P669 **ELASTODYNAMIC RECIPROCITY THEOREMS FOR TIME-LAPSE SEISMIC METHODS**

JACOB FOKKEMA[1], KEES WAPENAAR[1], MENNO DILLEN[1] and KARIN SCHALKWIJK[2]
[1]*Section of Applied Geophysics, Centre for Technical Geoscience, Delft University of Technology, The Netherlands*
[2]*Section of Seismics and Acoustics, Centre for Technical Geoscience, Delft University of Technology, The Netherlands*

Introduction

Reciprocity theorems play an important role in formulating true amplitude operations on seismic wave fields, such as multiple elimination, migration and characterization. In general, a reciprocity theorem interrelates the quantities that characterize two admissible physical states that could occur in one and the same domain (de Hoop, 1988). One state is identified with an actual measurement, while the other state can either be a computational state (e.g. migration operators), a desired state (e.g. multiple-free data) or another measurement (characterizing time-lapse differences in the reservoir). In previous work we discussed the application of acoustic reciprocity theorems for time-lapse seismic methods. In particular, in Fokkema et al. (1999) we discussed applications based on the full and one-way wave equations. Here we extend this to the elastodynamic situation.

Reciprocity theorem for the full elastodynamic wave field

In the space-frequency $(\mathbf{x}, \omega)$ domain, the equations that govern elastodynamic wave motion read

$$j\omega\varrho V_i - \partial_j T_{ij} = F_i \tag{1}$$

$$j\omega s_{ijkl}T_{kl} - \frac{1}{2}(\partial_i V_j + \partial_j V_i) = -H_{ij}, \tag{2}$$

where T_{ij} is the stress, V_i is the particle velocity, ϱ is the volume density of mass, s_{ijkl} is the compliance, F_i is the volume source density of volume force and H_{ij} is the volume source density of deformation rate. The Latin subscripts take on the values 1 to 3 and the summation convention applies to repeated subscripts. The stress obeys the symmetry relation $T_{ij} = T_{ji}$. The compliance obeys $s_{ijkl} = s_{jikl} = s_{jilk}$ and, assuming that the wave motion occurs adiabatically, $s_{ijkl} = s_{klij}$. We introduce two elastodynamic states (i.e., wave fields, medium parameters and sources), that will be distinguished by the subscripts A and B. For these two states we consider the interaction quantity $\partial_j\{T_{ij,A}V_{i,B} - V_{i,A}T_{ij,B}\}$. Applying the product rule for differentiation, substituting equations (1) and (2) for states A and B, integrating the result over a volume $\mathcal{V}$ with boundary $\partial\mathcal{V}$ and outward pointing normal vector $\mathbf{n} = (n_1, n_2, n_3)$ (see Figure 1) and applying the theorem of Gauss yields

$$\begin{aligned}&\int_{\mathbf{x}\in\partial\mathcal{V}} \{T_{ij,A}V_{i,B} - V_{i,A}T_{ij,B}\}n_j \mathrm{dA} = \\ &j\omega\int_{\mathbf{x}\in\mathcal{V}} \{T_{ij,A}(s_{ijkl,B} - s_{ijkl,A})T_{kl,B} - V_{i,A}(\varrho_B - \varrho_A)V_{i,B}\}\mathrm{dV} \\ &+\int_{\mathbf{x}\in\mathcal{V}} \{T_{ij,A}H_{ij,B} + V_{i,A}F_{i,B} - F_{i,A}V_{i,B} - H_{ij,A}T_{ij,B}\}\mathrm{dV}.\end{aligned} \tag{3}$$

Equation (3) is the Betti-Rayleigh reciprocity theorem.

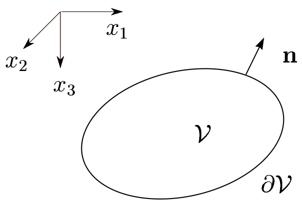


Figure 1: *Configuration for the Rayleigh-Betti reciprocity theorem.*

Reciprocity theorem for elastodynamic one-way wave fields

We introduce a system of coupled equations for the one-way wave fields $\mathbf{P}^+$ and $\mathbf{P}^-$, propagating in the positive and negative depth direction, respectively, according to

$$\partial_3 \mathbf{P} = \hat{\mathbf{B}}\mathbf{P} + \mathbf{S}, \tag{4}$$

(the hat denotes a pseudo-differential operator), with

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}^+ \\ \mathbf{P}^- \end{pmatrix}, \quad \mathbf{P}^\pm = \begin{pmatrix} \Phi^\pm \\ \Psi^\pm \\ \Upsilon^\pm \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} \mathbf{S}^+ \\ \mathbf{S}^- \end{pmatrix}, \quad \mathbf{S}^\pm = \begin{pmatrix} S_\phi^\pm \\ S_\psi^\pm \\ S_\upsilon^\pm \end{pmatrix}, \tag{5}$$

where $\Phi^\pm$, $\Psi^\pm$ and $\Upsilon^\pm$ represent the (flux-normalized) down- and upgoing quasi-P, quasi-S1 and quasi-S2 waves, respectively. $\mathbf{S}^+$ and $\mathbf{S}^-$ are source vectors for these one-way wave fields. The one-way operator matrix $\hat{\mathbf{B}}$ is defined as

$$\hat{\mathbf{B}} = \begin{pmatrix} -j\omega\hat{\Lambda}^+ & \mathbf{O} \\ \mathbf{O} & j\omega\hat{\Lambda}^- \end{pmatrix} + \begin{pmatrix} \hat{\mathbf{T}}^+ & \hat{\mathbf{R}}^- \\ -\hat{\mathbf{R}}^+ & \hat{\mathbf{T}}^- \end{pmatrix}, \tag{6}$$

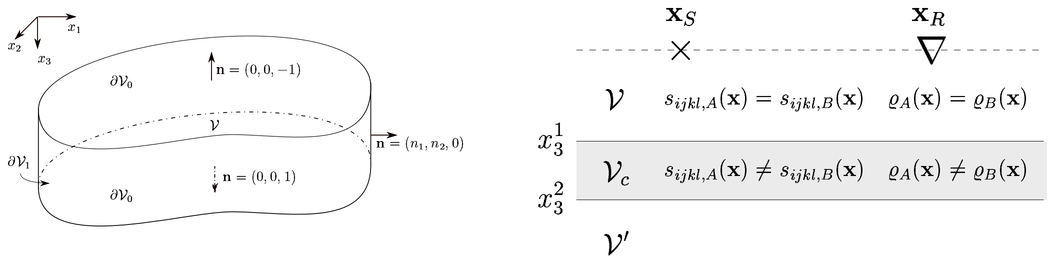
where $\hat{\Lambda}^\pm$ is the vertical slowness operator, and $\hat{\mathbf{R}}^\pm$ and $\hat{\mathbf{T}}^\pm$ are the reflection and transmission operators, respectively. We introduce two different states that will be distinguished by the subscripts A and B. For these two states we consider the interaction quantity $\partial_3\{\mathbf{P}_A^t\mathbf{N}\mathbf{P}_B\}$, with $\mathbf{N} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix}$ or, written alternatively, $\partial_3\{(\mathbf{P}_A^+)^t\mathbf{P}_B^- - (\mathbf{P}_A^-)^t\mathbf{P}_B^+\}$. The superscript ^{t} denotes transposition. Applying the product rule for differentiation, substituting the one-way wave equation (4) for states A and B, integrating the result over a cylindrical volume $\mathcal{V}$ with boundary $\partial\mathcal{V}_0 \cup \partial\mathcal{V}_1$ (see Figure 2a), applying the theorem of Gauss and using the symplectic relation $\hat{\mathbf{B}}^t\mathbf{N} = -\mathbf{N}\hat{\mathbf{B}}$, yields the following one-way reciprocity theorem

$$\int_{\mathbf{x}\in\partial\mathcal{V}_0} \mathbf{P}_A^t\mathbf{N}\mathbf{P}_B n_3 \mathrm{dA} = \int_{\mathbf{x}\in\mathcal{V}} \mathbf{P}_A^t\mathbf{N}\{\hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A\}\mathbf{P}_B \mathrm{dV} + \int_{\mathbf{x}\in\mathcal{V}} \{\mathbf{P}_A^t\mathbf{N}\mathbf{S}_B + \mathbf{S}_A^t\mathbf{N}\mathbf{P}_B\}\mathrm{dV}. \tag{7}$$

For elastodynamic one-way wave fields this reciprocity theorem has been strictly proven only for laterally invariant media; for acoustic one-way wave fields it has been proven for laterally varying media as well (Wapenaar and Grimbergen, 1996). In the following we will use it without further proof for elastodynamic one-way wave fields in laterally varying media.

Elastodynamic reciprocity theorems for time-lapse seismic

Since in a reciprocity theorem two states interact, it is optimally fitted to formulate the relation between two measurements in a time-lapse seismic experiment. State A is associated with the reference wave field at, say, $t = t_1$, while state B is associated with the monitoring wave field at, say, $t = t_2 > t_1$. It is noted that $t_2 - t_1$ is much longer than the seismic experiment time. In our analysis $\mathbb{R}^3$ is divided in three domains (Figure 2b): $\mathcal{V}_o$ is the domain where there are no differences between the material parameters in the two states, mostly associated with the domain above the reservoir (i.e., $x_3 \leq x_3^1$); the domain $\mathcal{V}_c$, for example associated with the reservoir ($x_3^1 < x_3 \leq x_3^2$), where there is a difference between the material parameters in the two states mostly due to the reservoir production history; and $\mathcal{V}'$ denotes the complement of $\mathcal{V} = \mathcal{V}_o \cup \mathcal{V}_c$ (i.e., $x_3 > x_3^2$); the material parameters in this domain may or may not be different.



(a) (b)

Figure 2: *(a) Modified configuration for the one-way reciprocity theorem. The combination of the two planar surfaces is denoted by $\partial\mathcal{V}_0$; the cylindrical surface is denoted by $\partial\mathcal{V}_1$. (b) Configuration for time-lapse seismic.*

Full wave equation

In order to simplify the analysis we only consider point sources of the volume force type. The source of state A is taken at $\mathbf{x} = \mathbf{x}_S$ in the x_m-direction, while the source of state B is taken at $\mathbf{x} = \mathbf{x}_R$ in the x_n-direction, according to

$$F_{i,A}(\mathbf{x},\omega) = f_A(\omega)\delta(\mathbf{x} - \mathbf{x}_S)\delta_{im}, \tag{8}$$

$$F_{i,B}(\mathbf{x},\omega) = f_B(\omega)\delta(\mathbf{x} - \mathbf{x}_R)\delta_{in}. \tag{9}$$

Application of reciprocity theorem (3) to domain $\mathcal{V} = \mathcal{V}_o \cup \mathcal{V}_c$ yields

$$\begin{aligned} & f_B(\omega)V_{n,A}(\mathbf{x}_R|\mathbf{x}_S) - f_A(\omega)V_{m,B}(\mathbf{x}_S|\mathbf{x}_R) \\ &= -j\omega \int_{\mathbf{x}\in\mathcal{V}} \{T_{ij,A}(s_{ijkl,B} - s_{ijkl,A})T_{kl,B} - V_{i,A}(\varrho_B - \varrho_A)V_{i,B}\}\mathrm{dV} \\ &+ \int_{x_3 = x_3^2} \{T_{i3,A}(\mathbf{x}|\mathbf{x}_S)V_{i,B}(\mathbf{x}|\mathbf{x}_R) - V_{i,A}(\mathbf{x}|\mathbf{x}_S)T_{i3,B}(\mathbf{x}|\mathbf{x}_R)\}\mathrm{dA}. \end{aligned} \tag{10}$$

The surface integral on the right-hand side of equation (10) takes into account a possible difference of the material parameters in $\mathcal{V}'$, below the reservoir; it vanishes when there is no difference between the two states in $\mathcal{V}'$.

One-way wave equation

In the one-way analysis we consider point-sources for downgoing waves in both states:

$$\mathbf{S}_A(\mathbf{x},\omega) = (\{\mathbf{s}_A^+(\omega)\}^t \;\; 0)^t\delta(\mathbf{x} - \mathbf{x}_S), \tag{11}$$

$$\mathbf{S}_B(\mathbf{x},\omega) = (\{\mathbf{s}_B^+(\omega)\}^t \;\; 0)^t\delta(\mathbf{x} - \mathbf{x}_R). \tag{12}$$

Application of reciprocity theorem (7) to domain $\mathcal{V} = \mathcal{V}_o \cup \mathcal{V}_c$ yields

$$\begin{aligned} & \{\mathbf{s}_B^+(\omega)\}^t\mathbf{P}_A^-(\mathbf{x}_R|\mathbf{x}_S) - \{\mathbf{s}_A^+(\omega)\}^t\mathbf{P}_B^-(\mathbf{x}_S|\mathbf{x}_R) \\ &= \int_{\mathbf{x}\in\mathcal{V}_c} \mathbf{P}_A^t(\mathbf{x}|\mathbf{x}_S)\mathbf{N}(\hat{\mathbf{B}}_B(\mathbf{x}) - \hat{\mathbf{B}}_A(\mathbf{x}))\mathbf{P}_B(\mathbf{x}|\mathbf{x}_R)\mathrm{dV} \\ &+ \int_{x_3 = x_3^2} [\{\mathbf{P}_A^-(\mathbf{x}|\mathbf{x}_S)\}^t\mathbf{P}_B^+(\mathbf{x}|\mathbf{x}_R) - \{\mathbf{P}_A^+(\mathbf{x}|\mathbf{x}_S)\}^t\mathbf{P}_B^-(\mathbf{x}|\mathbf{x}_R)]\mathrm{dA}. \end{aligned} \tag{13}$$

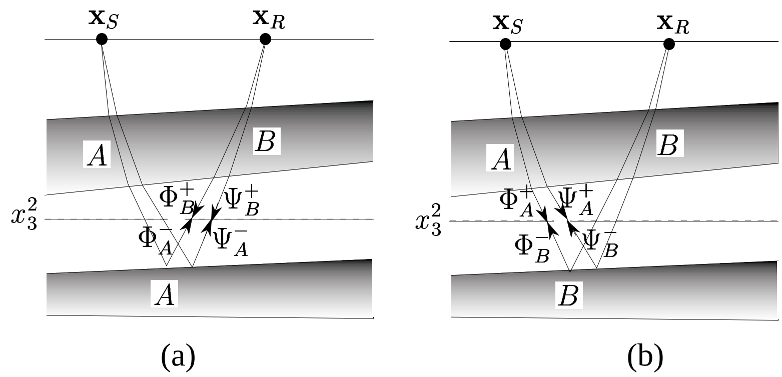


Figure 3: *Simplified representation of the two terms in the boundary integral in equation (13). Both terms accomplish a forward extrapolation of upgoing waves from x_3^2 to the surface.*

As in the previous case, the surface integral on the right-hand side of equation (13) vanishes when there is no difference between the two states in $\mathcal{V}'$ (i.e., below $x_3 = x_3^2$). Let us analyze this boundary integral, however, for the situation in which there are changes below $x_3 = x_3^2$. Figure 3 shows a configuration with two regions in which changes occur (the grey areas). Figure 3a shows some wavepaths in the first term of the boundary integral in equation (13), which can be written as

$$\int_{x_3=x_3^2} \{\mathbf{P}_A^-(\mathbf{x}|\mathbf{x}_S)\}^t \mathbf{P}_B^+(\mathbf{x}|\mathbf{x}_R)\mathrm{dA} = \int_{x_3=x_3^2} \{\Phi_A^-(\mathbf{x}|\mathbf{x}_S)\Phi_B^+(\mathbf{x}|\mathbf{x}_R) + \Psi_A^-(\mathbf{x}|\mathbf{x}_S)\Psi_B^+(\mathbf{x}|\mathbf{x}_R)\}\mathrm{dA}$$

(note that we ignore the contribution $\Upsilon_A^-\Upsilon_B^+$). If Φ_B^+ and Ψ_B^+ are interpreted as Green's functions for state B (multiplied by a source function), then it is understood that this integral performs an upward extrapolation of Φ_A^- and Ψ_A^- in state A from the depth level x_3^2 to $\mathbf{x}_R$ at the acquisition surface. This results in a virtual experiment in which the downgoing waves propagate from $\mathbf{x}_S$ through the medium in state A (before the changes took place), reflection at the second reservoir occurs in state A, and the upgoing waves propagate through the medium in state B (after the changes took place) to $\mathbf{x}_R$. The second term in the boundary integral in equation (13) (see Figure 3b) represents a similar virtual experiment with the same propagation paths, except with reflection taking place at the second reservoir in state B. Hence, since the traveltimes in these virtual experiments are the same, the difference of these terms (as expressed by the boundary integral in equation (13)) is proportional to the time-lapse changes of the elastodynamic reflectivity of the second reservoir.

Conclusions

We have formulated elastodynamic reciprocity theorems for time-lapse seismic, based on the full and the one-way wave equations. The latter form allows a straightforward physical interpretation of the various contributing terms. Its implementation requires wave field decomposition (Schalkwijk et al, 1998) and one-way wave field extrapolation of down- and upgoing P and S waves.

References

de Hoop, A.T., 1988, Time-domain reciprocity theorems for elastodynamic wave fields in fluids with relaxation: J. Ac. Soc. Am., **84**, 1877-1882

Fokkema, J.T., Wapenaar, C.P.A., and Dillen, M.W.P., 1999, Reciprocity theorems for time-lapse seismics based on the full and one-way wave equations: The 6th int. congr. of the Braz. Geoph. Soc., Rio de Janeiro

Schalkwijk, K.M., Wapenaar, C.P.A., and Verschuur, D.J., 1998, Decomposition of multicomponent ocean-bottom data in two steps: 68th annual SEG meeting, New Orleans, 1425-1428

Wapenaar, C.P.A., and Grimbergen, J.L.T., 1996, Reciprocity theorems for one-way wave fields: Geoph. J. Int. **127**, 169-177