the coupled Marchenko equations including free-surface multiples. The top windows show the results of the iterative substitution and the bottom windows contain the results of the sparse inversion.

To test whether the sparse inversion is indeed capable of solving the coupled Marchenko equations, we started by looking at model 1 for which the iterative scheme finds the correct solution. Both methods retrieve the correct downgoing focusing function, proving us that the sparse inversion is indeed capable of solving the coupled Marchenko equations. Next, we perform the sparse inversion for models 2 and

3, where iterative substitution does not find the correct result. The sparse inversion does find the correct solution and therefore performs better.

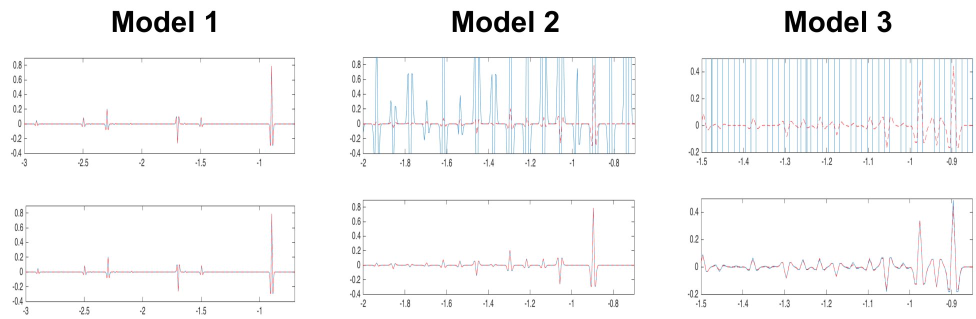


***Figure 2*** *Comparison of two methods: the iterative substitution in the upper figures, the sparse inverison with* $SPGl_1$ *in the lower figures. The red line indicates the modeled Green's functions and the blue line represents the retrieved Green's functions.*

## Conclusions

While iterative substitution provides a straightforward and natural way to solve the coupled Marchenko equations without free-surface multiples, it does not for the coupled Marchenko equations with free-surface multiples. Convergence of the iterative scheme is not guaranteed and incorrect Green's functions are often retrieved. The interferometric interpretation of both schemes has taught us that this difference can be attributed to the underlying mechanisms. A sparse inversion was suggested as an alternative method. While this method is computationally more expensive, we have demonstrated that it is capable of solving the coupled Marchenko equations including free-surface multiples where the iterative scheme fails. In addition, sparse inversion allows for more flexibility and is more robust. Using sparsity promotion and additional constraints, it might be able to deal better with noise and incomplete data. Simultaneous estimation of the source wavelet also belongs to the potential possibilities.

## Acknowledgements

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