THE INFINITE APERTURE PARADOX

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ABSTRACT

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Given an integral formulation for forward wave field extrapolation of data recorded on an *infinite aperture*, the formulation for inverse wave field extrapolation can be arrived at in two ways:
1. Exploiting the time symmetry of the wave equation, the forward-propagating Green's function may be replaced by a backward-propagating Green's function. This approach seems to be exact.
2. Alternatively, the spatial convolution operator for forward extrapolation may be replaced by a *de*convolution operator for inverse extrapolation. This deconvolution approach is spatially band-limited so it seems not to be exact.

Since both approaches lead to identical expressions for inverse extrapolation, a *paradox* has arisen. In this paper this paradox is unambiguously eliminated by a space-frequency domain analysis of the inverse wave field extrapolation integral.

KEY WORDS: inverse extrapolation, artefacts, backpropagation, infinite aperture.

INTRODUCTION

It is well known that wave field extrapolation of data recorded on a *finite* aperture is not exact. The extrapolated wave field contains artefacts which can be kinematically explained as ghost wave fields radiated by secondary sources located at the endpoints of the aperture (Fig. 1). Suppose now that the data would be recorded on an infinite aperture. Then *forward* wave field extrapolation would be exact: the ghost wave field of the secondary sources at the 'endpoints' would vanish when these endpoints were moved towards infinity.

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Since the wave equation is symmetrical in time one would expect a similar conclusion for *inverse* extrapolation.

Now, let us see whether or not this expectation is fulfilled. In a classical paper, Schneider (1978) presents an integral formulation for wave field extrapolation in the space-time domain. Starting with the exact Kirchhoff integral for forward extrapolation, he introduces inverse extrapolation essentially by changing the direction of time in the Green's function. The resulting backpropagating Green's function exploits the time symmetry of the acoustic wave equation and it seems that no approximations are introduced by this simple modification. In another classical paper, Berkhout and Van Wulfften Palthe (1979) derive inverse extrapolation as a spatial *de*convolution process in the space-frequency domain. In their appendix C they show that this deconvolution process is spatially bandlimited (evanescent waves are neglected). Hence, it seems to be not exact, even when the size of the aperture is infinite. It is interesting to note that their deconvolution operator, transformed back to the time domain, is identical to Schneider's operator.

We thus see that these two different interpretations of the same inverse extrapolation operator (an exact backpropagating Green's function versus a bandlimited deconvolution operator) constitute a paradox: inverse wave field extrapolation is either exact or not exact but obviously it cannot be both!
In our paper on inverse wave field extrapolation (Wapenaar et al., 1989) we eliminate this paradox as follows. Our starting point for the derivation of the inverse extrapolation operator is the Kirchhoff-Helmholtz integral with a backpropagating Green's function.

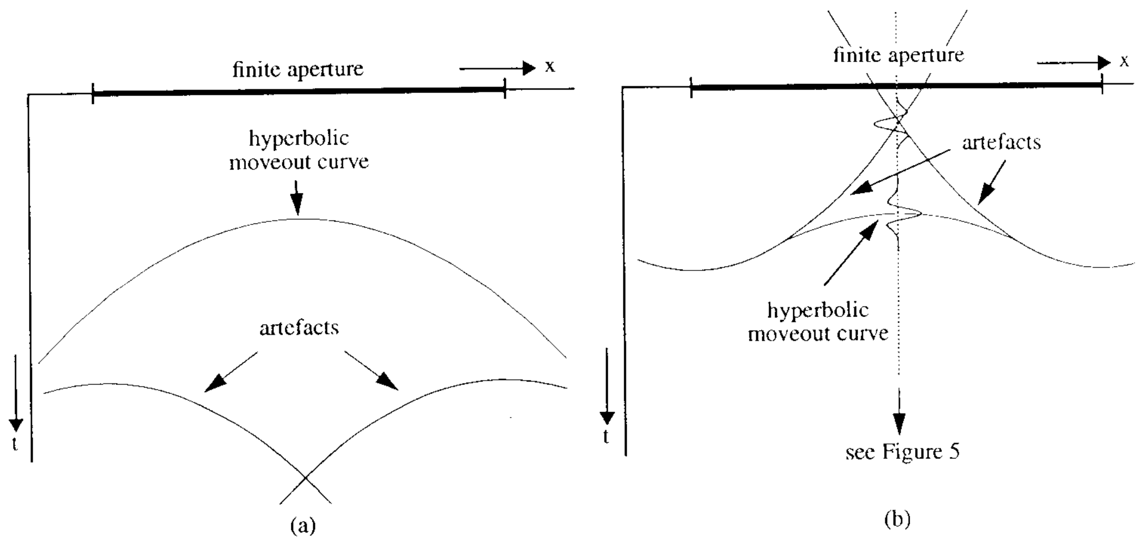

Fig. 1. Forward (a) and inverse (b) wave field extrapolation of a point source response recorded on a finite aperture. Note that both results contain artefacts as a result of the finite aperture.

This is essentially the space-frequency domain equivalent of Schneider's expression in the space-time domain. An important difference, however, is that we start with a *closed* surface integral, whereas (for inverse extrapolation) Schneider starts right away with an open surface integral over the infinite aperture. The closed surface integral is doubtlessly exact; however, by going from a closed to an open surface, a part of the surface integral is neglected. In our 1989 paper we prove (via the wavenumber-frequency domain), that the neglected part of the integral is not zero for evanescent waves. Omission thus introduces approximations that are equivalent with those of the bandlimited deconvolution approach of Berkhout and Van Wulfften Palthe. This eliminates the paradox and in 1989 we could not imagine any objections to our derivations. However, after the publication of our 1989 paper, we received a lot of comments from many colleagues in the geophysical community, who disagreed with our conclusions. Apparently, our derivation via the wavenumber-frequency domain was not convincing enough to settle the dispute. The aim of this paper is to present an alternative proof that inverse wave field extrapolation using backward propagating Green's functions is *not* exact. To avoid discussions concerning the branches of complex square roots in the wavenumber-frequency domain, the analysis is presented entirely in the *space*-frequency domain. It will turn out that the finite aperture artefacts pictured in Fig. 1b do *not* vanish for the situation of an infinite aperture. The remaining artefacts may be kinematically explained as (plane) ghost wave fields, backpropagated from secondary sources at infinity (Fig. 2b). It will also be shown that the amplitude of these ghost wave fields can be significant.

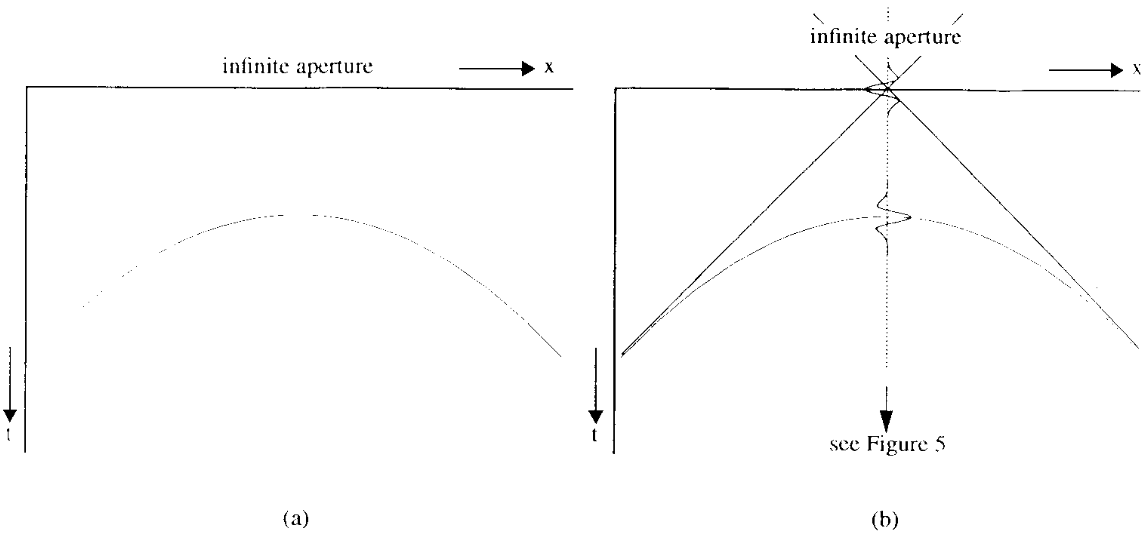

Fig. 2. Forward (a) and inverse (b) wave field extrapolation of a point source response recorded on an infinite aperture. Note that the inverse extrapolation results contains artefacts, in spite of the infinite aperture.

REPRESENTATION INTEGRALS

Throughout the analysis we consider for simplicity an infinite homogeneous lossless acoustic medium, with propagation velocity c. In this medium we consider a source-free volume V enclosed by a surface S with outward-pointing normal vector **n**. In the space-frequency domain, the acoustic pressure P at any point (x_r, y_r, z_r) in V can be expressed in terms of the acoustic pressure P and its gradient on S by either one of the following two representation integrals

$$P(x_r, y_r, z_r, \omega) = \oint_S [G\nabla P - P\nabla G] \cdot \mathbf{n} dS \tag{1a}$$

(Morse and Feshbach, 1953), or, exploiting the time symmetry of the wave equation

$$P(x_r, y_r, z_r, \omega) = \oint_S [G^*\nabla P - P\nabla G^*] \cdot \mathbf{n} dS \tag{1b}$$

(Bojarski, 1983), where G is the free-space Green's function, given by

$$G = e^{-jkr} / 4\pi r \ , \tag{2}$$

with $r = \sqrt{[(x_r - x)^2 + (y_r - y)^2 + (z_r - z)^2]}$ and $k = \omega/c$, ω being the angular frequency. Note that both expressions (1a) and (1b) are exact. They will be used to analyse the forward and inverse wave field extrapolation processes, respectively.

THE CONFIGURATION

In the following the closed surface S will consist of two horizontal infinite surfaces at depths $z = z_{-1}$ and $z = z_1$, respectively, and a cylindrical surface with a vertical axis through the origin of the coordinate system and infinite radius (Fig. 3). The contribution from the cylindrical surface to the integrals (1a) and (1b) vanishes (the area of this surface is proportional to the radius ϱ_{max}, the integrant is proportional to $1/\varrho_{max}^2$). Hence, for this configuration the integrals (1a) and (1b) can be written as

$$P(x_r, y_r, z_r, \omega) = P_{-1}^{(m)}(x_r, y_r, z_r, \omega) + P_1^{(m)}(x_r, y_r, z_r, \omega) \ , \tag{3a}$$

where

$$P_n^{(m)}(x_r, y_r, z_r, \omega) = n \int\int_{-\infty}^{\infty} [G^{(m)}\partial P/\partial z - P\partial G^{(m)}/\partial z]_{z_n} dxdy, \qquad (3b)$$

for m = −1, 1 and n = −1, 1, with

$$G^{(m)} = \begin{cases} G & (\text{for } m = 1) \\ G^* & (\text{for } m = -1) \end{cases} \qquad (3c)$$

Note that the integer n in front of the integral in (3b) accounts for the direction of the outward-pointing normal vector at depth levels z_n for n = −1, 1.

The integral in (3b) can be solved most conveniently if we can make use of cylindrical symmetry. To this end we choose a single monopole source for P on the z-axis at $(0, 0, z_s)$, with $z_s > z_1$ (Fig. 3). The source function is given by $S(\omega)$. Then, in cylindrical coordinates, we have

$$P = S(\omega)\, e^{-jks} / 4\pi s , \qquad (4)$$

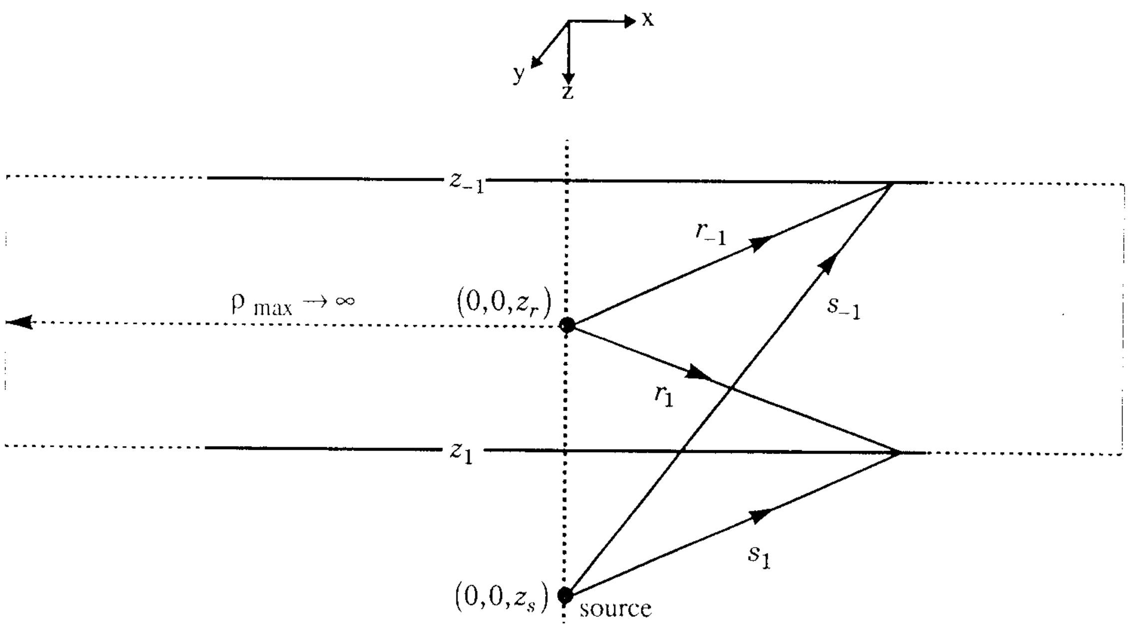

Fig. 3. Configuration for which the representation integrals are evaluated. Note that $z_{-1} < z_r < z_1 < z_s$.

with $s = \sqrt{[\varrho^2 + (z_s - z)^2]}$ and $\varrho = \sqrt{(x^2 + y^2)}$. Moreover, we evaluate the integral only for $(x_r, y_r, z_r) = (0, 0, z_r)$, with $z_{-1} < z_r < z_1$ (Fig. 3). Hence, for the Green's function we may write

$$G^{(m)} = e^{-jkmr} / 4\pi r \ , \tag{5}$$

with $r = \sqrt{[\varrho^2 + (z_r - z)^2]}$. For the configuration of Fig. 3 we speak of *forward* extrapolation when an integration over the lower surface $z = z_1$ describes the wave field at $(0, 0, z_r)$ of the source at $(0, 0, z_s)$. Similarly, we speak of *inverse* extrapolation when the wave field at $(0, 0, z_r)$ is obtained by an integration over the upper surface $z = z_{-1}$. Since equation (3) describes in general an integration over both surfaces, a careful analysis of the different contributions is required.

ANALYTICAL SOLUTION OF THE INTEGRALS

Using the cylindrical symmetry of P and $G^{(m)}$, equation (3) may be rewritten as

$$P(0, 0, z_r, \omega) = P_{-1}^{(m)}(0, 0, z_r, \omega) + P_1^{(m)}(0, 0, z_r, \omega) \ , \tag{6a}$$

where

$$P_n^{(m)}(0, 0, z_r, \omega) = 2\pi n \int_0^\infty [G^{(m)}\partial P/\partial z - P\partial G^{(m)}/\partial z]_{z_n} \varrho d\varrho \ . \tag{6b}$$

Upon substitution of equations (4) and (5) into (6b) we obtain

$$P_n^{(m)}(0, 0, z_r, \omega) = S(\omega)n/8\pi \int_0^\infty [\{(z_s - z_n)(1 + jks_n)/s_n^2$$
$$- (z_r - z_n)(1 + jkmr_n)/r_n^2\} \, e^{-jk(s_n + mr_n)}/s_n r_n] \, \varrho d\varrho, \tag{7}$$

with $s_n = \sqrt{[\varrho^2 + (z_s - z_n)^2]}$ and $r_n = \sqrt{[\varrho^2 + (z_r - z_n)^2]}$ (Fig. 3). The argument of the exponential is shown as a function of ϱ for $m = 1$ and $m = -1$ in Fig. 4. Note that for $m = 1$ it is stationary only for $\varrho = 0$, whereas for $m = -1$ it is stationary for $\varrho = 0$ *and* $\varrho = \infty$. This explains the fundamental difference between forward and inverse extrapolation, as will be shown in the following sections. The solution of the integral in equation (7) reads

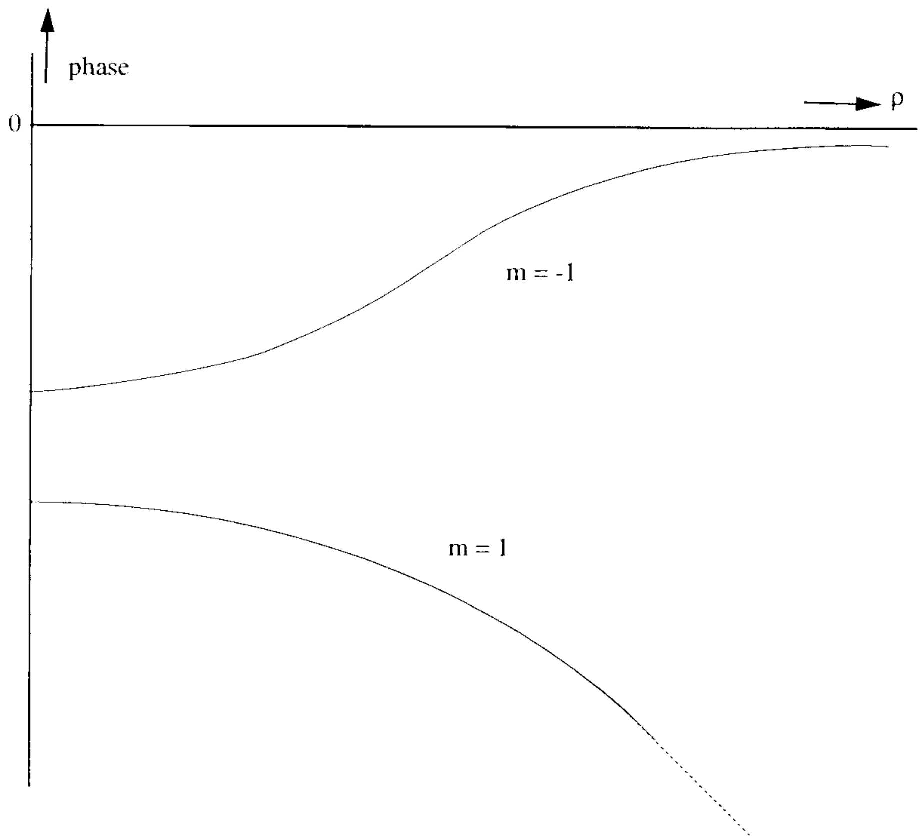

Fig. 4. Argument of the exponential in the integral (7) when using the forward (m = 1) and backward (m = − 1) propagating Green's function. The stationary behaviour at $\varrho = \infty$ for m = − 1 is responsible for the artefacts in inverse extrapolation.

$$P_n^{(m)}(0, 0, z_r, \omega) = S(\omega)n/8\pi \, [\{-m(z_s - z_n)/s_n + (z_r - z_n)/r_n\} \, e^{-jk(s_n + m r_n)}/(s_n + m r_n)]_{\varrho=0}^{\varrho=\infty} . \quad (8)$$

FORWARD EXTRAPOLATION

Choosing m = 1 means that the forward-propagating Green's function is employed (see equation 3c). Now in equation (8) the contribution for $\varrho = \infty$ vanishes, or, using the terminology of the introduction, the ghost wave field of the 'secondary sources at infinity' vanishes. For evaluating the contribution for $\varrho = 0$ we use the property $z_{-1} < z_r < z_1 < z_s$ (Fig. 3). The result is

$$P_n^{(1)}(0, 0, z_r, \omega) = S(\omega)n/8\pi \left[(1+n) \frac{e^{-jk[(z_s - z_n) - n(z_r - z_n)]}}{(z_s - z_n) - n(z_r - z_n)} \right] . \quad (9)$$

Hence,

$$P(0, 0, z_r, \omega) = P_{-1}^{(1)}(0, 0, z_r, \omega) + P_1^{(1)}(0, 0, z_r, \omega) , \qquad (10a)$$

with

$$P_{-1}^{(1)}(0, 0, z_r, \omega) = 0 \qquad (10b)$$

and

$$P_1^{(1)}(0, 0, z_r, \omega) = S(\omega)\, e^{-jk(z_s - z_r)} / 4\pi(z_s - z_r) . \qquad (10c)$$

From this result we may conclude that $P(0, 0, z_r, \omega)$ equals the exact monopole response at a distance $z_s - z_r$ from the source. Moreover, this exact result is fully obtained from the integral over the infinite horizontal surface at $z = z_1$ between the source at $(0, 0, z_s)$and the receiver at $(0, 0, z_r)$ (Fig. 3). This confirms our expectation of *forward* extrapolation from an infinite aperture.

INVERSE EXTRAPOLATION

Choosing $m = -1$ means that the backward propagating Green's function is employed (see equation 3c). Now in equation (8) the contribution for $\varrho = \infty$ does not vanish, or, using the terminology of the introduction, the backpropagated ghost wave field of the 'secondary sources at infinity' does not vanish. Evaluating this contribution involves an expansion of the square roots s_n and r_n for large ϱ. The result is

$$P_n^{(-1)}(0, 0, z_r, \omega) = S(\omega)n/8\pi \left[\frac{2(z_s - z_n) + 2(z_r - z_n)}{(z_s - z_n)^2 - (z_r - z_n)^2} - (1-n) \frac{e^{-jk[(z_s - z_n) + n(z_r - z_n)]}}{(z_s - z_n) + n(z_r - z_n)} \right] . \qquad (11)$$

Hence,

$$P(0, 0, z_r, \omega) = P_{-1}^{(-1)}(0, 0, z_r, \omega) + P_1^{(-1)}(0, 0, z_r, \omega) , \qquad (12a)$$

with

$$P_{-1}^{(-1)}(0, 0, z_r, \omega) = S(\omega)\, \{e^{-jk(z_s - z_r)} - 1\}/4\pi(z_s - z_r) \qquad (12b)$$

and

$$P_1^{(-1)}(0, 0, z_r, \omega) = S(\omega) / 4\pi(z_s - z_r) . \qquad (12c)$$

Note that again $P(0, 0, z_r, \omega)$ equals the exact monopole response at a distance $z_s - z_r$ from the source, as expected. However, for inverse extrapolation normally only the integral over the surface at $z = z_{-1}$ would be evaluated (Fig. 3). Hence, $P_{-1}^{(-1)}(0, 0, z_r, \omega)$, as given by equation (12b), represents the inverse extrapolation result for the situation of an infinite aperture at $z = z_{-1}$. Obviously, this result is *not* exact. The artefact has the same amplitude as the monopole response. Note that the time-domain equivalent of equation (12b) reads

$$p_{-1}^{(-1)}(0, 0, z_r, t) =$$

$$1 / 4\pi(z_s - z_r)\,[s\{t - (z_s - z_r)/c\} - s(t)]\,, \tag{13}$$

which means that the artefact shows up as a ghost event with zero time lag. This is seen in Fig. 2b at the point where the asymptotes of the hyperbola cross each other. The remaining part of the artefacts in Fig. 2b can be derived with a stationary phase analysis, which is beyond the scope of this paper.

EXAMPLE

We discuss a numerical experiment which shows the evolution of the artefact as a function of the aperture size. The upper limit of the integral in (7) is replaced by ϱ_{max} (the radius of the aperture) and the integral is evaluated *numerically* for different values of ϱ_{max}, ranging from 100 m to 51.2 km. Furthermore, we have chosen $m = -1$, $n = -1$, $c = 1200$ m/s, $z_{-1} = 0$ m, $z_r = 200$ m, $z_s = 400$ m, $\Delta\varrho = 10$ m. $S(\omega)$ is a zero-phase source function from 0 to 60 Hz. The results are shown in the time domain as a function of ϱ_{max} in Fig. 5. As expected, the result converges to the analytically derived result for $\varrho_{max} = \infty$ (equation 13).

MONOPOLE VERSUS DIPOLE SOURCES

In the examples that we considered so far, the source for the acoustic wave field P was a monopole source at $(0, 0, z_s)$. When we replace this source by a dipole source, we may expect that the 'secondary sources at infinity' are weaker and, consequently, that the inverse wave field extrapolation artefacts are less severe. (The directivity of the dipole source acts as a natural taper). To obtain the wave field $\hat{P}$ of a dipole source at $(0, 0, z_s)$, we apply the operator $-\partial/\partial z_s$ to the right-hand side of equation (4). Following the same steps as before, we eventually find that the inverse extrapolation result is obtained by applying the same operator to the right-hand side of equation (12b) (bear in mind that $G^{(m)}$ does not depend on z_s). This gives

$$\hat{P}^{(-1)}_{-1}(0, 0, z_r, \omega) = S(\omega)[\{1 + jk(z_s - z_r)\} e^{-jk(z_s - z_r)} - 1]$$
$$/ 4\pi(z_s - z_r)^2 . \quad (14)$$

Note that the artefact of this inverse extrapolated dipole response is relatively less severe. In the high frequency limit it vanishes completely.

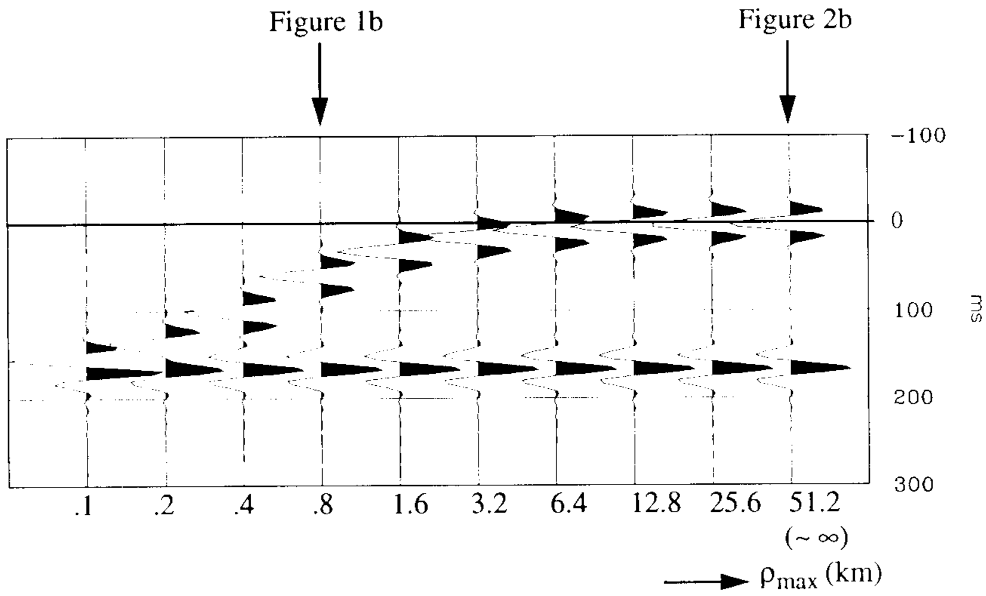

Fig. 5. Numerical evaluation of the integral (7), with the upper limit replaced by ϱ_{max}. The result converges to the analytically derived result for $\varrho_{max} = \infty$.

DISCUSSION

By considering the Sommerfeld integral (Aki and Richards, 1980), it can be shown that equations (12b) and (12c) represent the *propagating* and *evanescent* wave field contributions, respectively, of the monopole source at (0, 0, z_s). In other words, the integral over $z = z_{-1}$ accounts for the propagating wave field whereas the integral over $z = z_1$ accounts (in a stable manner) for the evanescent wave field (see Fig. 6). This is consistent with the analysis in our paper on inverse wave field extrapolation (Wapenaar et al., 1989) where we show that omitting the integral over $z = z_1$ is equivalent with neglecting the evanescent wave field. It is also consistent with the analysis of Berkhout and Van Wulfften Palthe (1979), who show that the spatial deconvolution process (i.e., the integral over $z = z_{-1}$) implies a spatial bandlimitation (propagating waves only) and, consequently, a limited spatial resolution. Hence, apart from the explanation in the foregoing sections, the artefacts that occur in inverse extrapolation can also be explained as a result of ignoring the evanescent wave field. In principle, a part of the evanescent wave field could be recovered from surface z_{-1}, provided that the distance to the source is small. This is what actually happens in near-field acoustic holography (Maynard et al., 1985). For this application the inverse operator is considered as a broad-band spatial deconvolution operator; it is derived by taking the *reciprocal* of the forward operator in the wavenumber-frequency domain. Applying this operator yields an improved lateral resolution. Following the arguments of Berkhout and Van Wulfften Palthe (1979), this approach is not recommended for seismic applications since in seismic data the evanescent wave field is generally far below the noise level.

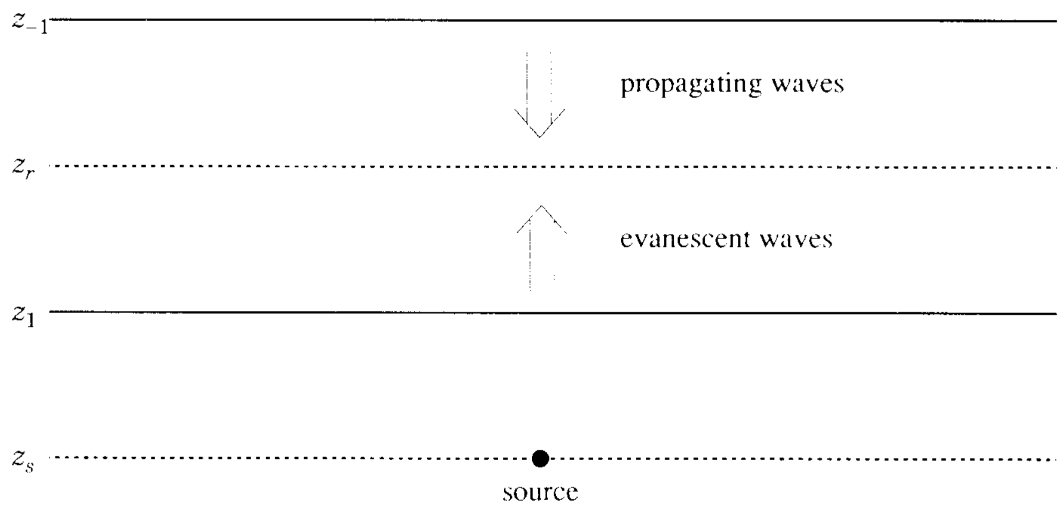

Fig. 6. The representation integral (with the backpropagating Green's function) over z_{-1} and z_1 yields the exact wave field at z_r. In inverse extrapolation from z_{-1} to z_r the integral over z_1 is ignored, which means that only the propagating part of the wave field is reconstructed (band limited extrapolation).

CONCLUSIONS

It has been shown that inverse wave field extrapolation using the backward propagating Green's function is not exact, even when an infinite aperture is available. Using a space-frequency domain analysis, it has been shown that the artefacts can be explained as ghost wave fields, backpropagated from secondary sources at the aperture 'endpoints' at infinity. An important consequence is that finite aperture artefacts that occur in practice cannot be removed by artificially extending the aperture towards infinity, unless this is accompanied by some taper. Note that although the analysis is carried out in the space-frequency domain, the conclusions apply equally well for inverse extrapolation techniques in the space-time domain. Hence, also the reverse time extrapolation approach, as advocated by McMechan (1983), suffers from the same artefacts when the size of the aperture is infinitely large. This is easily understood when one bears in mind that reverse time extrapolation is equivalent with Schneider's integral method with the time-reversed Green's function (see Esmersoy and Oristaglio, 1988). It has been argued that the artefacts can also be explained as a result of the negligence of the evanescent wave field in inverse wave field extrapolation (band-limited extrapolation). Apparently, employing the backpropagating Green's function for inverse wave field extrapolation is equivalent with ignoring the evanescent wave field. This explains the stability of the inverse wave field extrapolation process. For simplicity, we have restricted the analysis to the situation of a homogeneous medium. However, similar conclusions apply to the situation of an inhomogeneous medium where, in addition, the use of backpropagating Green's functions gives rise to amplitude errors that are proportional to the squared reflectivity of the interfaces in the medium (Wapenaar et al., 1989).

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