Z-99 SEISMIC INTERFEROMETRY, TIME-REVERSAL AND RECIPROCITY

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Introduction

Seismic interferometry is the process of generating new seismic responses by cross-correlating seismic observations at different receiver locations. A first version of this principle was derived by Claerbout [2], who showed that the reflection response of a horizontally layered medium can be synthesized from the autocorrelation of its transmission response. Later he conjectured a similar principle for cross-correlations of 3-D wave fields. In a similar fashion, Schuster [5] introduced the principle of interferometric imaging, i.e. forming an image of the subsurface from cross-correlated seismic traces. Wapenaar et al. [6, 8] used a one-way reciprocity theorem of the correlation type to derive relations between reflection and transmission responses of arbitrary 3-D inhomogeneous media. These relations form a basis for seismic interferometry and they prove Claerbout's conjecture. Moreover, despite the differences in assumptions, these relations show a strong resemblance with those of Weaver and Lobkis [9] and Derode et al. [3, 4] for the retrieval of the Green's function from cross-correlations of wave fields in closed and open systems, respectively. Derode et al. [3, 4] derive their expressions for the Green's function retrieval using physical arguments, exploiting the principle of time reversal invariance of the acoustic wave equation and source-receiver reciprocity. Their approach can be seen as the 'physical counterpart' of our derivation based on a reciprocity theorem of the correlation type, which implicitly also makes use of time reversal invariance. The relation between these two approaches is discussed in this paper.

Physical derivation of Green's function retrieval

In this section we summarize the 'physical approach' of Derode et al. [3, 4] for deriving expressions for Green's function retrieval. We freely modify their notation, so that it matches our notation in the next section. Consider the configuration in Figure 1a. The shaded area represents a lossless arbitrary inhomogeneous acoustic medium, which is embedded by a homogeneous medium. In the inhomogeneous medium we have denoted two points $\mathbf{x}_A$ and $\mathbf{x}_B$. Our aim is to show that the response at $\mathbf{x}_B$ due to an impulsive source at $\mathbf{x}_A$ [i.e., the Green's function $G(\mathbf{x}_B, \mathbf{x}_A, t)$] can be obtained by cross-correlating passive measurements of the wave fields at $\mathbf{x}_A$ and $\mathbf{x}_B$ due to sources in the embedding homogeneous medium. The derivation starts by considering another physical experiment, namely an impulsive source at $\mathbf{x}_A$ and receivers at $\mathbf{x}$ on a closed surface S in the embedding homogeneous medium. The response at one particular point $\mathbf{x}$ on S is illustrated in Figure 1a and is denoted by $G(\mathbf{x}, \mathbf{x}_A, t)$. Suppose that we record this response for all $\mathbf{x}$ on S, revert the time axis, and feed these time-reverted functions $G(\mathbf{x}, \mathbf{x}_A, -t)$ to sources at $\mathbf{x}$ on S, see Figure 1b. According to Huygens' principle the wave field at any point $\mathbf{x}_B$ in S due to these sources on S is then given by

$$p(\mathbf{x}_B, t) \propto \oint_S G(\mathbf{x}_B, \mathbf{x}, t) * G(\mathbf{x}, \mathbf{x}_A, -t) \mathrm{d}^2\mathbf{x}, \tag{1}$$

where $*$ denotes convolution, $\propto$ denotes 'proportional to' and $\mathbf{x}_B$ is considered as a variable. According to this equation, $G(\mathbf{x}_B, \mathbf{x}, t)$ propagates the source function $G(\mathbf{x}, \mathbf{x}_A, -t)$ from $\mathbf{x}$ to $\mathbf{x}_B$ and the result is integrated over all sources on S. Due to the invariance of the acoustic wave equation for time-reversal (assuming the medium is lossless), the wave field $p(\mathbf{x}_B, t)$ focusses for $\mathbf{x}_B = \mathbf{x}_A$ at $t = 0$. Hence, the wave field $p(\mathbf{x}_B, t)$ for arbitrary $\mathbf{x}_B$ and t can be seen as the response of a virtual source at $\mathbf{x}_A$ and $t = 0$, i.e., $G(\mathbf{x}_B, \mathbf{x}_A, t)$. Since the source at $\mathbf{x}_A$ is only virtual, causality is not obeyed and $p(\mathbf{x}_B, t)$ is symmetric in

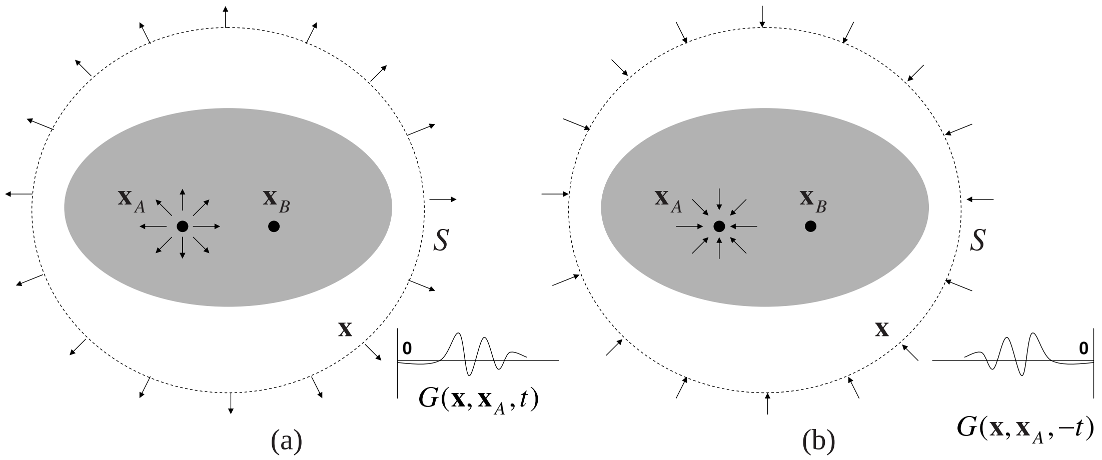

Figure 1: *Derivation of Green's function retrieval based on physical arguments (after Derode et al. [3, 4]). The aim is to find an expression for the Green's function between $\mathbf{x}_A$ and $\mathbf{x}_B$ in an inhomogeneous medium (the shaded area). The derivation starts with defining a source at $\mathbf{x}_A$ and receivers at $\mathbf{x}$ on S (Figure a). The responses on S are time-reverted and fed into sources on S (Figure b). The wave field inside S is then described according to Huygens' principle by equation (1). Since this wave field focusses for $\mathbf{x}_B = \mathbf{x}_A$ at $t = 0$, for arbitrary $\mathbf{x}_B$ it describes the response of a virtual source at $\mathbf{x}_A$ plus its time-reversed version (equation 2). Combining equations (1) and (2) and using source-receiver reciprocity for $G(\mathbf{x}_A, \mathbf{x}, -t)$ we obtain equation (3), which expresses $G(\mathbf{x}_B, \mathbf{x}_A, t)$ in terms of cross-correlations of observed wave fields at $\mathbf{x}_A$ and $\mathbf{x}_B$, due to sources at $\mathbf{x}$ on S.*

time, hence

$$p(\mathbf{x}_B, t) = G(\mathbf{x}_B, \mathbf{x}_A, t) + G(\mathbf{x}_B, \mathbf{x}_A, -t). \tag{2}$$

Equations (1) and (2) together give an interesting relation between Green's functions, but we are not there yet. In particular, the fact that we assume sources at S in terms of time-reversed Green's functions makes that these relations in their present form are of not much practical use. Let us now return to the situation that we described in the beginning of this section, namely cross-correlating passive measurements of the wave fields at $\mathbf{x}_A$ and $\mathbf{x}_B$ due to sources in the embedding homogeneous medium. If we choose a source with arbitrary source function $s(t)$ at a point $\mathbf{x}$ on S, the wave fields at $\mathbf{x}_A$ and $\mathbf{x}_B$ are given by $G(\mathbf{x}_A, \mathbf{x}, t) * s(t)$ and $G(\mathbf{x}_B, \mathbf{x}, t) * s(t)$ and their cross-correlation is given by $\{G(\mathbf{x}_B, \mathbf{x}, t) * s(t)\} * \{G(\mathbf{x}_A, \mathbf{x}, -t) * s(-t)\}$. We integrate this cross-correlation over all source positions $\mathbf{x}$ on S and use equations (1) and (2) and source-receiver reciprocity of the Green's function $G(\mathbf{x}_A, \mathbf{x}, -t)$ to obtain

$$\{G(\mathbf{x}_B, \mathbf{x}_A, t) + G(\mathbf{x}_B, \mathbf{x}_A, -t)\} * C_{ss}(t) \propto \oint_S \{G(\mathbf{x}_B, \mathbf{x}, t) * s(t)\} * \{G(\mathbf{x}_A, \mathbf{x}, -t) * s(-t)\} \mathrm{d}^2\mathbf{x}, \tag{3}$$

where $C_{ss}(t)$ is the autocorrelation of the source function $s(t)$, i.e. $C_{ss}(t) = s(t) * s(-t)$. Equation (3) constitutes a relation between the cross-correlation of passive measurements at $\mathbf{x}_A$ and $\mathbf{x}_B$ (the right-hand side), and the Green's function between these two points (the left-hand side). Since the Green's function $G(\mathbf{x}_B, \mathbf{x}_A, t)$ is causal, it can be obtained from the left-hand side of equation (3) by deconvolving for $C_{ss}(t)$ and then taking the causal part. Derode et al. [3, 4] rightfully remark that the reconstructed Green's function is not an ensemble average, so it properly contains the ballistic wave as well as the coda.

Finally we transform equation (3) to the space-frequency domain to facilitate the comparison with the results in the next section. We define the temporal Fourier transform of a space- and time-dependent quantity as $\hat{p}(\mathbf{x}, \omega) = \int \exp(-j\omega t) p(\mathbf{x}, t) \mathrm{d}t$, where j is the imaginary unit and ω the angular frequency. Hence, equation (3) transforms to

$$2\Re\{\hat{G}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}\hat{C}_{ss}(\omega) \propto \oint_S \{\hat{G}(\mathbf{x}_B, \mathbf{x}, \omega)\hat{s}(\omega)\}\{\hat{G}(\mathbf{x}_A, \mathbf{x}, \omega)\hat{s}(\omega)\}^* \mathrm{d}^2\mathbf{x}, \tag{4}$$

where $\Re$ denotes the real part and * complex conjugation. $\hat{C}_{ss}(\omega) = |\hat{s}(\omega)|^2$ is the power spectrum of $s(t)$.

Mathematical derivation of Green's function retrieval

In previous work we derived an expression for the reflection response of a 3-D inhomogeneous medium in terms of the transmission response of the same medium (Wapenaar et al. [6, 8]). The starting point for this derivation was a reciprocity theorem for coupled one-way wave equations; the reflection and transmission responses can be seen as Green's functions of the one-way wave equations. The reconstructed reflection response contained the ballistic wave as well as the coda, similar as the Green's function in the previous section. In order to make the relation of our work with the physical approach in the previous section more clear, here we follow a slightly different approach, using Rayleigh's reciprocity theorem as the starting point. Consider an acoustic wave field, characterized in the space-frequency domain by the acoustic pressure $\hat{p} = \hat{p}(\mathbf{x}, \omega)$ and the particle velocity $\hat{v}_i = \hat{v}_i(\mathbf{x}, \omega)$, and a source distribution $\hat{q} = \hat{q}(\mathbf{x}, \omega)$ representing the volume injection rate. Rayleigh's reciprocity theorem gives a relation between two independent acoustic states (i.e., wave fields and sources), which will be distinguished by subscripts A and B. For a domain V, enclosed by a boundary S with outward pointing normal vector $\mathbf{n} = (n_1, n_2, n_3)$ it reads

$$\int_V \{\hat{p}_A \hat{q}_B - \hat{q}_A \hat{p}_B\} \mathrm{d}^3\mathbf{x} = \oint_S \{\hat{p}_A \hat{v}_{i,B} - \hat{v}_{i,A} \hat{p}_B\} n_i \mathrm{d}^2\mathbf{x}. \tag{5}$$

The medium in V is arbitrary inhomogeneous and is the same in state A as in state B. Assuming the medium is lossless, we can again apply the principle of time-reversal invariance (Bojarski [1]). In the frequency domain time-reversal is replaced by complex conjugation. Hence, when $\hat{p}$ and $\hat{v}_i$ are a solution of the wave equation with source distribution $\hat{q}$, then $\hat{p}^*$ and $-\hat{v}_i^*$ obey the same wave equation with source distribution $-\hat{q}^*$. Making these substitutions for state A we obtain

$$\int_V \{\hat{p}_A^* \hat{q}_B + \hat{q}_A^* \hat{p}_B\} \mathrm{d}^3\mathbf{x} = \oint_S \{\hat{p}_A^* \hat{v}_{i,B} + \hat{v}_{i,A}^* \hat{p}_B\} n_i \mathrm{d}^2\mathbf{x}. \tag{6}$$

Next we choose point sources in both states, according to $\hat{q}_A(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_A)\hat{s}(\omega)$ and $\hat{q}_B(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_B)\hat{s}(\omega)$, with $\mathbf{x}_A$ and $\mathbf{x}_B$ both in V and $\hat{s}(\omega)$ the source spectrum. Then the wave fields in both states can be expressed in terms of Green's functions, according to

$$\hat{p}_A(\mathbf{x}, \omega) = \hat{G}(\mathbf{x}, \mathbf{x}_A, \omega)\hat{s}(\omega), \quad \hat{v}_{i,A}(\mathbf{x}, \omega) = -(j\omega\varrho(\mathbf{x}))^{-1} \partial_i \hat{G}(\mathbf{x}, \mathbf{x}_A, \omega)\hat{s}(\omega), \tag{7}$$

$$\hat{p}_B(\mathbf{x}, \omega) = \hat{G}(\mathbf{x}, \mathbf{x}_B, \omega)\hat{s}(\omega), \quad \hat{v}_{i,B}(\mathbf{x}, \omega) = -(j\omega\varrho(\mathbf{x}))^{-1} \partial_i \hat{G}(\mathbf{x}, \mathbf{x}_B, \omega)\hat{s}(\omega), \tag{8}$$

where $\varrho(\mathbf{x})$ is the mass density of the inhomogeneous medium. Substituting these expressions into equation (6) and using source-receiver reciprocity of the Green's functions gives

$$\begin{aligned} 2\Re\{\hat{G}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}\hat{C}_{ss}(\omega) \;=\; & \oint_S \frac{-1}{j\omega\varrho(\mathbf{x})} \Big(\{\partial_i \hat{G}(\mathbf{x}_B, \mathbf{x}, \omega)\hat{s}(\omega)\}\{\hat{G}(\mathbf{x}_A, \mathbf{x}, \omega)\hat{s}(\omega)\}^* \\ & - \{\hat{G}(\mathbf{x}_B, \mathbf{x}, \omega)\hat{s}(\omega)\}\{\partial_i \hat{G}(\mathbf{x}_A, \mathbf{x}, \omega)\hat{s}(\omega)\}^* \Big) n_i \mathrm{d}^2\mathbf{x}. \end{aligned} \tag{9}$$

This is an exact representation of the Green's function $\hat{G}(\mathbf{x}_B, \mathbf{x}_A, \omega)$ in terms of cross-correlations of observed wave fields at $\mathbf{x}_A$ and $\mathbf{x}_B$. Unlike in the previous section, we have not assumed that the surface S lies in a homogeneous embedding; in equation (9) S is an arbitrarily shaped surface in an arbitrary inhomogeneous lossless medium. The two terms under the integral ensure that waves propagating outward from the sources at $\mathbf{x}$ on S do not interact with those propagating inward and vice versa. Next we assume that the medium outside S is homogeneous. It can now be shown with stationary phase analysis that the two terms under the integral in equation (9) are approximately equal (but opposite in sign), hence

$$2\Re\{\hat{G}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}\hat{C}_{ss}(\omega) \approx \frac{-2}{j\omega\varrho} \oint_S \{\partial_i \hat{G}(\mathbf{x}_B, \mathbf{x}, \omega)\hat{s}(\omega)\}\{\hat{G}(\mathbf{x}_A, \mathbf{x}, \omega)\hat{s}(\omega)\}^* n_i \mathrm{d}^2\mathbf{x}, \tag{10}$$

where ϱ is the mass density of the homogeneous embedding. Finally, if we assume that S is a sphere with large enough radius such that the Fraunhofer far-field conditions apply, we obtain

$$2\Re\{\hat{G}(\mathbf{x}_B, \mathbf{x}_A, \omega)\}\hat{C}_{ss}(\omega) \approx \frac{2}{\varrho c} \oint_S \{\hat{G}(\mathbf{x}_B, \mathbf{x}, \omega)\hat{s}(\omega)\}\{\hat{G}(\mathbf{x}_A, \mathbf{x}, \omega)\hat{s}(\omega)\}^* \mathrm{d}^2\mathbf{x}, \tag{11}$$

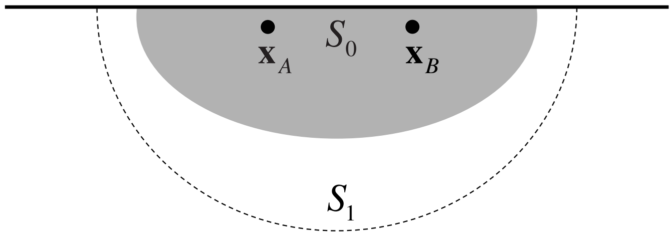

Figure 2: *In the seismic situation the free surface acts as a mirror. The Green's function $\hat{G}(\mathbf{x}_B, \mathbf{x}_A, \omega)$ can be retrieved by cross-correlating the responses of sources at S_1 only.*

where c is the propagation velocity of the homogeneous embedding. This concludes the comparison with the physical approach of Derode et al. [3, 4], discussed in the previous section.

Finally we consider a variant of our derivation. The Green's function introduced in equation (7) is the response of an impulsive point source of volume injection rate, which obeys the wave equation $\partial_i(\varrho^{-1}\partial_i\hat{G}) + (\omega^2/\varrho c^2)\hat{G} = -j\omega\delta(\mathbf{x} - \mathbf{x}_A)$. Let us define a Green's function $\hat{G}_0$ obeying the same wave equation, but with the source in the right-hand side replaced by $-\delta(\mathbf{x} - \mathbf{x}_A)$, hence $\hat{G}_0 = \frac{1}{j\omega}\hat{G}$. Following the same derivation as above, we obtain instead of equation (11)

$$2j\Im\{\hat{G}_0(\mathbf{x}_B, \mathbf{x}_A, \omega)\}\hat{C}_{ss}(\omega) \approx \frac{-2j\omega}{\varrho c} \oint_S \{\hat{G}_0(\mathbf{x}_B, \mathbf{x}, \omega)\hat{s}(\omega)\}\{\hat{G}_0(\mathbf{x}_A, \mathbf{x}, \omega)\hat{s}(\omega)\}^* \mathrm{d}^2\mathbf{x}, \tag{12}$$

where $\Im$ denotes the imaginary part. This result compares nicely with the results of Weaver and Lobkis [9, 10], who retrieve the imaginary part of the Green's function from the time-derivative of cross-correlations. A further comparison with the approach of Weaver and Lobkis [9, 10] is beyond the scope of this paper.

Modifications for the seismic situation

For the seismic situation we define the closed surface as $S = S_0 \cup S_1$, where S_0 is a part of the free surface and S_1 an arbitrarily shaped surface in the subsurface, see Figure 2. Since the acoustic pressure vanishes on S_0, the integral on the right-hand side of equations (5) to (12) needs only be evaluated over S_1. Hence, the Green's function $\hat{G}(\mathbf{x}_B, \mathbf{x}_A, \omega)$ can be recovered by cross-correlating the responses of sources on S_1 in the subsurface only. The free surface acts as a mirror which obviates the need of evaluating the integral over a closed surface. Other modifications are discussed in [7], such as generalizing the results for the full elastic situation, moving the points of observation $\mathbf{x}_A$ and $\mathbf{x}_B$ to the free surface and replacing the sources $\hat{s}(\omega)$ on S_1 by uncorrelated noise sources.

Conclusions

In the literature several derivations have been proposed for Green's function retrieval from cross-correlations (which is the basis for seismic interferometry). We have shown that the derivation by Derode et al. [3, 4], which is based on physical arguments, leads essentially to the same results as our derivation based on Rayleigh's reciprocity theorem [6, 8].

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