Z-99 RECIPROCITY AND POWER BALANCE FOR PARTIALLY COUPLED INTERFACES

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Introduction

The interaction of elastodynamic waves with imperfectly coupled interfaces has been investigated by many authors for various situations. Schoenberg [2] introduced the linear slip model for an interface between two elastic media. In this model it is assumed that the particle displacement of an elastic wave at an interface jumps by a finite amount, linearly proportional to the stress at the interface. The stress itself is assumed continuous across the interface. The ratio of the stress and the displacement jump is the specific boundary stiffness. Pyrak-Nolte et al. [1] extended this model with a specific boundary viscosity, which is the ratio of stress and a velocity jump across the interface. Other authors considered imperfect boundary conditions for amongst others, electromagnetic waves in matter and for Biot waves in porous media.

In this paper we analyze reciprocity theorems and power balances for various wave types in piecewise continuous inhomogeneous media, containing arbitrarily shaped interfaces with imperfect coupling. We illustrate the relations for the elastodynamic linear slip models of Schoenberg [2] and Pyrak-Nolte et al. [1].

General boundary conditions for wave fields at imperfectly coupled interfaces

We use a unified notation that applies to acoustic waves in fluids, elastodynamic waves in solids, electromagnetic waves in matter, poroelastic waves in porous solids and seismoelectric waves in porous solids. The general boundary conditions for each of these wave phenomena can be cast in a single matrix-vector equation, according to

$$\Delta(\mathbf{M}\hat{\mathbf{u}}) = \hat{\mathbf{Y}}\langle\mathbf{M}\hat{\mathbf{u}}\rangle, \tag{1}$$

where $\hat{\mathbf{u}}$ is the wave field vector, $\mathbf{M}$ is a matrix that contracts this wave vector to the components that are involved in the boundary conditions and $\hat{\mathbf{Y}}$ is a matrix containing the specific boundary parameters. $\Delta(\cdot)$ and $\langle\cdot\rangle$ represent the jump and the average across the interface, respectively. We illustrate equation (1) for the situation of elastodynamic waves in solids. An elastodynamic wave field is described in the space-frequency $(\mathbf{x},\omega)$ domain in terms of the stress $\hat{\tau}_{ij}(\mathbf{x},\omega)$ and the particle velocity $\hat{v}_i(\mathbf{x},\omega)$. We use a subscript notation for the components of vectorial and tensorial quantities. Lower-case Latin subscripts take on the values 1, 2 and 3 and Einstein's summation convention applies to repeated subscripts. Consider an interface with normal vector $\mathbf{n} = (n_1, n_2, n_3)^T$ between two solids with different (space-dependent) medium parameters (superscript ^{T} denotes transposition). The general boundary conditions for an elastodynamic wave field at an imperfectly coupled interface read

$$\Delta(\hat{\tau}_{ij}n_j) = \hat{\beta}_{ik}\langle\hat{v}_k\rangle, \tag{2}$$

$$\Delta\hat{v}_i = \hat{\gamma}_{ik}\langle\hat{\tau}_{kj}n_j\rangle. \tag{3}$$

Here $\hat{\beta}_{ik} = \hat{\beta}_{ik}(\mathbf{x},\omega)$ and $\hat{\gamma}_{ik} = \hat{\gamma}_{ik}(\mathbf{x},\omega)$ are anisotropic frequency-dependent parameters. Boundary conditions (2) and (3) can be captured by matrix-vector equation (1), with

$$\hat{\mathbf{u}} = \begin{pmatrix} \hat{\mathbf{v}} \\ -\hat{\boldsymbol{\tau}}_1 \\ -\hat{\boldsymbol{\tau}}_2 \\ -\hat{\boldsymbol{\tau}}_3 \end{pmatrix}, \mathbf{M} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & n_1\mathbf{I} & n_2\mathbf{I} & n_3\mathbf{I} \end{pmatrix}, \hat{\mathbf{Y}} = \begin{pmatrix} \mathbf{O} & -\hat{\boldsymbol{\gamma}} \\ -\hat{\boldsymbol{\beta}} & \mathbf{O} \end{pmatrix}, \tag{4}$$

where $(\hat{\mathbf{v}})_i = \hat{v}_i$, $(\hat{\boldsymbol{\tau}}_j)_i = \hat{\tau}_{ij}$, $(\hat{\boldsymbol{\beta}})_{ik} = \hat{\beta}_{ik}$, $(\hat{\boldsymbol{\gamma}})_{ik} = \hat{\gamma}_{ik}$, $(\mathbf{I})_{ik} = \delta_{ik}$ and $(\mathbf{O})_{ik} = 0$. We consider some special situations. When $\hat{\boldsymbol{\beta}} = \mathbf{O}$ (which is usually the case) and $\mathbf{n} = (0,0,1)^T$ (i.e., the interface is horizontal), then equations (1) and (4) yield $\Delta\hat{\mathbf{v}} = \hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\tau}}_3$. Furthermore, when $\hat{\boldsymbol{\gamma}}$ can be written as

$$\hat{\boldsymbol{\gamma}} = \begin{pmatrix} (\frac{K_1}{j\omega} + \eta)^{-1} & 0 & 0 \\ 0 & (\frac{K_2}{j\omega} + \eta)^{-1} & 0 \\ 0 & 0 & (\frac{K_3}{j\omega})^{-1} \end{pmatrix}, \tag{5}$$

this yields

$$\hat{\tau}_{13} = K_1 \frac{\Delta\hat{v}_1}{j\omega} + \eta\Delta\hat{v}_1, \quad \hat{\tau}_{23} = K_2 \frac{\Delta\hat{v}_2}{j\omega} + \eta\Delta\hat{v}_2, \quad \hat{\tau}_{33} = K_3 \frac{\Delta\hat{v}_3}{j\omega}. \tag{6}$$

These equations represent the frequency domain equivalent of the extended linear slip model of Pyrak-Nolte et al. [1]. The first term on the right-hand side of each of these equations represents the specific boundary stiffness K_i multiplied with the displacement jump $\Delta\hat{v}_i/j\omega$; the second term is the product of the specific boundary viscosity η and the velocity jump $\Delta\hat{v}_i$. When $\eta = 0$ and $K_1 = K_2$, these equations reduce to the linear slip model of Schoenberg [2].

Unified reciprocity theorems

For any of the wave phenomena mentioned in the previous section, we derived two unified reciprocity theorems (Wapenaar and Fokkema [3]), which are briefly reviewed here. We consider two physical states in a volume $\mathcal{V}$, enclosed by surface $\partial\mathcal{V}$ with outward pointing normal vector $\mathbf{n}$. The field quantities, the material parameters, as well as the source functions may be different in both states; they will be distinguished with subscripts A and B (of course the summation convention does not apply for these subscripts). For the moment we assume that there are no interfaces in $\mathcal{V}$. In the frequency domain, the reciprocity theorem of the convolution type reads

$$\oint_{\partial\mathcal{V}} \hat{\mathbf{u}}_A^T \mathbf{K}\mathbf{N}_{\mathbf{x}} \hat{\mathbf{u}}_B \,\mathrm{d}^2\mathbf{x} = \int_{\mathcal{V}} \Big[\hat{\mathbf{u}}_A^T \mathbf{K}\hat{\mathbf{s}}_B - \hat{\mathbf{s}}_A^T \mathbf{K}\hat{\mathbf{u}}_B\Big] \mathrm{d}^3\mathbf{x} + \int_{\mathcal{V}} \hat{\mathbf{u}}_A^T \mathbf{K}\big[j\omega(\mathbf{A}_A - \mathbf{A}_B) + (\mathbf{B}_A - \mathbf{B}_B)\big]\hat{\mathbf{u}}_B \,\mathrm{d}^3\mathbf{x}. \tag{7}$$

We speak of a convolution-type theorem, because the multiplications in the frequency domain correspond to convolutions in the time domain. This theorem interrelates the wave field quantities (contained in $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$), the material parameters (contained in $\mathbf{A}_A$, $\mathbf{B}_A$, $\mathbf{A}_B$ and $\mathbf{B}_B$) as well as the source functions (contained in $\hat{\mathbf{s}}_A$ and $\hat{\mathbf{s}}_B$) of states A and B. The material parameter matrices, the source vectors as well as matrices $\mathbf{N}_{\mathbf{x}}$ and $\mathbf{K}$ are given in [3] for the different wave phenomena discussed above. The reciprocity theorem of the correlation type reads

$$\oint_{\partial\mathcal{V}} \hat{\mathbf{u}}_A^H \mathbf{N}_{\mathbf{x}} \hat{\mathbf{u}}_B \,\mathrm{d}^2\mathbf{x} = \int_{\mathcal{V}} \Big[\hat{\mathbf{u}}_A^H \hat{\mathbf{s}}_B + \hat{\mathbf{s}}_A^H \hat{\mathbf{u}}_B\Big] \mathrm{d}^3\mathbf{x} + \int_{\mathcal{V}} \hat{\mathbf{u}}_A^H \big[j\omega(\mathbf{A}_A - \mathbf{A}_B) - (\mathbf{B}_A^H + \mathbf{B}_B)\big]\hat{\mathbf{u}}_B \,\mathrm{d}^3\mathbf{x}, \tag{8}$$

where superscript ^{H} denotes transposition and complex conjugation. We speak of correlation type, because the multiplications in the frequency domain correspond to correlations in the time domain. For elastodynamic waves, the matrices $\mathbf{N}_{\mathbf{x}}$ and $\mathbf{K}$ are given by

$$\mathbf{N}_{\mathbf{x}} = \begin{pmatrix} \mathbf{O} & n_1\mathbf{I} & n_2\mathbf{I} & n_3\mathbf{I} \\ n_1\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ n_2\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ n_3\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{I} \end{pmatrix}. \tag{9}$$

Reciprocity for imperfectly coupled interfaces

We now extend the reciprocity theorems for the situation in which $\mathcal{V}$ contains imperfectly coupled internal interfaces. To this end we subdivide $\mathcal{V}$ into L continuous regions, according to $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cdots \cup \mathcal{V}_L$, see Figure 1. Region $\mathcal{V}_l$ is enclosed by surface $\partial\mathcal{V}_l$ with outward pointing normal vector $\mathbf{n}_l$. The boundaries between these regions represent the imperfectly coupled internal interfaces. Note that each internal interface is part of two surfaces $\partial\mathcal{V}_l$, with oppositely pointing normal vectors $\mathbf{n}_l$, see Figure 1.

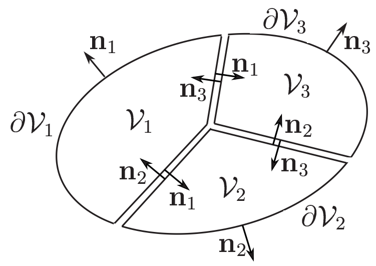

Figure 1: Piecewise continuous volume $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cdots \cup \mathcal{V}_L$.

Since the medium parameters in region $\mathcal{V}_l$ are continuous, the reciprocity theorems (7) and (8) apply to each of these regions. Summing both sides of these equations over l yields again equations (7) and (8) for the total volume $\mathcal{V}$, with in the left-hand sides extra integrals over the internal interfaces, given by

$$\int_{\Sigma_{\rm int}} \Big[\big(\hat{\mathbf{u}}_A^T \mathbf{K} \mathbf{N}_{\mathbf{x}} \hat{\mathbf{u}}_B \big)_1 + \big(\hat{\mathbf{u}}_A^T \mathbf{K} \mathbf{N}_{\mathbf{x}} \hat{\mathbf{u}}_B \big)_2 \Big] \,{\rm d}^2\mathbf{x} \quad \text{and} \quad \int_{\Sigma_{\rm int}} \Big[\big(\hat{\mathbf{u}}_A^H \mathbf{N}_{\mathbf{x}} \hat{\mathbf{u}}_B \big)_1 + \big(\hat{\mathbf{u}}_A^H \mathbf{N}_{\mathbf{x}} \hat{\mathbf{u}}_B \big)_2 \Big] \,{\rm d}^2\mathbf{x}, \tag{10}$$

respectively, where $\Sigma_{\rm int}$ constitutes the total of all internal interfaces; the subscripts $_1$ and $_2$ denote the two sides of the internal interfaces. In the following, we evaluate these integrals, using the general boundary condition of equation (1). To this end we first introduce matrices $\mathbf{N}$ and $\mathbf{J}$, such that

$$\mathbf{K}\mathbf{N}_{\mathbf{x}} = \mathbf{M}^T \mathbf{N} \mathbf{M} \quad \text{and} \quad \mathbf{N}_{\mathbf{x}} = \mathbf{M}^H \mathbf{J} \mathbf{M}. \tag{11}$$

For example, for elastodynamic waves in solids, matrices $\mathbf{N}$ and $\mathbf{J}$ are given by

$$\mathbf{N} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}. \tag{12}$$

Since the normal vectors have different signs at opposite sides of an interface (see Figure 1), we have

$$\mathbf{M}_2^T \mathbf{N} \mathbf{M}_2 = -\mathbf{M}_1^T \mathbf{N} \mathbf{M}_1 \quad \text{and} \quad \mathbf{M}_2^H \mathbf{J} \mathbf{M}_2 = -\mathbf{M}_1^H \mathbf{J} \mathbf{M}_1. \tag{13}$$

We use equations (11) and (13) to rewrite the interface integrals of equation (10) as

$$\int_{\Sigma_{\rm int}} \big(\hat{\mathbf{u}}_{1,A}^T \mathbf{M}^T \mathbf{N} \mathbf{M} \hat{\mathbf{u}}_{1,B} - \hat{\mathbf{u}}_{2,A}^T \mathbf{M}^T \mathbf{N} \mathbf{M} \hat{\mathbf{u}}_{2,B} \big) \,{\rm d}^2\mathbf{x} \tag{14}$$

and

$$\int_{\Sigma_{\rm int}} \big(\hat{\mathbf{u}}_{1,A}^H \mathbf{M}^H \mathbf{J} \mathbf{M} \hat{\mathbf{u}}_{1,B} - \hat{\mathbf{u}}_{2,A}^H \mathbf{M}^H \mathbf{J} \mathbf{M} \hat{\mathbf{u}}_{2,B} \big) \,{\rm d}^2\mathbf{x}, \tag{15}$$

respectively, where $\mathbf{M}$ stands for $\mathbf{M}_1$. In case of perfect coupling, we have $\mathbf{M}\hat{\mathbf{u}}_2 = \mathbf{M}\hat{\mathbf{u}}_1$ for state A as well as state B, hence, the internal interface integrals vanish. This means that the reciprocity theorems (7) and (8) are valid for a piecewise continuous medium (as in Figure 1) with perfectly coupled interfaces. Of course the more interesting case is the one in which the interfaces are partially coupled. For this situation we rewrite the general boundary condition (1) as

$$\mathbf{M}(\hat{\mathbf{u}}_2 - \hat{\mathbf{u}}_1) = \hat{\mathbf{Y}} \mathbf{M} (\hat{\mathbf{u}}_2 + \hat{\mathbf{u}}_1)/2, \tag{16}$$

or

$$\mathbf{M}\hat{\mathbf{u}}_2 = \hat{\mathbf{Z}}\,\mathbf{M}\hat{\mathbf{u}}_1, \quad \text{with} \quad \hat{\mathbf{Z}} = (\mathbf{I} - \hat{\mathbf{Y}}/2)^{-1} (\mathbf{I} + \hat{\mathbf{Y}}/2). \tag{17}$$

We substitute equation (17) for states A and B into the interface integrals (14) and (15), which yields

$$\int_{\Sigma_{\rm int}} \hat{\mathbf{u}}_A^T \mathbf{M}^T \big(\mathbf{N} - \hat{\mathbf{Z}}_A^T \mathbf{N} \hat{\mathbf{Z}}_B \big) \mathbf{M} \hat{\mathbf{u}}_B \,{\rm d}^2\mathbf{x} \quad \text{and} \quad \int_{\Sigma_{\rm int}} \hat{\mathbf{u}}_A^H \mathbf{M}^H \big(\mathbf{J} - \hat{\mathbf{Z}}_A^H \mathbf{J} \hat{\mathbf{Z}}_B \big) \mathbf{M} \hat{\mathbf{u}}_B \,{\rm d}^2\mathbf{x}, \tag{18}$$

respectively. This is the final form of the integrals that have to be added to the left-hand sides of equations (7) and (8), respectively. Note that $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$ stand for $\hat{\mathbf{u}}_{1,A}$ and $\hat{\mathbf{u}}_{1,B}$, respectively (similar as $\mathbf{M}$ stands for $\mathbf{M}_1$). It is arbitrary which side of the interface is designated 'side 1'. All that matters is that $\hat{\mathbf{u}}_A$, $\hat{\mathbf{u}}_B$ and $\mathbf{M}$ all refer to the same side of the interface.

Analysis of the internal interface integral in the reciprocity theorem of the convolution type

In the following analysis we take $\hat{\mathbf{Z}}_A = \hat{\mathbf{Z}}_B = \hat{\mathbf{Z}}$. The internal interface integral in the reciprocity theorem of the convolution type (equation 18, first integral) vanishes when

$$\hat{\mathbf{Z}}^T \mathbf{N} \hat{\mathbf{Z}} = \mathbf{N}, \tag{19}$$

or, substituting equation (17), reorganizing some terms and using the property $\mathbf{N}^{-1} = -\mathbf{N}$, when

$$(\mathbf{I} - \hat{\mathbf{Y}}/2)\mathbf{N}(\mathbf{I} - \hat{\mathbf{Y}}^T/2) = (\mathbf{I} + \hat{\mathbf{Y}}/2)\mathbf{N}(\mathbf{I} + \hat{\mathbf{Y}}^T/2). \tag{20}$$

Hence, reciprocity theorem (7) is valid in all situations where equation (20) is fulfilled. As a consequence, in those situations source-receiver reciprocity remains valid when the source and receiver are separated by imperfectly coupled interfaces.

As an example, we substitute $\hat{\mathbf{Y}}$ and $\mathbf{N}$ as defined in equations (4) and (12) for elastodynamic waves in solids into equation (20). It thus follows that the interface integral vanishes when $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}^T$ and $\hat{\boldsymbol{\gamma}} = \hat{\boldsymbol{\gamma}}^T$. In the linear slip model of Schoenberg [2] as well as the extended linear slip model of Pyrak-Nolte et al. [1] these conditions are fulfilled since $\hat{\boldsymbol{\beta}} = \mathbf{O}$ and $\hat{\boldsymbol{\gamma}}$ is diagonal (equation 5).

Analysis of the internal interface integral in the reciprocity theorem of the correlation type

The internal interface integral in the reciprocity theorem of the correlation type (equation 18, second integral) vanishes when

$$\hat{\mathbf{Z}}^H \mathbf{J} \hat{\mathbf{Z}} = \mathbf{J}, \tag{21}$$

or, substituting equation (17), reorganizing some terms and using the property $\mathbf{J}^{-1} = \mathbf{J}$, when

$$(\mathbf{I} - \hat{\mathbf{Y}}/2)\mathbf{J}(\mathbf{I} - \hat{\mathbf{Y}}^H/2) = (\mathbf{I} + \hat{\mathbf{Y}}/2)\mathbf{J}(\mathbf{I} + \hat{\mathbf{Y}}^H/2). \tag{22}$$

Hence, reciprocity theorem (8) is valid in all situations where equation (22) is fulfilled. As a consequence, in those situations no power dissipation occurs at the imperfectly coupled interfaces.

As an example, we substitute $\hat{\mathbf{Y}}$ and $\mathbf{J}$ as defined in equations (4) and (12) for elastodynamic waves in solids into equation (22). It thus follows that the interface integral vanishes when $\hat{\boldsymbol{\beta}} = -\hat{\boldsymbol{\beta}}^H$ and $\hat{\boldsymbol{\gamma}} = -\hat{\boldsymbol{\gamma}}^H$. In the linear slip model of Schoenberg [2] these conditions are fulfilled, since $\hat{\boldsymbol{\beta}} = \mathbf{O}$ and $\hat{\boldsymbol{\gamma}}$ is a purely imaginary diagonal matrix (equation (5), with $\eta = 0$). However, in the extended linear slip model of Pyrak-Nolte et al. [1] these conditions are not fulfilled since the diagonal matrix $\hat{\boldsymbol{\gamma}}$ (equation 5) is not purely imaginary.

Conclusions

We have formulated general boundary conditions at imperfectly coupled interfaces for acoustic waves in fluids, elastodynamic waves in solids, electromagnetic waves in matter, poroelastic waves in porous solids and seismoelectric waves in porous solids. These boundary conditions are captured by the general matrix-vector equation $\Delta(\mathbf{M}\hat{\mathbf{u}}) = \hat{\mathbf{Y}}\langle\mathbf{M}\hat{\mathbf{u}}\rangle$, where matrix $\hat{\mathbf{Y}}$ contains the specific interface parameters. Using this equation, we have extended two unified reciprocity theorems (one of the convolution-type and one of the correlation-type) with an extra integral over the imperfectly coupled interfaces. We have formulated conditions for the matrix $\hat{\mathbf{Y}}$ under which the extra integrals vanish [equations (20) and (22)]. It appears that the extra integral in the convolution-type reciprocity theorem vanishes in the considered cases, which means amongst others that source-receiver reciprocity remains valid when the source and receiver are separated by imperfectly coupled interfaces. The extra integral in the correlation-type reciprocity theorem vanishes only in a limited number of cases. In those situations where it does not vanish, the imperfectly coupled interfaces dissipate power.

References

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