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Waves in space-dependent and time-dependent materials: A systematic comparison

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ABSTRACT

Waves in space-dependent and in time-dependent materials obey similar wave equations, with interchanged time- and space-coordinates. However, since the causality conditions are the same in both types of material (i.e., without interchangement of time- and space-coordinates), the solutions are dissimilar.

We present a systematic treatment of wave propagation and scattering in 1D space-dependent and in 1D time-dependent materials. After formulating unified equations, we discuss Green's functions and simple wave field representations for both types of material. Next we discuss propagation invariants, i.e., quantities that are independent of the space coordinate in a space-dependent material (such as the net power-flux density) or of the time coordinate in a time-dependent material (such as the net field-momentum density). A discussion of general reciprocity theorems leads to the well-known source–receiver reciprocity relation for the Green's function of a space-dependent material and a new source–receiver reciprocity relation for the Green's function of a time-dependent material. A discussion of general wave field representations leads to the well-known expression for Green's function retrieval from the correlation of passive measurements in a space-dependent material and a new expression for Green's function retrieval in a time-dependent material.

After an introduction of a matrix–vector wave equation, we discuss propagator matrices for both types of material. Since the initial condition for a propagator matrix in a time-dependent material follows from the boundary condition for a propagator matrix in a space-dependent material by interchanging the time- and space-coordinates, the propagator matrices for both types of material are interrelated in the same way. This also applies to representations and reciprocity theorems involving propagator matrices, and to Marchenko-type focusing functions.

1. Introduction

A wave that encounters a temporal change of material parameters (a so-called time boundary) undergoes reflection and transmission [1], similar to a wave that is incident on a spatial change of material parameters (a space boundary). Although research on wave propagation and scattering in time-dependent materials has been around for several decades [2,3], recent advances in the construction of dynamic metamaterials have given this field of research a significant boost [4]. Whereas most applications concern electromagnetic waves [5–7], mechanical waves show a similar scattering behaviour when confronted with a temporal change of parameters [8–11]. In particular, Fink and coworkers [8,9] show that water waves propagate back to their point of origin when the

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Table 1
Quantities in unified equations (1)–(4).

	U	V	P	Q	α	β	a	b
1. TE waves	D_y	B_z	E_y	H_z	ε	μ	$-J_y^{\rm e}$	$-J_z^{\rm m}$
2. TM waves	B_y	$-D_z$	H_y	$-E_z$	μ	ε	$-J_y^{\rm m}$	$J_z^{\rm e}$
3. Acoustic waves	$-\Theta$	m_x	p	v_x	κ	ρ	q	f_x
4. SH waves	m_y	$-2e_{yx}$	v_y	$-\tau_{yx}$	ρ	μ^{-1}	f_y	$2h_{yx}$

restoring force responsible for wave propagation (gravity), and hence the propagation velocity, is temporarily changed by a vertical acceleration.

Several authors have discussed the analogy between the underlying equations for space-dependent and for time-dependent materials [1,12–15]. For example, the roles of the time- and space-coordinates in the 1D wave equation for a space-dependent material are interchanged in the 1D wave equation for a time-dependent material. Despite the simple relations between the wave equations, the relation between the solutions of these equations (i.e., the wave fields in space-dependent and in time-dependent materials) is less straightforward. The reason for this is that the causality conditions are the same in both types of material. Only when the initial conditions and boundary conditions would be interchanged (along with the interchangement of time- and space-coordinates in the wave equations), the solutions would obey a simple relation as well.

The aim of this paper is to discuss a number of fundamental aspects of wave propagation and scattering in space-dependent and in time-dependent materials and compare these in a systematic way. Our discussion partly overlaps with earlier reviews, such as the excellent paper by Caloz and Deck-Léger [15], but we also discuss new results. We use a unified notation for different wave phenomena (electromagnetic, acoustic, elastodynamic), so that all relations discussed in this paper hold simultaneously for these phenomena. For simplicity, we restrict ourselves to 1D waves only. We discuss Green's functions, propagation invariants, reciprocity theorems, wave field representations and expressions for Green's function retrieval. In most of these cases, the derived solutions for space-dependent and time-dependent materials are not exchangeable as a result of non-exchangeable causality conditions. We also discuss propagator matrices for space-dependent and time-dependent materials and show that they are completely exchangeable as a result of interchangeable boundary and initial conditions. Finally, we discuss Marchenko-type focusing functions for both types of material and show that they are also exchangeable.

2. Unified basic equations and constitutive relations for 1D wave fields

Throughout this paper, we consider 1D wave fields as a function of space (denoted by x) and time (denoted by t). We take x increasing towards the right. Using analogies between electromagnetic, acoustic and elastodynamic waves [16–21], the basic equations in a unified notation are

$$\partial_t U + \partial_x Q = a, \tag{1}$$

$$\partial_t V + \partial_x P = b, \tag{2}$$

where ∂_x and ∂_t denote partial derivatives with respect to space and time, respectively, $U(x,t)$, $V(x,t)$, $P(x,t)$ and $Q(x,t)$ are space- and time-dependent wave-field quantities and $a(x,t)$ and $b(x,t)$ are space- and time-dependent source quantities, see Table 1. The wave-field quantities are mutually related via the following constitutive equations

$$U = \alpha P, \tag{3}$$

$$V = \beta Q, \tag{4}$$

where $\alpha(x,t)$ and $\beta(x,t)$ are the parameters of space- and time-dependent materials, see Table 1. Rows 1 and 2 contain the quantities for electromagnetic wave propagation, with TE standing for transverse electric and TM for transverse magnetic. The quantities are electric flux densities $D_y(x,t)$ and $D_z(x,t)$, magnetic flux densities $B_y(x,t)$ and $B_z(x,t)$, electric field strengths $E_y(x,t)$ and $E_z(x,t)$, magnetic field strengths $H_y(x,t)$ and $H_z(x,t)$, permittivity $\varepsilon(x,t)$, permeability $\mu(x,t)$, external electric current densities $J_y^{\rm e}(x,t)$ and $J_z^{\rm e}(x,t)$ and external magnetic current densities $J_y^{\rm m}(x,t)$ and $J_z^{\rm m}(x,t)$. The quantities in row 3, associated to acoustic wave propagation in a fluid material, are dilatation $\Theta(x,t)$, longitudinal mechanical momentum density $m_x(x,t)$, acoustic pressure $p(x,t)$, longitudinal particle velocity $v_x(x,t)$, compressibility $\kappa(x,t)$, mass density $\rho(x,t)$, volume-injection rate density $q(x,t)$ and external longitudinal force density $f_x(x,t)$. For horizontally polarised shear (SH) waves in a solid material, we have in row 4 transverse mechanical momentum density $m_y(x,t)$, shear strain $e_{yx}(x,t)$, transverse particle velocity $v_y(x,t)$, shear stress $\tau_{yx}(x,t)$, mass density $\rho(x,t)$, shear modulus $\mu(x,t)$, external transverse force density $f_y(x,t)$ and external shear deformation rate density $h_{yx}(x,t)$.

In the following we consider either space-dependent parameters ($\alpha(x)$ and $\beta(x)$) or time-dependent parameters ($\alpha(t)$ and $\beta(t)$). For discussions on wave propagation and scattering in materials that are both space- and time-dependent, we refer to [6,15,22]; for non-reciprocal wave propagation due to "travelling wave modulation", see Refs. [23–30].

3. Wave equations and Green's functions

In this and subsequent sections, the first subsection reviews a specific subject for a space-dependent material. This serves as an introduction to the second subsection, which discusses the same subject for a time-dependent material, including the analogies and differences.

3.1. Space-dependent material

We consider a space-dependent material that is constant over time, with parameters $\alpha(x)$ and $\beta(x)$. Substituting the constitutive Eqs. (3) and (4) into the basic Eqs. (1) and (2), using the fact that $\alpha(x)$ and $\beta(x)$ are independent of time, gives

$$\alpha\partial_t P + \partial_x Q = a, \tag{5}$$

$$\beta\partial_t Q + \partial_x P = b. \tag{6}$$

For a space-dependent material with piecewise continuous parameters, these equations are supplemented with boundary conditions at all points where $\alpha(x)$ and $\beta(x)$ undergo a finite jump. The boundary conditions are that $P(x,t)$ and $Q(x,t)$ are continuous at those points (see Appendix A.1 for a review of reflection and transmission coefficients at a space boundary).

We obtain a second order wave equation for the field $P(x,t)$ by eliminating $Q(x,t)$ from Eqs. (5) and (6), according to

$$\frac{1}{\beta c^2}\partial_t^2 P - \partial_x\Big(\frac{1}{\beta}\partial_x P\Big) = \partial_t a - \partial_x\Big(\frac{b}{\beta}\Big), \tag{7}$$

with propagation velocity $c(x)$ given by

$$c = \frac{1}{\sqrt{\alpha\beta}}, \tag{8}$$

with space-dependent parameters $\alpha(x)$ and $\beta(x)$. We also define

$$\eta = \sqrt{\frac{\beta}{\alpha}} = \beta c = \frac{1}{\alpha c}, \tag{9}$$

where η stands for impedance in the case of TE and acoustic waves (rows 1 and 3 of Table 1) or admittance in the case of TM and SH waves (rows 2 and 4 of Table 1).

We define the Green's function $\mathcal{G}_x(x,x_0,t)$ as the response to an impulsive point source $\delta(x-x_0)\delta(t)$, hence

$$\frac{1}{\beta c^2}\partial_t^2 \mathcal{G}_x - \partial_x\Big(\frac{1}{\beta}\partial_x \mathcal{G}_x\Big) = \delta(x-x_0)\delta(t), \tag{10}$$

with causality condition

$$\mathcal{G}_x(x,x_0,t) = 0 \quad \text{for} \quad t<0. \tag{11}$$

This condition implies that $\mathcal{G}_x(x,x_0,t)$ is outward propagating for $|x|\to\infty$. The subscript x in $\mathcal{G}_x$ denotes that this is the Green's function of a space-dependent material.

A simple representation for $P(x,t)$ is obtained when P and $\mathcal{G}_x$ are defined in the same material and both are outward propagating for $|x|\to\infty$. Whereas $P(x,t)$ is the response to an arbitrary source distribution $\partial_t a(x,t)$ (Eq. (7), assuming $b=0$), $\mathcal{G}_x(x,x_0,t)$ is the response to an impulsive point source at an arbitrary location x_0 at $t=0$ (Eq. (10)). Because Eqs. (7) and (10) are linear, a representation for $P(x,t)$ follows by applying Huygens' superposition principle. Assuming $\partial_t a(x,t)$ is causal, i.e., $\partial_t a(x,t)=0$ for $t<0$, this gives [31,32]

$$P(x,t) = \int_{-\infty}^{\infty} \mathrm{d}x' \int_0^t \mathcal{G}_x(x,x',t-t')\partial_{t'} a(x',t')\mathrm{d}t'. \tag{12}$$

This representation is a special case of the more general representation for a space-dependent material, derived in Section 6.1.

We discuss a numerical example of an acoustic Green's function for a piecewise homogeneous material, consisting of five homogeneous slabs, each with a thickness of 40 mm. The propagation velocities are 1.0, 1.0, 2.0, 1.0 and 2.2 km/s, respectively. The half-spaces to the left and the right of the space-dependent material are homogeneous, with the same velocities as the first and last slab, respectively. The parameter β is constant throughout. The source is located between the first and the second slab, at $x_0 = 40$ mm. We use a recursive "layer-code" method [33] to model the response to this source. Fig. 1 shows an x,t-diagram of $\mathcal{G}_x(x,x_0,t)$, convolved in time with a temporal wavelet with a central frequency $\omega_0/2\pi = 300$ kHz. The causality condition of Eq. (11) implies that the Green's function is zero above the green line at $t=0$. The red arrows indicate the rightward propagating primary wave and the blue arrows the leftward propagating primary reflections. Multiply scattered waves are also clearly visible. Note that the field is outward propagating for $x=0$ and $x=200$ mm (and hence for $|x|\to\infty$, since the left and right half-spaces are homogeneous).

3.2. Time-dependent material

We consider a time-dependent homogeneous material with parameters $\alpha(t)$ and $\beta(t)$. Substituting the constitutive Eqs. (3) and (4) into the basic Eqs. (1) and (2), using the fact that $\alpha(t)$ and $\beta(t)$ are independent of space, gives

$$\partial_t U + \frac{1}{\beta}\partial_x V = a, \tag{13}$$

$$\partial_t V + \frac{1}{\alpha}\partial_x U = b. \tag{14}$$

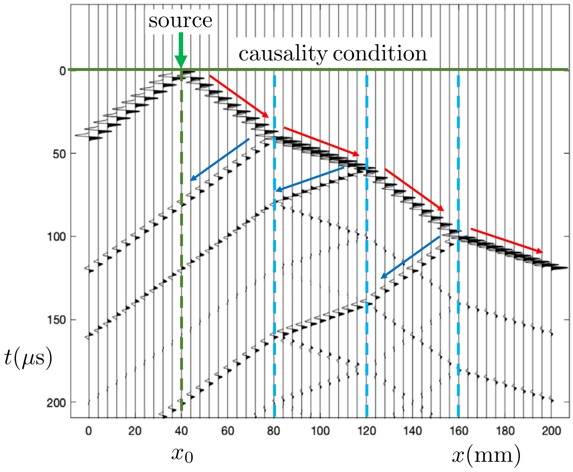


Fig. 1. Green's function $\mathcal{G}_x(x, x_0, t)$ (convolved with a temporal wavelet) for a piecewise homogeneous space-dependent material.

For a time-dependent material with piecewise continuous parameters, these equations are supplemented with boundary conditions at all time instants where $\alpha(t)$ and $\beta(t)$ undergo a finite jump. The boundary conditions are that $U(x, t)$ and $V(x, t)$ are continuous at those time instants (see Appendix A.2 for a review of reflection and transmission coefficients at a time boundary).

We obtain a second order wave equation for the field $U(x, t)$ by eliminating $V(x, t)$ from Eqs. (13) and (14), according to

$$\partial_t(\beta\partial_t U) - \beta c^2 \partial_x^2 U = \partial_t(\beta a) - \partial_x b, \tag{15}$$

with propagation velocity $c(t)$ again given by Eq. (8), this time with time-dependent parameters $\alpha(t)$ and $\beta(t)$.

Note that Eqs. (5), (6) and (7) can be transformed into Eqs. (14), (13) and (15) and vice-versa, by the following mapping

$$\{P, Q, x, t, \alpha, \beta, a, b, c\} \leftrightarrow \{U, V, t, x, \alpha^{-1}, \beta^{-1}, b, a, c^{-1}\}. \tag{16}$$

We define the Green's function $\mathcal{G}_t(x, t, t_0)$ as the response to an impulsive point source $\delta(x)\delta(t - t_0)$, hence

$$\partial_t(\beta\partial_t \mathcal{G}_t) - \beta c^2 \partial_x^2 \mathcal{G}_t = \delta(x)\delta(t - t_0), \tag{17}$$

with causality condition

$$\mathcal{G}_t(x, t, t_0) = 0 \quad \text{for} \quad t < t_0. \tag{18}$$

This condition implies that $\mathcal{G}_t(x, t, t_0)$ is outward propagating for $|x| \to \infty$. The subscript t in $\mathcal{G}_t$ denotes that this is the Green's function of a time-dependent material.

A simple representation for $U(x, t)$ is obtained when U and $\mathcal{G}_t$ are defined in the same material and both are outward propagating for $|x| \to \infty$. Whereas $U(x, t)$ is the response to a source distribution $-\partial_x b(x, t)$ (Eq. (15), assuming $a = 0$), $\mathcal{G}_t(x, t, t_0)$ is the response to an impulsive point source at $x = 0$ at an arbitrary time t_0 (Eq. (17)). Because Eqs. (15) and (17) are linear, a representation for $U(x, t)$ follows by applying Huygens' superposition principle. Assuming $\partial_x b(x, t)$ is causal, i.e., $\partial_x b(x, t) = 0$ for $t < 0$, this gives

$$U(x, t) = -\int_{-\infty}^{\infty} \mathrm{d}x' \int_0^t \mathcal{G}_t(x - x', t, t')\partial_{x'} b(x', t')\mathrm{d}t'. \tag{19}$$

Using Eq. (3), a representation for $P(x, t)$ follows from $P(x, t) = \frac{1}{\alpha(t)} U(x, t)$, with $U(x, t)$ given by Eq. (19). The representation of Eq. (19) is a special case of the more general representation for a time-dependent material, derived in Section 6.2.

We discuss a numerical example of an acoustic Green's function for a piecewise constant material, consisting of five time-independent slabs. Following the mapping of Eq. (16), we "construct" this material from the material used for the numerical example in Section 3.1, with time and space interchanged and with the reciprocal propagation velocities. For convenience, we define 1 km as the unit of distance and 1 km/s as the unit of velocity. With this definition, the reciprocal propagation velocities are 1.0, 1.0, 0.5, 1.0 and 0.455 km/s, respectively. The half-times before and after the time-dependent material are constant, with the same velocities as the first and last slab, respectively. The parameter β is again constant throughout. Note that 1 mm, which is actually 1 μkm, maps to 1 μs and vice-versa. Hence, the slab thickness of 40 mm is mapped to a slab duration of 40 μs. The source is located between the first and second slab, at $t_0 = 40\,\mu\text{s}$. Fig. 2a shows an x, t-diagram of $\mathcal{G}_t(x, t, t_0)$, convolved in space with a spatial wavelet with a central wavenumber $k_0/2\pi = 300 * 10^3\ \text{km}^{-1}$. The causality condition of Eq. (18) implies that the Green's function is zero left of the green line at $t = t_0$. The red arrows indicate the rightward propagating primary wave (i.e., in the $+x$ direction) and the blue arrows the leftward propagating primary reflections (in the $-x$ direction). Multiply scattered waves are also clearly visible.

Since the causality conditions (Eqs. (11) and (18)) do not follow the mapping of Eq. (16), the x, t-diagrams of the Green's functions for space-dependent and time-dependent materials (Figs. 1 and 2a) are very different. For the specially designed case

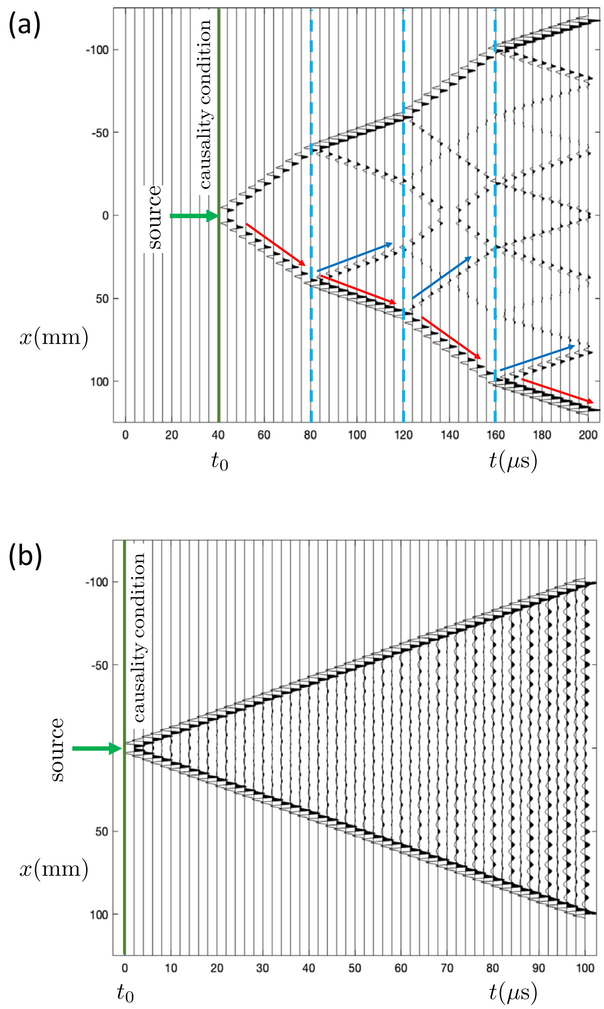


Fig. 2. Green's function $\mathcal{G}_t(x,t,t_0)$ (convolved with a spatial wavelet) for (a) a piecewise constant and (b) a sinusoidally modulated time-dependent material.

considered here (with reciprocal velocities), only the rightward propagating primary waves (indicated by the red arrows) exhibit interchangeable kinematical behaviour between the two cases (but they have different amplitudes). All other events are different in these figures. Whereas the multiply scattered waves in $\mathcal{G}_x(x,x_0,t)$ in Fig. 1 consist of ongoing reverberations between space boundaries, the multiply scattered waves in $\mathcal{G}_t(x,t,t_0)$ in Fig. 2a are the result of "forward-in-time" reflections and transmissions at time boundaries (see also Fig. A.5 in the Appendix); their total number is finite. In Section 9 we discuss propagator matrices for space-dependent and time-dependent materials and show that these follow the mapping of Eq. (16) for all events. Moreover, in Section 9.2 we show how $\mathcal{G}_t(x,t,t_0)$ is related to one of the elements of the propagator matrix for a time-dependent material.

Finally, Fig. 2b is an example of an acoustic Green's function $\mathcal{G}_t(x,t,t_0)$ (convolved with the same spatial wavelet as in Fig. 2a) of a sinusoidally modulated time-dependent material, with propagation velocity $c(t) = 1 + \frac{1}{10}\sin(\pi t/2)$ km/s (with t in µs, ranging from $t_0 = 0\,\mu s$ to $t_N = 100\,\mu s$) and constant β. For the modelling we divided the velocity profile into 4000 constant velocity slabs with a duration of 0.025 µs each. Note the complex scattering behaviour and the increasing amplitudes of the scattered events with time. To compensate for the increasing amplitudes, Ref. [12] introduces a dissipative time-dependent material.

4. Propagation invariants

4.1. Space-dependent material

We review propagation invariants for a space-dependent material with parameters $\alpha(x)$ and $\beta(x)$. Given a space- and time-dependent function $P(x,t)$, we define its temporal Fourier transform as

$$\hat{P}(x,\omega) = \int_{-\infty}^{\infty} P(x,t)\exp\{i\omega t\}\mathrm{d}t, \tag{20}$$

with ω denoting the angular frequency and i the imaginary unit. With this definition, derivatives with respect to time transform to multiplications with $-i\omega$. Hence, Eqs. (5) and (6) transform to

$$-i\omega\alpha\hat{P} + \partial_x\hat{Q} = \hat{a}, \tag{21}$$

$$-i\omega\beta\hat{Q} + \partial_x\hat{P} = \hat{b}. \tag{22}$$

In the following we consider two independent states, indicated by subscripts A and B, obeying Eqs. (21) and (22). In the most general case, sources, material parameters and wave fields may be different in the two states. We derive relations between these states. First we consider the quantity $\partial_x\{\hat{P}_A\hat{Q}_B - \hat{Q}_A\hat{P}_B\}$. Applying the product rule for differentiation, using Eqs. (21) and (22) for states A and B to get rid of the derivatives, we obtain

$$\begin{aligned}\partial_x\{\hat{P}_A\hat{Q}_B - \hat{Q}_A\hat{P}_B\} =\ & i\omega(\alpha_B-\alpha_A)\hat{P}_A\hat{P}_B - i\omega(\beta_B-\beta_A)\hat{Q}_A\hat{Q}_B\\ & -\hat{a}_A\hat{P}_B + \hat{b}_A\hat{Q}_B + \hat{P}_A\hat{a}_B - \hat{Q}_A\hat{b}_B.\end{aligned} \tag{23}$$

This is the local reciprocity theorem of the time-convolution type [17,34], in which products like $\hat{P}_A\hat{Q}_B$ correspond to convolutions along the time coordinate in the x,t-domain. Next, we consider the quantity $\partial_x\{\hat{P}_A^*\hat{Q}_B + \hat{Q}_A^*\hat{P}_B\}$ (where the asterisk denotes complex conjugation) and apply the same operations, yielding

$$\begin{aligned}\partial_x\{\hat{P}_A^*\hat{Q}_B + \hat{Q}_A^*\hat{P}_B\} =\ & i\omega(\alpha_B-\alpha_A)\hat{P}_A^*\hat{P}_B + i\omega(\beta_B-\beta_A)\hat{Q}_A^*\hat{Q}_B\\ & +\hat{a}_A^*\hat{P}_B + \hat{b}_A^*\hat{Q}_B + \hat{P}_A^*\hat{a}_B + \hat{Q}_A^*\hat{b}_B.\end{aligned} \tag{24}$$

This is the local reciprocity theorem of the time-correlation type [17,35], in which products like $\hat{P}_A^*\hat{Q}_B$ correspond to correlations along the time coordinate in the x,t-domain. In Section 5.1 we will use Eqs. (23) and (24) as the basis for deriving global reciprocity theorems of the time-convolution and time-correlation type. Here we use these equations to derive propagation invariants for a space-dependent material. To this end we take identical material parameters in states A and B and we assume that there are no sources. With this, Eqs. (23) and (24) simplify to

$$\partial_x\{\hat{P}_A\hat{Q}_B - \hat{Q}_A\hat{P}_B\} = 0, \tag{25}$$

$$\partial_x\{\hat{P}_A^*\hat{Q}_B + \hat{Q}_A^*\hat{P}_B\} = 0, \tag{26}$$

hence, $\hat{P}_A\hat{Q}_B - \hat{Q}_A\hat{P}_B$ and $\hat{P}_A^*\hat{Q}_B + \hat{Q}_A^*\hat{P}_B$ are space-propagation invariants, i.e., they are independent of the space coordinate x. This holds for continuously varying material parameters $\alpha(x)$ and $\beta(x)$. For a material with piecewise continuous parameters, the boundary conditions state that $\hat{P}$ and $\hat{Q}$ are continuous at all points where $\alpha(x)$ and $\beta(x)$ are discontinuous. This implies that the space-propagation invariants also hold for a space-dependent material with piecewise continuous parameters. Propagation invariants find applications in the analysis of symmetry properties of reflection and transmission responses and have been used for the design of efficient numerical modelling schemes [36–39].

For the special case that the wave fields in states A and B are identical, we may drop the subscripts A and B. The first space-propagation invariant then vanishes and is no longer useful. The second space-propagation invariant simplifies to $2\Re\{\hat{P}^*\hat{Q}\}$, where $\Re$ denotes the real part. We define $\hat{\jmath}(x,\omega) = \frac{1}{2}\Re\{\hat{P}^*\hat{Q}\}$ as the net power-flux density in the x-direction in the x,ω-domain. Hence, the net power-flux density is conserved (i.e., it is independent of x) in a space-dependent material with piecewise continuous parameters.

4.2. Time-dependent material

We derive propagation invariants for a time-dependent material with parameters $\alpha(t)$ and $\beta(t)$. Given a space- and time-dependent function $U(x,t)$, we define its spatial Fourier transform as

$$\check{U}(k,t) = \int_{-\infty}^{\infty} U(x,t)\exp\{-ikx\}\mathrm{d}x, \tag{27}$$

with k denoting the wavenumber. Following common conventions, we use opposite signs in the exponentials in the temporal and spatial Fourier transforms (Eqs. (20) and (27)). With this definition, derivatives with respect to space transform to multiplications with ik. Hence, Eqs. (13) and (14) transform to

$$\partial_t\check{U} + \frac{ik}{\beta}\check{V} = \check{a}, \tag{28}$$

$$\partial_t\check{V} + \frac{ik}{\alpha}\check{U} = \check{b}. \tag{29}$$

Note that Eqs. (21) and (22) can be transformed into Eqs. (29) and (28) and vice-versa, by the following modified mapping

$$\{\hat{P},\hat{Q},x,\omega,\alpha,\beta,\hat{a},\hat{b},c\} \leftrightarrow \{\check{U},\check{V},t,-k,\alpha^{-1},\beta^{-1},\check{b},\check{a},c^{-1}\}. \tag{30}$$

In the following we consider two independent states, indicated by subscripts A and B, obeying Eqs. (28) and (29). In the most general case, sources, material parameters and wave fields may be different in the two states. Applying the mapping of Eq. (30) to Eqs. (23) and (24) yields the local reciprocity theorems of the space-convolution and space-correlation type, in which products like $\check{U}_A\check{V}_B$ and $\check{U}_A^*\check{V}_B$ correspond to convolutions and correlations, respectively, along the space coordinate in the x,t-domain. In Section 5.2 we derive global reciprocity theorems of the space-convolution and space-correlation type. Here we derive propagation invariants for a time-dependent material. To this end we take identical material parameters in states A and B and we assume that there are no sources. Applying the mapping of Eq. (30) to Eqs. (25) and (26) yields

$$\partial_t\{\check{U}_A\check{V}_B - \check{V}_A\check{U}_B\} = 0, \tag{31}$$

$$\partial_t\{\check{U}_A^*\check{V}_B + \check{V}_A^*\check{U}_B\} = 0, \tag{32}$$

hence, $\check{U}_A\check{V}_B - \check{V}_A\check{U}_B$ and $\check{U}_A^*\check{V}_B + \check{V}_A^*\check{U}_B$ are time-propagation invariants, i.e., they are independent of the time coordinate t. This holds for continuously varying material parameters $\alpha(t)$ and $\beta(t)$. For a material with piecewise continuous parameters, the boundary conditions state that $\check{U}$ and $\check{V}$ are continuous at all time instants where $\alpha(t)$ and $\beta(t)$ are discontinuous. This implies that the time-propagation invariants also hold for a time-dependent material with piecewise continuous parameters.

For the special case that the wave fields in states A and B are identical, we may drop the subscripts A and B. The first time-propagation invariant then vanishes and is no longer useful. The second time-propagation invariant simplifies to $2\Re\{\check{U}^*\check{V}\}$. We define $\check{M}(k,t) = \frac{1}{2}\Re\{\check{U}^*\check{V}\}$ as the net field-momentum density [21,40] in the x-direction in the k,t-domain (to be distinguished from the mechanical momentum densities m_x and m_y in Table 1). Hence, the net field-momentum density is conserved (i.e., it is independent of t) in a time-dependent material with piecewise continuous parameters.

Using Eqs. (3), (4) and (8), we obtain for the net power-flux density $\check{\jmath}(k,t)$ in the x-direction in the k,t-domain, defined as $\frac{1}{2}\Re\{\check{P}^*\check{Q}\}$,

$$\check{\jmath}(k,t) = c^2(t)\check{M}(k,t). \tag{33}$$

Hence, whereas the net field-momentum density $\check{M}(k,t)$ is conserved, the net power-flux density $\check{\jmath}(k,t)$ is not conserved (i.e., it is dependent on t) in a time-dependent material. This is explained as the result of energy being added to or extracted from the wave field by the external source that modulates the material parameters [2,13,15].

5. Reciprocity theorems

5.1. Space-dependent material

We review general reciprocity theorems for a space-dependent material with piecewise continuous parameters $\alpha(x)$ and $\beta(x)$. Integrating both sides of Eqs. (23) and (24) from x_b to x_e (with subscripts b and e standing for "begin" and "end"), taking into account that $\hat{P}$ and $\hat{Q}$ are continuous at points where $\alpha(x)$ and $\beta(x)$ are discontinuous, yields

$$\{\hat{P}_A\hat{Q}_B - \hat{Q}_A\hat{P}_B\}\Big|_{x_b}^{x_e} = \int_{x_b}^{x_e}\Big(i\omega(\alpha_B-\alpha_A)\hat{P}_A\hat{P}_B - i\omega(\beta_B-\beta_A)\hat{Q}_A\hat{Q}_B - \hat{a}_A\hat{P}_B + \hat{b}_A\hat{Q}_B + \hat{P}_A\hat{a}_B - \hat{Q}_A\hat{b}_B\Big)\mathrm{d}x, \tag{34}$$

$$\{\hat{P}_A^*\hat{Q}_B + \hat{Q}_A^*\hat{P}_B\}\Big|_{x_b}^{x_e} = \int_{x_b}^{x_e}\Big(i\omega(\alpha_B-\alpha_A)\hat{P}_A^*\hat{P}_B + i\omega(\beta_B-\beta_A)\hat{Q}_A^*\hat{Q}_B + \hat{a}_A^*\hat{P}_B + \hat{b}_A^*\hat{Q}_B + \hat{P}_A^*\hat{a}_B + \hat{Q}_A^*\hat{b}_B\Big)\mathrm{d}x. \tag{35}$$

These are the global reciprocity theorems of the time-convolution and time-correlation type, respectively, for a space-dependent material [17,34,35,41,42]. We use Eq. (34) in Section 6.1 to derive a general wave field representation and in Section 7.1 we use Eq. (35) to derive an expression for Green's function retrieval, both for space-dependent materials. Here we discuss two special cases of Eqs. (34) and (35).

First we derive an expression for source–receiver reciprocity of the Green's function of a space-dependent material from Eq. (34). To this end we take identical material parameters in both states, i.e., $\alpha_A = \alpha_B = \alpha$ and $\beta_A = \beta_B = \beta$ and we assume that the material is homogeneous for $x \le x_b$ and for $x \ge x_e$. For state A we take a Green's state with a unit source at x_A between x_b and x_e, hence, we substitute $-i\omega\hat{a}_A(x,\omega) = \delta(x-x_A)$, $\hat{b}_A(x,\omega) = 0$, $\hat{P}_A(x,\omega) = \hat{\mathcal{G}}_x(x,x_A,\omega)$ and, using Eq. (22), $\hat{Q}_A(x,\omega) = \frac{1}{i\omega\beta(x)}\partial_x\hat{\mathcal{G}}_x(x,x_A,\omega)$. For state B we take a Green's state with a unit source at x_B between x_b and x_e, and we substitute similar expressions. At x_b and x_e the field is leftward and rightward propagating, respectively (see for example Fig. 1, with $x_b = 0$ and $x_e = 200$ mm), i.e., proportional to $\exp(-i\omega x/c(x_b))$ and $\exp(i\omega x/c(x_e))$, respectively. Hence

$$\hat{Q}_A(x_b,\omega) = \frac{1}{i\omega\beta(x_b)}\partial_x\hat{\mathcal{G}}_x(x,x_A,\omega)|_{x=x_b} = -\frac{1}{\eta(x_b)}\hat{\mathcal{G}}_x(x_b,x_A,\omega), \tag{36}$$

$$\hat{Q}_A(x_e,\omega) = \frac{1}{i\omega\beta(x_e)}\partial_x\hat{\mathcal{G}}_x(x,x_A,\omega)|_{x=x_e} = +\frac{1}{\eta(x_e)}\hat{\mathcal{G}}_x(x_e,x_A,\omega), \tag{37}$$

with η defined in Eq. (9), and similar expressions for state B. With this, the left-hand side of Eq. (34) vanishes. From the remaining terms on the right-hand side we obtain

$$\hat{\mathcal{G}}_x(x_B,x_A,\omega) = \hat{\mathcal{G}}_x(x_A,x_B,\omega), \tag{38}$$

or, in the space–time domain,

$$\mathcal{G}_x(x_B,x_A,t) = \mathcal{G}_x(x_A,x_B,t), \tag{39}$$

which formulates the classical source–receiver reciprocity relation for a space-dependent material [17,31,43]. These expressions remain valid for arbitrary x_A and x_B when taking $x_b \to -\infty$ and $x_e \to \infty$.

Second, we derive a power balance for a space-dependent material from Eq. (35). Taking identical states A and B (hence, identical sources, material parameters and wave fields), we drop the subscripts A and B. Eq. (35) thus yields [17,34]

$$\hat{\jmath}(x,\omega)\Big|_{x_b}^{x_e} = \frac{1}{2}\Re\int_{x_b}^{x_e}\left(\hat{a}^*\hat{P} + \hat{b}^*\hat{Q}\right)\mathrm{d}x. \tag{40}$$

This equation states that the power generated by sources in the region between x_b and x_e (the right-hand side) is equal to the power leaving this region (the left-hand side). Hence, this equation formulates the power balance for a space-dependent material.

5.2. Time-dependent material

We derive general reciprocity theorems for a time-dependent material with piecewise continuous parameters $\alpha(t)$ and $\beta(t)$. Applying the mapping of Eq. (30) to Eqs. (34) and (35) yields

$$\{\check{U}_A\check{V}_B - \check{V}_A\check{U}_B\}\Big|_{t_b}^{t_e} = \int_{t_b}^{t_e}\Big(-ik(\alpha_B^{-1}-\alpha_A^{-1})\check{U}_A\check{U}_B + ik(\beta_B^{-1}-\beta_A^{-1})\check{V}_A\check{V}_B$$
$$-\check{b}_A\check{U}_B + \check{a}_A\check{V}_B + \check{U}_A\check{b}_B - \check{V}_A\check{a}_B\Big)\mathrm{d}t, \tag{41}$$

$$\{\check{U}_A^*\check{V}_B + \check{V}_A^*\check{U}_B\}\Big|_{t_b}^{t_e} = \int_{t_b}^{t_e}\Big(-ik(\alpha_B^{-1}-\alpha_A^{-1})\check{U}_A^*\check{U}_B - ik(\beta_B^{-1}-\beta_A^{-1})\check{V}_A^*\check{V}_B$$
$$+\check{b}_A^*\check{U}_B + \check{a}_A^*\check{V}_B + \check{U}_A^*\check{b}_B + \check{V}_A^*\check{a}_B\Big)\mathrm{d}t. \tag{42}$$

These are the global reciprocity theorems of the space-convolution and space-correlation type, respectively, for a time-dependent material. We use Eq. (41) in Section 6.2 to derive a general wave field representation and in Section 7.2 we use Eq. (42) to derive an expression for Green's function retrieval, both for time-dependent materials. Here we discuss two special cases of Eqs. (41) and (42).

First we derive an expression for source–receiver reciprocity of the Green's function of a time-dependent material from Eq. (41). Since the causality conditions for the Green's functions do not obey the mapping of Eq. (16), the derivation of source–receiver reciprocity is different from that in Section 5.1. In particular, for the Green's function of a time-dependent material there is not an equivalent of leftward and rightward propagating waves at t_b and t_e, respectively (see for example Fig. 2a, with $t_b = 0$ and $t_e = 200\,\mu$s). We take again $\alpha_A = \alpha_B = \alpha$ and $\beta_A = \beta_B = \beta$. For state A we take a Green's state with an impulse at t_A between t_b and t_e, according to $-ik\check{b}_A(k,t) = \delta(t-t_A)$ and $\check{a}_A(k,t) = 0$. However, we define $\check{U}_A(k,t) = \check{\mathcal{G}}_t^a(k,t,t_A)$, where $\check{\mathcal{G}}_t^a(k,t,t_A)$ is the acausal Green's function, i.e., $\check{\mathcal{G}}_t^a(k,t,t_A) = 0$ for $t > t_A$ (hence, the impulse at t_A is actually a sink). Using Eq. (28) we have $\check{V}_A(k,t) = -\frac{\beta(t)}{ik}\partial_t\check{\mathcal{G}}_t^a(k,t,t_A)$. For state B we take a Green's state with an impulsive source at t_B between t_b and t_e, according to $-ik\check{b}_B(k,t) = \delta(t-t_B)$ and $\check{a}_B(k,t) = 0$. We define $\check{U}_B(k,t) = \check{\mathcal{G}}_t(k,t,t_B)$, where $\check{\mathcal{G}}_t(k,t,t_B)$ is the causal Green's function, i.e., $\check{\mathcal{G}}_t(k,t,t_B) = 0$ for $t < t_B$. Moreover, $\check{V}_B(k,t) = -\frac{\beta(t)}{ik}\partial_t\check{\mathcal{G}}_t(k,t,t_B)$. With these definitions, the acausal Green's function is zero for $t = t_e$ and the causal Green's function is zero for $t = t_b$ (the latter is seen for example in Fig. 2a, with $t_b = 0$). With this, the left-hand side of Eq. (41) vanishes. From the remaining terms on the right-hand side we obtain

$$\check{\mathcal{G}}_t^a(k,t_B,t_A) = \check{\mathcal{G}}_t(k,t_A,t_B), \tag{43}$$

or, in the space–time domain,

$$\mathcal{G}_t^a(x,t_B,t_A) = \mathcal{G}_t(x,t_A,t_B), \tag{44}$$

which formulates source–receiver reciprocity for a time-dependent material. These expressions remain valid for arbitrary t_A and t_B when taking $t_b \to -\infty$ and $t_e \to \infty$. Note that for $t_A < t_B$ these expressions reduce to the trivial relation $0 = 0$.

Second, we derive a field-momentum balance for a time-dependent material from Eq. (42). Taking identical states A and B, we drop the subscripts A and B. Eq. (42) thus yields

$$\check{M}(k,t)\Big|_{t_b}^{t_e} = \frac{1}{2}\Re\int_{t_b}^{t_e}\left(\check{b}^*\check{U} + \check{a}^*\check{V}\right)\mathrm{d}t. \tag{45}$$

This equation states that the field momentum generated by sources in the interval between t_b and t_e (the right-hand side) is equal to the field momentum leaving this interval (the left-hand side). Hence, this equation formulates the field-momentum balance for a time-dependent material.

6. Wave field representations

6.1. Space-dependent material

We review a general wave field representation for a space-dependent material with piecewise continuous parameters $\alpha(x)$ and $\beta(x)$. Our starting point is the global reciprocity theorem of the time-convolution type for a space-dependent material, formulated by Eq. (34). For state A we take the Green's state, hence, we substitute $-i\omega\hat{a}_A(x,\omega) = \delta(x - x_A)$, $\hat{b}_A(x,\omega) = 0$, $\hat{P}_A(x,\omega) = \hat{\mathcal{G}}_x(x, x_A, \omega)$ and $\hat{Q}_A(x,\omega) = \frac{1}{i\omega\beta_A(x)}\partial_x\hat{\mathcal{G}}_x(x, x_A, \omega)$. For state B we take the actual field and drop the subscripts B. Substitution into Eq. (34), using reciprocity relation (38), this yields the following classical representation

$$\chi(x_A)\hat{P}(x_A,\omega) = \int_{x_b}^{x_e}\Big(-i\omega\hat{\mathcal{G}}_x(x_A,x,\omega)\hat{a}(x,\omega) + \frac{1}{\beta_A(x)}\{\partial_x\hat{\mathcal{G}}_x(x_A,x,\omega)\}\hat{b}(x,\omega)\Big)\mathrm{d}x$$
$$+\int_{x_b}^{x_e}\Big(\omega^2\hat{\mathcal{G}}_x(x_A,x,\omega)\Delta\alpha(x)\hat{P}(x,\omega) + \frac{i\omega}{\beta_A(x)}\{\partial_x\hat{\mathcal{G}}_x(x_A,x,\omega)\}\Delta\beta(x)\hat{Q}(x,\omega)\Big)\mathrm{d}x$$
$$+\Big(i\omega\hat{\mathcal{G}}_x(x_A,x,\omega)\hat{Q}(x,\omega) - \frac{1}{\beta_A(x)}\{\partial_x\hat{\mathcal{G}}_x(x_A,x,\omega)\}\hat{P}(x,\omega)\Big)\Big|_{x_b}^{x_e}, \tag{46}$$

with

$$\Delta\alpha(x) = \alpha(x) - \alpha_A(x), \tag{47}$$
$$\Delta\beta(x) = \beta(x) - \beta_A(x), \tag{48}$$

and where $\chi(x_A)$ is the characteristic function, defined as

$$\chi(x_A) = \begin{cases} 1 & \text{for} \quad x_b < x_A < x_e, \\ \frac{1}{2} & \text{for} \quad x_A = x_b \quad \text{or} \quad x_A = x_e, \\ 0 & \text{for} \quad x_A < x_b \quad \text{or} \quad x_A > x_e. \end{cases} \tag{49}$$

Eq. (46) is a generalisation of Eq. (12), transformed to the frequency domain. It expresses the wave field at any point x_A between x_b and x_e (including these points). The first term on the right-hand side accounts for the contribution of the sources between x_b and x_e, the second term describes scattering due to the material contrast functions $\Delta\alpha(x)$ and $\Delta\beta(x)$, and the last term describes contributions from the fields at x_b and x_e. The representation of Eq. (46) finds applications in the analysis of wave scattering problems in space-dependent materials [17,31,32,44,45].

6.2. Time-dependent material

We derive a general wave field representation for a time-dependent material with piecewise continuous parameters $\alpha(t)$ and $\beta(t)$. Our starting point is the global reciprocity theorem of the space-convolution type for a time-dependent material, formulated by Eq. (41). In anticipation of using the reciprocity relation (43), for state A we take the acausal Green's state, hence, we substitute $-ik\check{b}_A(k,t) = \delta(t-t_A)$, $\check{a}_A(k,t) = 0$, $\check{U}_A(k,t) = \check{\mathcal{G}}_t^a(k,t,t_A)$ and $\check{V}_A(k,t) = -\frac{\beta_A(t)}{ik}\partial_t\check{\mathcal{G}}_t^a(k,t,t_A)$. For state B we take the actual field and drop the subscripts B. Substitution into Eq. (41), using reciprocity relation (43), this yields the following representation

$$\chi(t_A)\check{U}(k,t_A) = \int_{t_b}^{t_e}\Big(-ik\check{\mathcal{G}}_t(k,t_A,t)\check{b}(k,t) - \beta_A(t)\{\partial_t\check{\mathcal{G}}_t(k,t_A,t)\}\check{a}(k,t)\Big)\mathrm{d}t$$
$$+\int_{t_b}^{t_e}\Big(-k^2\check{\mathcal{G}}_t(k,t_A,t)\Delta\alpha^{-1}(t)\check{U}(k,t) + ik\beta_A(t)\{\partial_t\check{\mathcal{G}}_t(k,t_A,t)\}\Delta\beta^{-1}(t)\check{V}(k,t)\Big)\mathrm{d}t$$
$$+\Big(ik\check{\mathcal{G}}_t(k,t_A,t)\check{V}(k,t) + \beta_A(t)\{\partial_t\check{\mathcal{G}}_t(k,t_A,t)\}\check{U}(k,t)\Big)\Big|_{t_b}^{t_e}, \tag{50}$$

with

$$\Delta\alpha^{-1}(t) = \alpha^{-1}(t) - \alpha_A^{-1}(t), \tag{51}$$
$$\Delta\beta^{-1}(t) = \beta^{-1}(t) - \beta_A^{-1}(t), \tag{52}$$

and where $\chi(t_A)$ is the characteristic function, defined as

$$\chi(t_A) = \begin{cases} 1 & \text{for} \quad t_b < t_A < t_e, \\ \frac{1}{2} & \text{for} \quad t_A = t_b \quad \text{or} \quad t_A = t_e, \\ 0 & \text{for} \quad t_A < t_b \quad \text{or} \quad t_A > t_e. \end{cases} \tag{53}$$

Eq. (50) is a generalisation of Eq. (19), transformed to the wavenumber domain. It expresses the wave field at any time t_A between t_b and t_e (including these time instants). The first term on the right-hand side accounts for the contribution of the sources between t_b and t_e, the second term describes scattering due to the material contrast functions $\Delta\alpha^{-1}(t)$ and $\Delta\beta^{-1}(t)$, and the last term describes contributions from the fields at t_b and t_e (with the contribution from the field at t_e being zero when $t_A < t_e$). The representation of Eq. (50) finds potential applications in the analysis of wave scattering problems in time-dependent materials.

7. Green's function retrieval

7.1. Space-dependent material

Under specific circumstances, the correlation of passive wave measurements at two receivers yields the response to a virtual impulsive source at the position of one of these receivers, observed by the other receiver (i.e., the Green's function between the receivers). This concept has found numerous applications in ultrasonics [46–48], seismology [49–56], ocean acoustics [57,58], infrasound [59,60], medical imaging [61,62] and engineering [63,64].

Following the approach of Ref. [65], we use the global reciprocity theorem of the time-correlation type (Eq. (35)) to derive an expression for Green's function retrieval for a space-dependent material with piecewise continuous parameters $\alpha(x)$ and $\beta(x)$. To this end, we take identical material parameters in both states, i.e., $\alpha_A = \alpha_B = \alpha$ and $\beta_A = \beta_B = \beta$ and we assume that the material is homogeneous for $x \le x_b$ and for $x \ge x_e$. For state A we take a Green's state with a unit source at x_A between x_b and x_e, and we substitute $-i\omega\hat{a}_A(x,\omega) = \delta(x - x_A)$, $\hat{b}_A(x,\omega) = 0$ and $\hat{P}_A(x,\omega) = \hat{\mathcal{G}}_x(x, x_A, \omega)$; for $\hat{Q}_A(x,\omega)$ we use Eqs. (36) and (37). For state B we take a Green's state with a unit source at x_B between x_b and x_e, and we substitute similar expressions. Furthermore, we use the source–receiver reciprocity relation of Eq. (38). This yields

$$
\begin{aligned}
&2i\Im\{\hat{\mathcal{G}}_x(x_B, x_A, \omega)\} = \\
&\quad \frac{2i\omega}{\eta(x_b)}\hat{\mathcal{G}}_x^*(x_A, x_b, \omega)\hat{\mathcal{G}}_x(x_B, x_b, \omega) + \frac{2i\omega}{\eta(x_e)}\hat{\mathcal{G}}_x^*(x_A, x_e, \omega)\hat{\mathcal{G}}_x(x_B, x_e, \omega),
\end{aligned} \tag{54}
$$

where $\Im$ denotes the imaginary part. Applying an inverse temporal Fourier transform yields

$$
\begin{aligned}
&\mathcal{G}_x(x_B, x_A, t) - \mathcal{G}_x(x_B, x_A, -t) = \\
&\quad -2\partial_t \sum_{x = x_b, x_e} \frac{1}{\eta(x)} \int_{-\infty}^{\infty} \mathcal{G}_x(x_A, x, t')\mathcal{G}_x(x_B, x, t + t')\mathrm{d}t'.
\end{aligned} \tag{55}
$$

The right-hand side is the time derivative of the superposition of time correlations of measurements by receivers at positions x_A and x_B, in response to impulsive sources at x_b and x_e. The left-hand side is the causal Green's function $\mathcal{G}_x(x_B, x_A, t)$ between x_A and x_B, minus its time-reversed version. Hence, the Green's function $\mathcal{G}_x(x_B, x_A, t)$ is retrieved by evaluating the right-hand side of this expression and taking the causal part. Note that this is independent of the positions x_b and x_e of the sources, as long as the receivers are located between these sources and the material left of x_b and right of x_e is homogeneous. When the impulsive sources at x_b and x_e are replaced by uncorrelated noise sources, the retrieved response is the Green's function $\mathcal{G}_x(x_B, x_A, t)$, convolved with the autocorrelation of the noise.

7.2. Time-dependent material

Following a similar approach as in Section 7.1, we use the global reciprocity theorem of the space-correlation type (Eq. (42)) to derive an expression for Green's function retrieval for a time-dependent material with piecewise continuous parameters $\alpha(t)$ and $\beta(t)$. We take again $\alpha_A = \alpha_B = \alpha$ and $\beta_A = \beta_B = \beta$. For state A we take an acausal Green's state with a unit sink at t_A between t_b and t_e and we substitute $-ik\check{b}_A(k,t) = \delta(t - t_A)$, $\check{a}_A(k,t) = 0$, $\check{U}_A(k,t) = \check{\mathcal{G}}_t^a(k, t, t_A)$ and $\check{V}_A(k,t) = -\frac{\beta(t)}{ik}\partial_t\check{\mathcal{G}}_t^a(k, t, t_A)$. For state B we take an acausal Green's state with a unit sink at t_B between t_b and t_e, and we substitute similar expressions. Furthermore, we use the source–receiver reciprocity relation of Eq. (43). This yields

$$
\begin{aligned}
&\check{\mathcal{G}}_t(k, t_B, t_A) - \{\check{\mathcal{G}}_t^a(k, t_B, t_A)\}^* = \\
&\quad \beta(t_b)\Big[\check{\mathcal{G}}_t^*(k, t_A, t_b)\partial_{t_b}\check{\mathcal{G}}_t(k, t_B, t_b) - \{\partial_{t_b}\check{\mathcal{G}}_t^*(k, t_A, t_b)\}\check{\mathcal{G}}_t(k, t_B, t_b)\Big].
\end{aligned} \tag{56}
$$

Applying an inverse spatial Fourier transform yields

$$
\begin{aligned}
&\mathcal{G}_t(x, t_B, t_A) - \mathcal{G}_t^a(-x, t_B, t_A) = \\
&\quad \beta(t_b)\int_{-\infty}^{\infty}\Big[\mathcal{G}_t(x', t_A, t_b)\partial_{t_b}\mathcal{G}_t(x + x', t_B, t_b) - \{\partial_{t_b}\mathcal{G}_t(x', t_A, t_b)\}\mathcal{G}_t(x + x', t_B, t_b)\Big]\mathrm{d}x'.
\end{aligned} \tag{57}
$$

The right-hand side is the superposition of space correlations of measurements by receivers at time instants t_A and t_B, in response to an impulsive source at t_b and its time derivative and vice-versa. The left-hand side is the causal Green's function $\mathcal{G}_t(x, t_B, t_A)$ between t_A and t_B, minus its space-reversed acausal counterpart. Hence, the Green's function $\mathcal{G}_t(x, t_B, t_A)$ (when $t_B > t_A$) or $-\mathcal{G}_t^a(-x, t_B, t_A)$ (when $t_B < t_A$) is retrieved by evaluating the right-hand side of this expression. Note that this is independent of the time instant t_b of the source, as long t_A and t_B are both larger than t_b. Unlike the two-sided representation of Eq. (55), which requires sources at x_b and x_e, this is a single-sided representation, which requires sources at t_b only. Note that the time-derivatives in Eq. (57) act on a superposition of left- and right-going waves at t_b (see for example Fig. 2, with $t_b = t_0$), hence, we cannot use an expression similar to Eq. (36) to simplify the right-hand side of Eq. (57) further.

8. Matrix–vector wave equation

8.1. Space-dependent material

For a space-dependent material with continuous parameters $\alpha(x)$ and $\beta(x)$, Eqs. (5) and (6) can be combined into the following matrix–vector wave equation in the x,t-domain [66–71]

$$
\partial_x\mathbf{q}_x = \mathbf{A}_x\mathbf{q}_x + \mathbf{d}_x, \tag{58}
$$

with wave field vector $\mathbf{q}_x(x,t)$, operator matrix $\mathbf{A}_x(x,t)$ and source vector $\mathbf{d}_x(x,t)$ defined as

$$\mathbf{q}_x = \begin{pmatrix} P \\ Q \end{pmatrix}, \quad \mathbf{A}_x = \begin{pmatrix} 0 & -\beta\partial_t \\ -\alpha\partial_t & 0 \end{pmatrix}, \quad \mathbf{d}_x = \begin{pmatrix} b \\ a \end{pmatrix}. \tag{59}$$

For a space-dependent material with piecewise continuous parameters, this equation is supplemented with boundary conditions at all points where $\alpha(x)$ and $\beta(x)$ are discontinuous. The boundary condition is that $\mathbf{q}_x(x,t)$ is continuous at those points.

Using the temporal Fourier transform defined in Eq. (20), we obtain the following matrix–vector wave equation in the x,ω-domain

$$\partial_x \hat{\mathbf{q}}_x = \hat{\mathbf{A}}_x \hat{\mathbf{q}}_x + \hat{\mathbf{d}}_x, \tag{60}$$

with wave field vector $\hat{\mathbf{q}}_x(x,\omega)$, matrix $\hat{\mathbf{A}}_x(x,\omega)$ and source vector $\hat{\mathbf{d}}_x(x,\omega)$ defined as

$$\hat{\mathbf{q}}_x = \begin{pmatrix} \hat{P} \\ \hat{Q} \end{pmatrix}, \quad \hat{\mathbf{A}}_x = \begin{pmatrix} 0 & i\omega\beta \\ i\omega\alpha & 0 \end{pmatrix}, \quad \hat{\mathbf{d}}_x = \begin{pmatrix} \hat{b} \\ \hat{a} \end{pmatrix}. \tag{61}$$

Note that matrix $\hat{\mathbf{A}}_x(x,\omega)$ obeys the following symmetry properties

$$\hat{\mathbf{A}}_x^t \mathbf{N} = -\mathbf{N}\hat{\mathbf{A}}_x, \tag{62}$$

$$\hat{\mathbf{A}}_x^\dagger \mathbf{K} = -\mathbf{K}\hat{\mathbf{A}}_x, \tag{63}$$

$$\hat{\mathbf{A}}_x^* \mathbf{J} = \mathbf{J}\hat{\mathbf{A}}_x, \tag{64}$$

where superscript t denotes transposition, superscript $\dagger$ denotes transposition and complex conjugation, and where

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{65}$$

8.2. *Time-dependent material*

Applying the mapping of Eq. (16) to Eqs. (58) and (59) yields the following matrix–vector wave equation in the x,t-domain for a time-dependent material with continuous parameters $\alpha(t)$ and $\beta(t)$

$$\partial_t \mathbf{q}_t = \mathbf{A}_t \mathbf{q}_t + \mathbf{d}_t, \tag{66}$$

with wave field vector $\mathbf{q}_t(x,t)$, operator matrix $\mathbf{A}_t(x,t)$ and source vector $\mathbf{d}_t(x,t)$ defined as

$$\mathbf{q}_t = \begin{pmatrix} U \\ V \end{pmatrix}, \quad \mathbf{A}_t = \begin{pmatrix} 0 & -\frac{1}{\beta}\partial_x \\ -\frac{1}{\alpha}\partial_x & 0 \end{pmatrix}, \quad \mathbf{d}_t = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{67}$$

For a time-dependent material with piecewise continuous parameters, this equation is supplemented with boundary conditions at all time instants where $\alpha(t)$ and $\beta(t)$ are discontinuous. The boundary condition is that $\mathbf{q}_t(x,t)$ is continuous at those time instants.

Applying the mapping of Eq. (30) to Eqs. (60) and (61), we obtain the following matrix–vector wave equation in the k,t-domain

$$\partial_t \check{\mathbf{q}}_t = \check{\mathbf{A}}_t \check{\mathbf{q}}_t + \check{\mathbf{d}}_t, \tag{68}$$

with

$$\check{\mathbf{q}}_t = \begin{pmatrix} \check{U} \\ \check{V} \end{pmatrix}, \quad \check{\mathbf{A}}_t = \begin{pmatrix} 0 & -\frac{ik}{\beta} \\ -\frac{ik}{\alpha} & 0 \end{pmatrix}, \quad \check{\mathbf{d}}_t = \begin{pmatrix} \check{a} \\ \check{b} \end{pmatrix}. \tag{69}$$

Matrix $\check{\mathbf{A}}_t$ obeys the same symmetries as $\hat{\mathbf{A}}_x$, as formulated by Eqs. (62)–(64).

9. Propagator matrices and representations

9.1. *Space-dependent material*

For a space-dependent material, a propagator matrix "propagates" a wave field (represented as a vectorial quantity) from one plane in space to another [72–74]. It has found many applications, particularly in elastodynamic wave problems [33,75,76].

We define the propagator matrix $\mathbf{W}_x(x,x_0,t)$ for a space-dependent material with continuous parameters $\alpha(x)$ and $\beta(x)$ as the solution of matrix–vector Eq. (58) without the source term, hence

$$\partial_x \mathbf{W}_x = \mathbf{A}_x \mathbf{W}_x, \tag{70}$$

with operator matrix $\mathbf{A}_x(x,t)$ defined in Eq. (59) and with boundary condition

$$\mathbf{W}_x(x_0,x_0,t) = \mathbf{I}\delta(t), \tag{71}$$

where $\mathbf{I}$ is the identity matrix.

A simple representation for the wave field vector $\mathbf{q}_x(x,t)$ obeying Eq. (58) is obtained when $\mathbf{q}_x$ and $\mathbf{W}_x$ are defined in the same source-free material between x_0 and x (where x can be either larger or smaller than x_0). Whereas $\mathbf{q}_x(x,t)$ can have any time-dependency at $x = x_0$, $\mathbf{W}_x(x,x_0,t)$ collapses to $\mathbf{I}\delta(t)$ at $x = x_0$. Because Eqs. (58) and (70) are linear, a representation for $\mathbf{q}_x(x,t)$ follows by applying Huygens' superposition principle, according to

$$\mathbf{q}_x(x,t) = \int_{-\infty}^{\infty} \mathbf{W}_x(x,x_0,t-t')\mathbf{q}_x(x_0,t')\mathrm{d}t'. \tag{72}$$

Note that $\mathbf{W}_x(x,x_0,t)$ propagates the wave field vector $\mathbf{q}_x$ from x_0 to x, hence the name "propagator matrix". We partition $\mathbf{W}_x(x,x_0,t)$ as follows

$$\mathbf{W}_x(x,x_0,t) = \begin{pmatrix} W_x^{P,P} & W_x^{P,Q} \\ W_x^{Q,P} & W_x^{Q,Q} \end{pmatrix}(x,x_0,t). \tag{73}$$

The first and second superscripts refer to the wave field quantities in vector $\mathbf{q}_x$, defined in Eq. (59), at x and x_0, respectively. For more general representations with propagator matrices, including source terms and differences in material parameters (analogous to the representation with Green's functions discussed in Section 6.1), see Ref. [77].

Using the temporal Fourier transform defined in Eq. (20), we obtain the following space-frequency domain equation for $\hat{\mathbf{W}}_x(x,x_0,\omega)$

$$\partial_x \hat{\mathbf{W}}_x = \hat{\mathbf{A}}_x \hat{\mathbf{W}}_x, \tag{74}$$

with matrix $\hat{\mathbf{A}}_x(x,\omega)$ defined in Eq. (61) and with boundary condition

$$\hat{\mathbf{W}}_x(x_0,x_0,\omega) = \mathbf{I}. \tag{75}$$

The representation of Eq. (72) transforms to

$$\hat{\mathbf{q}}_x(x,\omega) = \hat{\mathbf{W}}_x(x,x_0,\omega)\hat{\mathbf{q}}_x(x_0,\omega). \tag{76}$$

By applying this equation recursively, it follows that $\hat{\mathbf{W}}_x$ obeys the following recursive expression

$$\hat{\mathbf{W}}_x(x_N,x_0,\omega) = \hat{\mathbf{W}}_x(x_N,x_{N-1},\omega)\cdots\hat{\mathbf{W}}_x(x_n,x_{n-1},\omega)\cdots\hat{\mathbf{W}}_x(x_1,x_0,\omega), \tag{77}$$

where $x_1 \cdots x_n \cdots x_{N-1}$ are points where the material parameters may be discontinuous. As a special case of Eq. (77) we obtain

$$\hat{\mathbf{W}}_x(x_{n-1},x_n,\omega)\hat{\mathbf{W}}_x(x_n,x_{n-1},\omega) = \hat{\mathbf{W}}_x(x_{n-1},x_{n-1},\omega) = \mathbf{I}, \tag{78}$$

from which it follows that $\hat{\mathbf{W}}_x(x_{n-1},x_n,\omega)$ is the inverse of $\hat{\mathbf{W}}_x(x_n,x_{n-1},\omega)$. For a homogeneous slab between x_{n-1} and x_n, with parameters α_n, β_n, $c_n = 1/\sqrt{\alpha_n\beta_n}$, $\eta_n = \sqrt{\beta_n/\alpha_n}$ and thickness $\Delta x_n = x_n - x_{n-1}$, we have

$$\hat{W}_x^{P,P}(x_n,x_{n-1},\omega) = \cos(\omega\Delta x_n/c_n), \tag{79}$$

$$\hat{W}_x^{P,Q}(x_n,x_{n-1},\omega) = i\eta_n\sin(\omega\Delta x_n/c_n), \tag{80}$$

$$\hat{W}_x^{Q,P}(x_n,x_{n-1},\omega) = \frac{i}{\eta_n}\sin(\omega\Delta x_n/c_n), \tag{81}$$

$$\hat{W}_x^{Q,Q}(x_n,x_{n-1},\omega) = \cos(\omega\Delta x_n/c_n). \tag{82}$$

From Eq. (77), we obtain a similar recursive expression in the space–time domain, according to

$$\mathbf{W}_x(x_N,x_0,t) = \mathbf{W}_x(x_N,x_{N-1},t) *_t \cdots *_t \mathbf{W}_x(x_n,x_{n-1},t) *_t \cdots *_t \mathbf{W}_x(x_1,x_0,t), \tag{83}$$

where $*_t$ denotes a time convolution (more formally defined in Eq. (72)). For a homogeneous slab between x_{n-1} and x_n, we find from Eqs. (79)–(82)

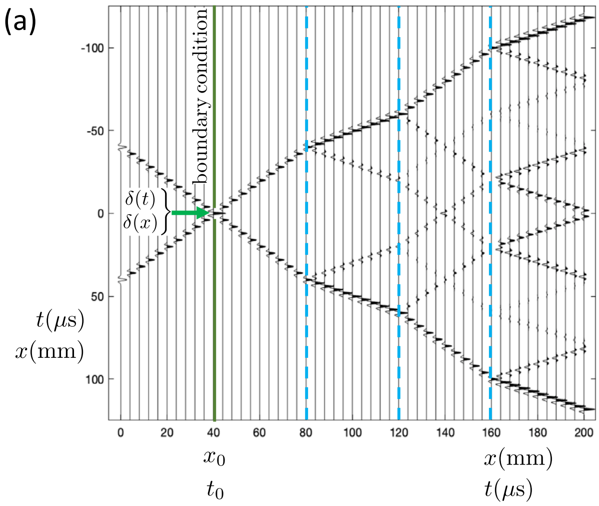
$$W_x^{P,P}(x_n,x_{n-1},t) = \frac{1}{2}\{\delta(t-\Delta x_n/c_n) + \delta(t+\Delta x_n/c_n)\}, \tag{84}$$

$$W_x^{P,Q}(x_n,x_{n-1},t) = \frac{\eta_n}{2}\{\delta(t-\Delta x_n/c_n) - \delta(t+\Delta x_n/c_n)\}, \tag{85}$$

$$W_x^{Q,P}(x_n,x_{n-1},t) = \frac{1}{2\eta_n}\{\delta(t-\Delta x_n/c_n) - \delta(t+\Delta x_n/c_n)\}, \tag{86}$$

$$W_x^{Q,Q}(x_n,x_{n-1},t) = \frac{1}{2}\{\delta(t-\Delta x_n/c_n) + \delta(t+\Delta x_n/c_n)\}. \tag{87}$$

For the same piecewise homogeneous material as used for the numerical example in Section 3.1, the elements $W_x^{P,P}(x,x_0,t)$ and $W_x^{P,Q}(x,x_0,t)$ for $x_0 = 40$ mm (convolved with a temporal wavelet with a central frequency $\omega_0/2\pi = 300$ kHz) are shown as x,t-diagrams in Fig. 3a and b. The green lines indicate the boundary conditions $W_x^{P,P}(x_0,x_0,t) = \delta(t)$ and $W_x^{P,Q}(x_0,x_0,t) = 0$ (Eqs. (71) and (73)). Note that these figures clearly exhibit the recursive character, described by Eq. (83).



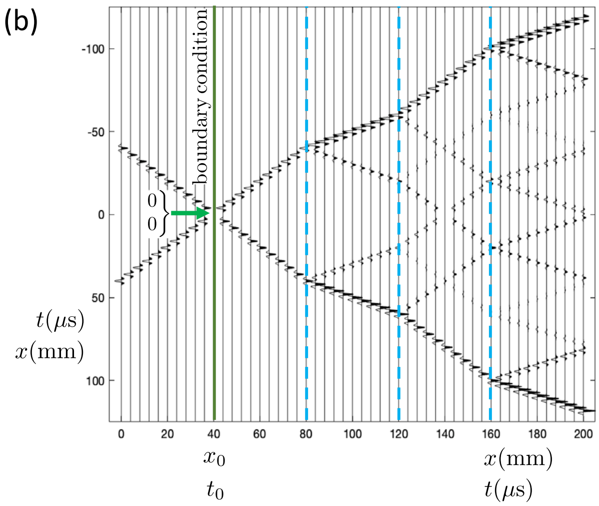


Fig. 3. Propagator matrix elements $W_x^{P,P}(x,x_0,t)$ (a) and $W_x^{P,Q}(x,x_0,t)$ (b) (convolved with a temporal wavelet) for a piecewise homogeneous space-dependent material. The labels at the vertical axes denote time (in μs) and those at the horizontal axes denote space (in mm). With interchanged labels (and "boundary condition" replaced by "initial condition") these figures can be interpreted as $W_t^{U,U}(x,t,t_0)$ (a) and $W_t^{U,V}(x,t,t_0)$ (b) (convolved with a spatial wavelet) for a piecewise constant time-dependent material.

9.2. Time-dependent material

For a time-dependent material, a propagator matrix propagates a wave field from one instant in time to another. In the literature on time-dependent materials this matrix is usually called the transfer matrix [12,14,78], but for consistency with Section 9.1, we hold on to the name propagator matrix.

We define the propagator matrix $\mathbf{W}_t(x,t,t_0)$ for a time-dependent material with continuous parameters $\alpha(t)$ and $\beta(t)$ as the solution of matrix–vector Eq. (66) without the source term, hence

$$\partial_t \mathbf{W}_t = \mathbf{A}_t \mathbf{W}_t, \tag{88}$$

with operator matrix $\mathbf{A}_t(x,t)$ defined in Eq. (67) and with initial condition

$$\mathbf{W}_t(x,t_0,t_0) = \mathbf{I}\delta(x). \tag{89}$$

Note that the mapping of Eq. (16) not only applies to the wave equations (Eqs. (70) and (88)), but also to the boundary and initial conditions (Eqs. (71) and (89)). Consequently, the mappings of Eqs. (16) and (30) also apply to all expressions for the propagator matrix in the space–time and Fourier-domains, respectively. We discuss a few of these mappings explicitly.

The representation of Eq. (72) maps to

$$\mathbf{q}_t(x,t) = \int_{-\infty}^{\infty} \mathbf{W}_t(x-x',t,t_0)\mathbf{q}_t(x',t_0)\mathrm{d}x'. \tag{90}$$

Note that $\mathbf{W}_t(x,t,t_0)$ propagates the wave field vector $\mathbf{q}_t$ from t_0 to t. We partition $\mathbf{W}_t(x,t,t_0)$ as follows

$$\mathbf{W}_t(x,t,t_0) = \begin{pmatrix} W_t^{U,U} & W_t^{U,V} \\ W_t^{V,U} & W_t^{V,V} \end{pmatrix}(x,t,t_0). \tag{91}$$

The first and second superscripts refer to the wave field quantities in vector $\mathbf{q}_t$, defined in Eq. (67), at t and t_0, respectively. The recursive expression of Eq. (83) maps to

$$\mathbf{W}_t(x,t_N,t_0) = \mathbf{W}_t(x,t_N,t_{N-1}) *_x \cdots *_x \mathbf{W}_t(x,t_n,t_{n-1}) *_x \cdots *_x \mathbf{W}_t(x,t_1,t_0), \tag{92}$$

where $*_x$ denotes a space convolution (more formally defined in Eq. (90)). For a constant slab between t_{n-1} and t_n with duration $\Delta t_n = t_n - t_{n-1}$, we find from Eqs. (84)–(87)

$$W_t^{U,U}(x,t_n,t_{n-1}) = \frac{1}{2}\{\delta(x-c_n\Delta t_n)+\delta(x+c_n\Delta t_n)\}, \tag{93}$$

$$W_t^{U,V}(x,t_n,t_{n-1}) = \frac{1}{2\eta_n}\{\delta(x-c_n\Delta t_n)-\delta(x+c_n\Delta t_n)\}, \tag{94}$$

$$W_t^{V,U}(x,t_n,t_{n-1}) = \frac{\eta_n}{2}\{\delta(x-c_n\Delta t_n)-\delta(x+c_n\Delta t_n)\}, \tag{95}$$

$$W_t^{V,V}(x,t_n,t_{n-1}) = \frac{1}{2}\{\delta(x-c_n\Delta t_n)+\delta(x+c_n\Delta t_n)\}. \tag{96}$$

For the same piecewise constant material as used for the first numerical example in Section 3.2, the elements $W_t^{U,U}(x,t,t_0)$ and $W_t^{U,V}(x,t,t_0)$ for $t_0 = 40$ μs (convolved with a spatial wavelet with a central wavenumber $k_0/2\pi = 300 * 10^3$ km^{-1}) are shown as x,t-diagrams in Fig. 3a and b. The green lines indicate the initial conditions $W_t^{U,U}(x,t_0,t_0) = \delta(x)$ and $W_t^{U,V}(x,t_0,t_0) = 0$ (Eqs. (89) and (91)). Note that these figures clearly exhibit the recursive character, described by Eq. (92).

Finally we show that the Green's function $\mathcal{G}_t(x,t,t_0)$, obeying Eq. (17) with causality condition (18), is related to $W_t^{U,V}(x,t,t_0)$ via

$$-\partial_x\mathcal{G}_t(x,t,t_0) = H(t-t_0)W_t^{U,V}(x,t,t_0). \tag{97}$$

Due to the Heaviside function $H(t-t_0)$, the causality condition (18) is fulfilled, so we only need to show that $H(t-t_0)W_t^{U,V}(x,t,t_0)$ obeys the same wave equation as $-\partial_x\mathcal{G}_t(x,t,t_0)$. For the first derivative with respect to time we obtain, using the product rule for differentiation and Eqs. (67), (88), (89) and (91),

$$\partial_t\{H(t-t_0)W_t^{U,V}(x,t,t_0)\} = -\frac{1}{\beta(t)}H(t-t_0)\partial_x W_t^{V,V}(x,t,t_0). \tag{98}$$

Multiplying both sides with $\beta(t)$ and differentiating again with respect to time, we obtain in a similar way

$$\begin{aligned}\partial_t\big(\beta(t)\partial_t\{H(t-t_0)W_t^{U,V}(x,t,t_0)\}\big) &= -\partial_x\delta(x)\delta(t-t_0) \\ &+ \frac{1}{\alpha(t)}\partial_x^2\{H(t-t_0)W_t^{U,V}(x,t,t_0)\}.\end{aligned} \tag{99}$$

Comparing this with Eq. (17), with $c(t)$ defined in Eq. (8), we observe that $H(t-t_0)W_t^{U,V}(x,t,t_0)$ indeed obeys the same wave equation as $-\partial_x\mathcal{G}_t(x,t,t_0)$. Hence, $\mathcal{G}_t(x,t,t_0)$ is obtained by integrating $-H(t-t_0)W_t^{U,V}(x,t,t_0)$ with respect to x (see Figs. 2a and 3b). Note that a relation similar to Eq. (97) does not exist for a space-dependent material (since the causality conditions (Eqs. (11) and (18)) do not follow the mapping of Eq. (16)).

10. Matrix–vector reciprocity theorems

10.1. Space-dependent material

We review matrix–vector reciprocity theorems for a space-dependent material with piecewise continuous parameters $\alpha(x)$ and $\beta(x)$. We consider two independent states, indicated by subscripts A and B, obeying Eq. (60), and we derive relations between these states. In the most general case, sources, material parameters and wave fields may be different in the two states. We consider the quantities $\partial_x\{\hat{\mathbf{q}}_{x,A}^t(x,\omega)\mathbf{N}\hat{\mathbf{q}}_{x,B}(x,\omega)\}$ and $\partial_x\{\hat{\mathbf{q}}_{x,A}^\dagger(x,\omega)\mathbf{K}\hat{\mathbf{q}}_{x,B}(x,\omega)\}$. Applying the product rule for differentiation, using wave Eq. (60) and symmetry relations (62) and (63) for states A and B, yields

$$\partial_x\{\hat{\mathbf{q}}_{x,A}^t\mathbf{N}\hat{\mathbf{q}}_{x,B}\} = \hat{\mathbf{q}}_{x,A}^t\mathbf{N}\Delta\hat{\mathbf{A}}_x\hat{\mathbf{q}}_{x,B} + \hat{\mathbf{d}}_{x,A}^t\mathbf{N}\hat{\mathbf{q}}_{x,B} + \hat{\mathbf{q}}_{x,A}^t\mathbf{N}\hat{\mathbf{d}}_{x,B}, \tag{100}$$

$$\partial_x\{\hat{\mathbf{q}}_{x,A}^\dagger\mathbf{K}\hat{\mathbf{q}}_{x,B}\} = \hat{\mathbf{q}}_{x,A}^\dagger\mathbf{K}\Delta\hat{\mathbf{A}}_x\hat{\mathbf{q}}_{x,B} + \hat{\mathbf{d}}_{x,A}^\dagger\mathbf{K}\hat{\mathbf{q}}_{x,B} + \hat{\mathbf{q}}_{x,A}^\dagger\mathbf{K}\hat{\mathbf{d}}_{x,B}, \tag{101}$$

with $\Delta\hat{\mathbf{A}}_x = \hat{\mathbf{A}}_{x,B} - \hat{\mathbf{A}}_{x,A}$. Eqs. (100) and (101) are the matrix–vector forms of the local reciprocity theorems of the time-convolution and time-correlation type, respectively, as formulated by Eqs. (23) and (24). Integration of both sides of Eqs. (100) and (101) from

x_b to x_e yields [79,80]

$$\hat{\mathbf{q}}^t_{x,A}\mathbf{N}\hat{\mathbf{q}}_{x,B}\Big|_{x_b}^{x_e} = \int_{x_b}^{x_e}\{\hat{\mathbf{q}}^t_{x,A}\mathbf{N}\Delta\hat{\mathbf{A}}_x\hat{\mathbf{q}}_{x,B} + \hat{\mathbf{d}}^t_{x,A}\mathbf{N}\hat{\mathbf{q}}_{x,B} + \hat{\mathbf{q}}^t_{x,A}\mathbf{N}\hat{\mathbf{d}}_{x,B}\}\mathrm{d}x, \tag{102}$$

$$\hat{\mathbf{q}}^\dagger_{x,A}\mathbf{K}\hat{\mathbf{q}}_{x,B}\Big|_{x_b}^{x_e} = \int_{x_b}^{x_e}\{\hat{\mathbf{q}}^\dagger_{x,A}\mathbf{K}\Delta\hat{\mathbf{A}}_x\hat{\mathbf{q}}_{x,B} + \hat{\mathbf{d}}^\dagger_{x,A}\mathbf{K}\hat{\mathbf{q}}_{x,B} + \hat{\mathbf{q}}^\dagger_{x,A}\mathbf{K}\hat{\mathbf{d}}_{x,B}\}\mathrm{d}x. \tag{103}$$

These are the matrix–vector forms of the global reciprocity theorems of the time-convolution and time-correlation type, respectively, as formulated by Eqs. (34) and (35). When there are no sources and the material parameters are identical in both states, the right-hand sides of Eqs. (100)–(103) are zero. From Eqs. (100) and (101) it then follows that $\hat{\mathbf{q}}^t_{x,A}\mathbf{N}\hat{\mathbf{q}}_{x,B}$ and $\hat{\mathbf{q}}^\dagger_{x,A}\mathbf{K}\hat{\mathbf{q}}_{x,B}$ are space-propagation invariants [36–39].

We use Eqs. (102) and (103) (with zeroes on the right-hand sides) to derive reciprocity relations for the propagator matrix. For state A we substitute $\hat{\mathbf{q}}_{x,A}(x,\omega) = \hat{\mathbf{W}}_x(x,x_A,\omega)$ and, using Eq. (75), $\hat{\mathbf{q}}_{x,A}(x_A,\omega) = \mathbf{I}$. Similarly, For state B we substitute $\hat{\mathbf{q}}_{x,B}(x,\omega) = \hat{\mathbf{W}}_x(x,x_B,\omega)$ and $\hat{\mathbf{q}}_{x,B}(x_B,\omega) = \mathbf{I}$. Taking x_A and x_B equal to x_b and x_e (in arbitrary order) yields

$$\hat{\mathbf{W}}^t_x(x_B,x_A,\omega)\mathbf{N} = \mathbf{N}\hat{\mathbf{W}}_x(x_A,x_B,\omega), \tag{104}$$

$$\hat{\mathbf{W}}^\dagger_x(x_B,x_A,\omega)\mathbf{K} = \mathbf{K}\hat{\mathbf{W}}_x(x_A,x_B,\omega). \tag{105}$$

Using $\mathbf{N}^{-1}\mathbf{K} = -\mathbf{J}$ we find from Eqs. (104) and (105)

$$\hat{\mathbf{W}}^*_x(x_A,x_B,\omega)\mathbf{J} = \mathbf{J}\hat{\mathbf{W}}_x(x_A,x_B,\omega), \tag{106}$$

or, in the space–time domain

$$\mathbf{W}_x(x_A,x_B,-t)\mathbf{J} = \mathbf{J}\mathbf{W}_x(x_A,x_B,t). \tag{107}$$

10.2. Time-dependent material

The matrix–vector reciprocity theorems for a time-dependent material with piecewise continuous parameters $\alpha(t)$ and $\beta(t)$ follow from those in Section 10.1 by applying the mappings of Eqs. (16) and (30). In particular, the reciprocity relations for the propagator matrix are

$$\check{\mathbf{W}}^t_t(k,t_B,t_A)\mathbf{N} = \mathbf{N}\check{\mathbf{W}}_t(k,t_A,t_B), \tag{108}$$

$$\check{\mathbf{W}}^\dagger_t(k,t_B,t_A)\mathbf{K} = \mathbf{K}\check{\mathbf{W}}_t(k,t_A,t_B), \tag{109}$$

$$\check{\mathbf{W}}^*_t(k,t_A,t_B)\mathbf{J} = \mathbf{J}\check{\mathbf{W}}_t(k,t_A,t_B), \tag{110}$$

$$\mathbf{W}_t(-x,t_A,t_B)\mathbf{J} = \mathbf{J}\mathbf{W}_t(x,t_A,t_B). \tag{111}$$

11. Marchenko-type focusing functions

11.1. Space-dependent material

Building on work by Rose [81,82], geophysicists used the Marchenko equation to develop methods for retrieving the wave field inside a space-dependent material from reflection measurements at its boundary [83–88]. Focusing functions play a central role in this methodology. For a 1D space-dependent material, a Marchenko-type focusing function $F_x(x,x_0,t)$ is defined as a specific solution of the wave equation, which focuses at the focal point $x = x_0$ (i.e., $F_x(x_0,x_0,t) \propto \delta(t)$) and which propagates unidirectionally through the focal point. It has recently been shown that there exists a close relation between focusing functions and the propagator matrix [89,90]. Here we briefly review this relation for a space-dependent material with parameters $\alpha(x)$ and $\beta(x)$. We start by noting that the elements $W_x^{P,P}(x,x_0,t)$ and $W_x^{Q,Q}(x,x_0,t)$ are symmetric functions of time, whereas $W_x^{P,Q}(x,x_0,t)$ and $W_x^{Q,P}(x,x_0,t)$ are asymmetric functions of time. This follows simply from Eq. (107) and the expressions for matrices $\mathbf{W}_x$ and $\mathbf{J}$ in Eqs. (73) and (65). For elements $W_x^{P,P}(x,x_0,t)$ and $W_x^{P,Q}(x,x_0,t)$ these symmetry properties are also clearly seen in Fig. 3a and b, respectively. Exploiting these symmetries, Marchenko-type focusing functions can be expressed in terms of the elements of the propagator matrix, according to [90]

$$F_x^P(x,x_0,t) = W_x^{P,P}(x,x_0,t) - \frac{1}{\eta(x_0)}W_x^{P,Q}(x,x_0,t), \tag{112}$$

$$F_x^Q(x,x_0,t) = W_x^{Q,P}(x,x_0,t) - \frac{1}{\eta(x_0)}W_x^{Q,Q}(x,x_0,t). \tag{113}$$

From these expressions and Eqs. (71) and (73) we obtain the following focusing conditions for $x = x_0$

$$F_x^P(x_0,x_0,t) = \delta(t), \tag{114}$$

$$F_x^Q(x_0,x_0,t) = -\frac{1}{\eta(x_0)}\delta(t). \tag{115}$$

Fig. 4 shows an x,t-diagram of $F_x^P(x,x_0,t)$ (convolved with a temporal wavelet with a central frequency $\omega_0/2\pi = 300$ kHz) for a focal point at $x_0 = 40$ mm. The interpretation is as follows. The four blue arrows in the right-most slab indicate leftward propagating

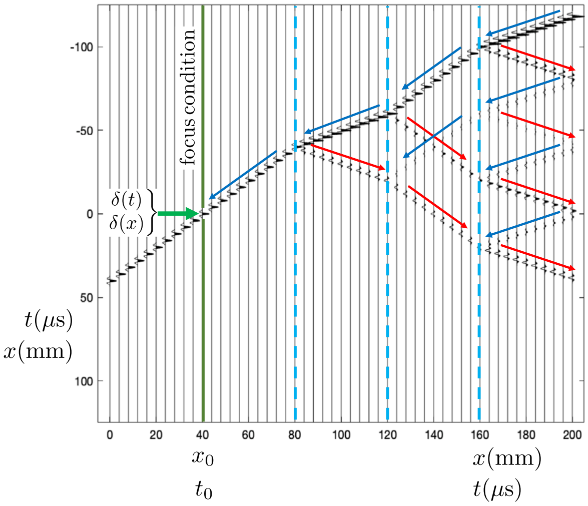


Fig. 4. Focusing function $F_x^P(x, x_0, t)$ (convolved with a temporal wavelet) for a piecewise homogeneous space-dependent material. The label at the vertical axis denotes time (in μs) and that at the horizontal axis denotes space (in mm). With interchanged labels and reversed blue arrows this figure can be interpreted as $F_t^U(x, t, t_0)$ (convolved with a spatial wavelet) for a piecewise constant time-dependent material.

waves that are emitted into the material from the right, at $x = x_N = 200$ mm. After interaction with the boundaries between the homogeneous slabs, a single leftward propagating wave arrives at $x = x_0 = 40$ mm, where it obeys the focusing condition of Eq. (114). The red arrows indicate the rightward propagating scattered part of the focusing function $F_x^P(x, x_0, t)$.

Finally, note that the elements of the propagator matrix can be expressed in terms of the Marchenko-type focusing functions, according to

$$W_x^{P,P}(x, x_0, t) = \frac{1}{2}\{F_x^P(x, x_0, t) + F_x^P(x, x_0, -t)\}, \tag{116}$$

$$W_x^{P,Q}(x, x_0, t) = -\frac{\eta(x_0)}{2}\{F_x^P(x, x_0, t) - F_x^P(x, x_0, -t)\}, \tag{117}$$

$$W_x^{Q,P}(x, x_0, t) = \frac{1}{2}\{F_x^Q(x, x_0, t) - F_x^Q(x, x_0, -t)\}, \tag{118}$$

$$W_x^{Q,Q}(x, x_0, t) = -\frac{\eta(x_0)}{2}\{F_x^Q(x, x_0, t) + F_x^Q(x, x_0, -t)\}. \tag{119}$$

11.2. Time-dependent material

The relations between the Marchenko-type focusing functions and the propagator matrix for a time-dependent material with parameters $\alpha(t)$ and $\beta(t)$ follow from those in Section 11.1 by applying the mapping of Eq. (16), hence

$$F_t^U(x, t, t_0) = W_t^{U,U}(x, t, t_0) - \eta(t_0)W_t^{U,V}(x, t, t_0), \tag{120}$$

$$F_t^V(x, t, t_0) = W_t^{V,U}(x, t, t_0) - \eta(t_0)W_t^{V,V}(x, t, t_0), \tag{121}$$

with focusing conditions for $t = t_0$

$$F_t^U(x, t_0, t_0) = \delta(x), \tag{122}$$

$$F_t^V(x, t_0, t_0) = -\eta(t_0)\delta(x). \tag{123}$$

Fig. 4 shows an x, t-diagram of $F_t^U(x, t, t_0)$ (convolved with a spatial wavelet with a central wavenumber $k_0/2\pi = 300 * 10^3$ km^{-1}) for a focal time at $t_0 = 40$ μs. For a correct interpretation, the direction of the blue arrows should be reversed. Starting with a leftward propagating wave at $t = t_0 = 40$ μs, the response at $t = t_N = 200$ μs, i.e., $F_t^U(x, t_N, t_0)$, consists of leftward (blue) and rightward (red) propagating waves. Using the mapping of Eq. (16), this response is identical to the focusing function $F_x^P(x_N, x_0, t)$ of the complimentary space-dependent material. Van Manen et al. [22] exploit this property to design an acoustic space–time material which computes its own inverse.

12. Conclusions

We have discussed and compared some fundamental aspects of waves in space-dependent and in time-dependent materials. The basic equations for a 1D space-dependent material can be transformed into those for a time-dependent material and vice-versa by interchanging the space and time coordinates and by applying a specific mapping of wave field components, material parameters and source terms. When the boundary and initial conditions can be transformed in the same way, then also the solutions of the equations for space-dependent and time-dependent materials can be transformed into one another by the same mapping. When the boundary and initial conditions cannot be transformed in the same way, then the solutions are different.

Green's functions in space-dependent and in time-dependent materials obey the same causality condition (i.e., they are zero before the action of the source) and therefore they cannot be transformed into one another. We have derived a source–receiver reciprocity relation for time-dependent materials, which relates a causal Green's function to an acausal Green's function with interchanged time coordinates. This is different from the source–receiver reciprocity relation for space-dependent materials, which interrelates two causal Green's functions with interchanged space coordinates. We have also derived a new representation for retrieving Green's functions from the space correlation of passive measurements in time-dependent materials. Unlike the corresponding representation for space-dependent materials it is single-sided, meaning that it suffices to correlate two responses (at two time instants) to sources at a single time instant.

Propagator matrices in space-dependent and in time-dependent materials obey boundary and initial conditions which can be transformed into one another in the same way as the underlying wave equations. Hence, these propagator matrices are interrelated in the same way. This also applies to representations and reciprocity theorems involving propagator matrices, and to Marchenko-type focusing functions, which can be expressed as combinations of elements of the propagator matrix.

CRediT authorship contribution statement

Kees Wapenaar: Writing –original draft, Visualization, Software, Methodology, Investigation, Conceptualization. **Johannes Aichele:** Writing –review & editing, Investigation, Conceptualization. **Dirk-Jan van Manen:** Writing –review & editing, Investigation, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix. Reflection and transmission coefficients

A.1. Space-dependent material

We review reflection and transmission coefficients for a wave incident on a space boundary between two homogeneous time-invariant half-spaces. Although the derivation is well-known, it also serves as an introduction for the derivation of reflection and transmission coefficients of a time boundary in the next subsection.

For a homogeneous time-invariant source-free material, the wave equation is derived by substituting the constitutive Eqs. (3) and (4) with constant parameters α and β into Eqs. (1) and (2), and subsequently eliminating Q from these equations. We thus obtain

$$\frac{1}{c^2}\partial_t^2 P - \partial_x^2 P = 0, \tag{A.1}$$

with propagation velocity c given by Eq. (8). Once P is resolved from Eq. (A.1), Q follows from $\beta\partial_t Q = -\partial_x P$, and U and V follow from the constitutive Eqs. (3) and (4).

Consider two homogeneous time-invariant source-free half-spaces, separated by a space boundary, which we take for convenience at $x = 0$, see Fig. A.5a. The parameters of the half-spaces $x < 0$ and $x > 0$ are denoted with subscripts 1 and 2, respectively. In the half-space $x < 0$ we define a rightward propagating monochromatic incident field with unit amplitude and angular frequency ω_1. Using complex notation, we have

$$P_I(x,t) = \exp i(k_1 x - \omega_1 t), \quad \text{with} \quad k_1 = \frac{\omega_1}{c_1}, \tag{A.2}$$

$$Q_I(x,t) = \frac{1}{\eta_1} P_I(x,t), \quad \text{with} \quad \eta_1 = \beta_1 c_1 = \sqrt{\frac{\beta_1}{\alpha_1}}. \tag{A.3}$$

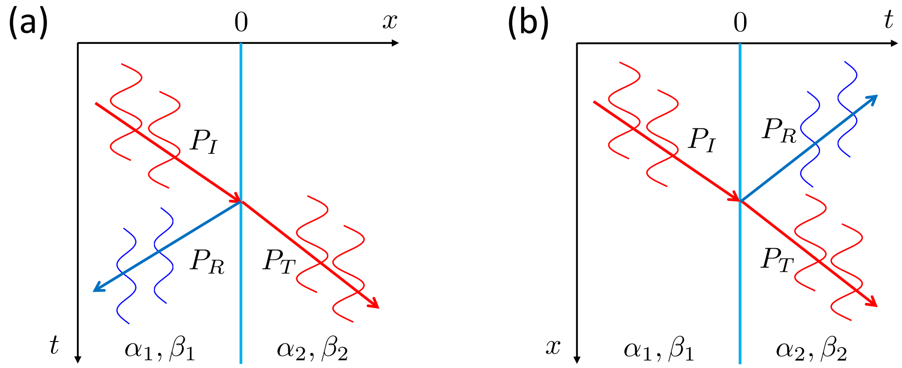

Fig. A.5. Incident (P_I), transmitted (P_T) and reflected (P_R) fields for the situation of a space boundary (a) and a time boundary (b). Note that the axes are interchanged between (a) and (b). Hence, waves propagating rightward and leftward along the x-axis are represented by rightward and leftward pointing arrows in (a) and by downward and upward pointing arrows in (b).

The rightward propagating transmitted field in the half-space $x > 0$ is defined as

$$P_T(x,t) = T_x \exp i(k_2 x - \omega_2 t), \quad \text{with} \quad k_2 = \frac{\omega_2}{c_2}, \tag{A.4}$$

$$Q_T(x,t) = \frac{1}{\eta_2} P_T(x,t), \quad \text{with} \quad \eta_2 = \beta_2 c_2 = \sqrt{\frac{\beta_2}{\alpha_2}}, \tag{A.5}$$

where T_x is the transmission coefficient. The subscript x denotes that this coefficient belongs to a space boundary. The leftward propagating reflected field in the half-space $x < 0$ is defined as

$$P_R(x,t) = R_x \exp i(-k_1 x - \omega_1 t), \tag{A.6}$$

$$Q_R(x,t) = -\frac{1}{\eta_1} P_R(x,t), \tag{A.7}$$

where R_x is the reflection coefficient. The boundary conditions at $x = 0$ state that P and Q are continuous, hence

$$P_I(0,t) + P_R(0,t) = P_T(0,t), \tag{A.8}$$

$$Q_I(0,t) + Q_R(0,t) = Q_T(0,t). \tag{A.9}$$

These equations should hold for all t, from which it follows that $\omega_2 = \omega_1$ and hence $k_2 = \frac{c_1}{c_2} k_1$, meaning that the wavenumber k_2 of the transmitted field is different from the wavenumber k_1 of the incident and reflected fields (unless of course when $c_2 = c_1$). Moreover, it follows that

$$R_x = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \tag{A.10}$$

$$T_x = \frac{2\eta_2}{\eta_2 + \eta_1}, \tag{A.11}$$

which are the well-known expressions for the reflection and transmission coefficients of a space boundary. Note that

$$\frac{1}{\eta_1}(1 - R_x^2) = \frac{1}{\eta_2} T_x^2. \tag{A.12}$$

We define the net power-flux density in the x-direction as

$$j = \frac{1}{2} \Re\{P^* Q\}. \tag{A.13}$$

Substituting $P = P_I + P_R$, $Q = Q_I + Q_R$ for $x < 0$ and $P = P_T$, $Q = Q_T$ for $x > 0$ into Eq. (A.13) we find, using Eq. (A.12), that j is constant. Hence, the net power-flux density of a monochromatic wave field is conserved when traversing a space boundary.

A.2. Time-dependent material

We review reflection and transmission coefficients for a wave incident on a time boundary between two homogeneous time-invariant "half-times" [1,2,13,15]. We take the time boundary for convenience at $t = 0$, see Fig. A.5b. The parameters for $t < 0$ and $t > 0$ are denoted with subscripts 1 and 2, respectively. The monochromatic incident and transmitted fields are again given by Eqs. (A.2)–(A.5), this time for $t < 0$ and $t > 0$, respectively, and with T_x replaced by T_t, with subscript t denoting that this coefficient belongs to a time boundary. Due to causality, the reflected field will not propagate back in time. Instead, the leftward propagating reflected field for $t > 0$ (indicated by the upward pointing arrow in Fig. A.5b) is defined as

$$P_R(x,t) = R_t \exp i(k_2 x + \omega_2 t), \tag{A.14}$$

$$Q_R(x,t) = -\frac{1}{\eta_2} P_R(x,t), \tag{A.15}$$

where R_t is the reflection coefficient. In the literature on waves in time-dependent materials there has been debate whether the tangential electric and magnetic flux densities (D_y, D_z, B_y and B_z) or the tangential electric and magnetic field strengths (E_y, E_z, H_y and H_z) should be continuous at a time boundary. Refs. [1,2,13,15] are proponents of continuity of the flux densities and Ref. [3] is a proponent of continuity of the field strengths. Today the consensus is that the flux densities are continuous at a time boundary, see Refs. [1,13,15] for a clear explanation. According to Table 1 the flux densities are examples of the U and V fields. Taking U and V continuous at $t = 0$ we obtain (using Eqs. (3) and (4))

$$\alpha_1 P_I(x,0) = \alpha_2\{P_R(x,0) + P_T(x,0)\}, \tag{A.16}$$

$$\beta_1 Q_I(x,0) = \beta_2\{Q_R(x,0) + Q_T(x,0)\}. \tag{A.17}$$

These equations should hold for all x, from which it follows that $k_2 = k_1$ and hence $\omega_2 = \frac{c_2}{c_1}\omega_1$, meaning that the frequency ω_2 of the transmitted and reflected fields is different from the frequency ω_1 of the incident field [6,13,15] (unless of course when $c_2 = c_1$). Moreover, it follows that

$$R_t = \frac{1}{2}\left(\frac{\alpha_1}{\alpha_2} - \frac{c_2}{c_1}\right), \tag{A.18}$$

$$T_t = \frac{1}{2}\left(\frac{\alpha_1}{\alpha_2} + \frac{c_2}{c_1}\right). \tag{A.19}$$

These are expressions for the reflection and transmission coefficients of a time boundary. For TE waves, substituting the material parameters from row 1 of Table 1, it is easily seen that these expressions are the same as those given by Eqs. (4) and (5) in [1]. Note that

$$\frac{\alpha_1}{c_1} = \frac{\alpha_2}{c_2}(T_t^2 - R_t^2). \tag{A.20}$$

We define the net field-momentum density in the x-direction as [21,40]

$$M = \frac{1}{2}\Re\{U^* V\}. \tag{A.21}$$

Substituting $U = \alpha_1 P_I$, $V = \beta_1 Q_I$ for $t < 0$ and $U = \alpha_2(P_R + P_T)$, $V = \beta_2(Q_R + Q_T)$ for $t > 0$ into Eq. (A.21) we find, using Eq. (A.20), that M is constant. Hence, the net field-momentum density of a monochromatic wave field is conserved when traversing a time boundary.

Using Eqs. (3), (4) and (8), the net power-flux density in the x-direction can be expressed as $j = c^2 M$. Since M is constant, we find $j_2 = \frac{c_2^2}{c_1^2} j_1$. This is the discrete counterpart of Eq. (33) for a continuously varying time-dependent material.

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