JASA ARTICLE

On the relation between time-reversed acoustics and Green's function retrieval in space-variant and in time-variant materials

Kees Wapenaar,[1,a)] Johannes Aichele,[2] and Dirk-Jan van Manen[2]
[1]Delft University of Technology, Department of Geoscience and Engineering, Stevinweg 1, 2628 CN Delft, The Netherlands
[2]ETH Zürich, Institute of Geophysics, Sonneggstrasse 5, 8092 Zürich, Switzerland

ABSTRACT:
The methods of time-reversed acoustics and Green's function retrieval are traditionally deployed for classical inhomogeneous, time-invariant materials. The mutual relation between these methods is well established. Recently, similar methods have been proposed for homogeneous, time-variant materials. Here, we investigate their mutual relation and their relation with the corresponding methods in classical materials. For this analysis, we make use of the fact that the wave equations for both classes of material are similar, with the roles of time and space interchanged. However, the principle of causality holds for both classes of material; hence, here the roles of time and space are not interchanged. We find that (1) whereas classical time-reversed acoustics involves emission of a time-reversed single-component wave field from a (ideally closed) boundary into the inhomogeneous material, its idealized counterpart involves emission of a sign-reversed two-component wave field, recorded in a time-reversed material, from a single time instant into the actual time-variant material; and (2) whereas classical Green's function retrieval involves temporal cross-correlation of wave fields at two space locations in response to single-component sources on a (ideally closed) boundary, its counterpart involves spatial cross-correlation of wave fields at two time instants in response to two-component sources at a single time instant.
 https://doi.org/10.1121/10.0044236

(Received 26 January 2026; revised 28 April 2026; accepted 6 June 2026; published online 2 July 2026)

[Editor: Michael R. Haberman] Pages: 201–219

I. INTRODUCTION

Waves propagating through time-variant materials are scattered by time boundaries, just as waves propagating through space-variant materials are scattered by spatial boundaries. For electromagnetic waves, scattering by time boundaries has been studied for decades.[1–3] The research in this area expanded significantly since the introduction of dynamic metamaterials.[4–10] For acoustic waves, the phenomenon of time scattering has been introduced by Fink and co-workers.[11–15] In particular, they demonstrated theoretically and experimentally that a time boundary sends waves back in space to focus at their origin, similar to time-reversed waves that are sent back from a spatial boundary.[16]

Whether we consider acoustic or electromagnetic waves, there exists much analogy between wave phenomena in homogeneous, time-variant materials, and those in "classical" inhomogeneous, time-invariant materials.[3,17–23] In particular, the roles of time and space are interchanged between the wave equations for both classes of material. However, the analogy also has its limitations. For example, waves reflected by a time boundary propagate forward in time and backward in space, similar as waves reflected by a spatial boundary.[17,19,21] Hence, here the roles of time and space are not interchanged. More generally, the principle of causality is a time-related property, whether we consider time-variant or space-variant materials.[23] In other words, whereas the roles of time and space are interchanged between the wave equations for both classes of material, they are in general not interchanged in the initial and boundary conditions, and hence the relation between the solutions of the wave equations (i.e., the wave fields in both classes of material) is in general not straightforward. Another aspect that complicates the analogy is the fact that the number of time dimensions (one) is different from the number of space dimensions (two or three), except of course for 1D situations.

Despite the aforementioned limitations, we investigate the relation between time-reversed acoustics and Green's function retrieval in space-variant and in time-variant materials. For classical inhomogeneous, time-invariant materials, the research fields of time-reversed acoustics[16,24,25] and Green's function retrieval by temporal cross-correlation[26–31] originated independently. However, it was soon discovered that these fields are closely related.[32,33] It is reasonable to assume that a similar close relationship exists between time-reversed acoustics[11,12] and Green's function retrieval by spatial cross-correlation[23,34] in homogeneous, time-variant materials. In a companion paper,[35] Aichele investigates the relations between temporal and spatial cross-correlation,

a)Email: C.P.A.Wapenaar@TUDelft.nl

underlying time-reversed acoustics and Green's function retrieval. In that analysis, the material is kept space- and time-invariant, except for an impulsive temporal disruption, representing a time boundary. In this paper, we extend the analysis for materials that are either arbitrarily space-variant or arbitrarily time-variant.

After a brief discussion of the general equations for waves in space- and time-variant materials (Sec. II), in Sec. III we review classical time-reversed acoustics, Green's function retrieval and their mutual relation for arbitrarily inhomogeneous, time-invariant materials. This serves as an introduction to Sec. IV, where we discuss time-reversed acoustics, Green's function retrieval, and their mutual relation for homogeneous, arbitrarily time-variant materials. In that section, we also discuss the relation with the classical methods reviewed in Sec. III. We present conclusions in Sec. V.

II. GENERAL EQUATIONS FOR WAVES IN SPACE- AND TIME-VARIANT MATERIALS

Although in this paper we deal primarily with 2D and 3D acoustic waves, we use a unified notation that also covers 2D horizontally polarized shear (SH) waves, 2D transverse electric (TE), and 2D transverse magnetic (TM) waves (the latter two are relevant because much of the theory for waves in time-variant materials was initially developed for electromagnetic waves). We denote position in space by the Cartesian coordinate vector $\mathbf{x} = (x, y, z)$ (for the 3D situation) or $\mathbf{x} = (x, z)$ (for the 2D situation), and time by t. With reference to Table I, we consider space- and time-variant wave field quantities $U(\mathbf{x}, t)$, $\mathbf{V}(\mathbf{x}, t)$, $P(\mathbf{x}, t)$, and $\mathbf{Q}(\mathbf{x}, t)$, space- and time-variant material parameters $\alpha(\mathbf{x}, t)$ and $\beta(\mathbf{x}, t)$, and space- and time-variant source quantities $a(\mathbf{x}, t)$, $\mathbf{b}(\mathbf{x}, t)$. The boldface quantities $\mathbf{V}$, $\mathbf{Q}$, and $\mathbf{b}$ denote vectors with three components, i.e., $\mathbf{V} = (V_x, V_y, V_z)$ etc. (for the 3D situation), or with two components, i.e., $\mathbf{V} = (V_x, V_z)$ etc. (for the 2D situation). The wave field quantities obey the following unified equations:

$$\partial_t U + \nabla \cdot \mathbf{Q} = a, \tag{1}$$

$$\partial_t \mathbf{V} + \nabla P = \mathbf{b}, \tag{2}$$

where ∂_t stands for the partial differential operator $\partial/\partial t$, and ∇ is the nabla operator, defined as $\nabla = (\partial_x, \partial_y, \partial_z)$ (for the 3D situation) or $\nabla = (\partial_x, \partial_z)$ (for the 2D situation). For acoustic waves, using the quantities defined in row 1 of Table I, Eq. (1) is the acoustic deformation equation, $-\partial_t \Theta + \nabla \cdot \mathbf{v} = q$, and Eq. (2) stands for equilibrium of mechanical momentum, $\partial_t \mathbf{m} + \nabla p = \mathbf{f}$.[20,36]

Assuming the material is isotropic and instantaneously and locally reacting, the unified wave field quantities in Eqs. (1) and (2) are related via the constitutive relations:

$$U = \alpha P, \tag{3}$$

$$\mathbf{V} = \beta \mathbf{Q}. \tag{4}$$

For acoustic waves, these relations are $-\Theta = \kappa p$ and $\mathbf{m} = \rho \mathbf{v}$, respectively.

Equations (1)–(4) hold in materials of which the parameters $\alpha(\mathbf{x}, t)$ and $\beta(\mathbf{x}, t)$ vary continuously with space and time. They need to be supplemented with boundary conditions at those positions and times where these parameters are discontinuous. At a spatial boundary with normal $\mathbf{n}(\mathbf{x})$, where the parameters are discontinuous in space but continuous in time, the boundary conditions demand that P and $\mathbf{Q} \cdot \mathbf{n}$ are continuous (for acoustic waves, these are the well-known boundary conditions for the acoustic pressure p and the normal component of the particle velocity $\mathbf{v} \cdot \mathbf{n}$). At a time boundary, where the parameters are discontinuous in time but continuous in space, the boundary conditions demand that U and $\mathbf{V}$ are continuous.[1,3,17,18,21] Several authors discuss customized approaches for materials with specific combinations of spatial and time boundaries.[7,21,22] Note that at the intersection of a spatial boundary with a time boundary, the boundary conditions for the spatial boundary are inconsistent with those for the time boundary. This can be solved by the introduction of a field-dependent point-source at the intersection.[10] Other authors discuss "travelling-wave modulated" materials, which exhibit non-reciprocal wave behaviour.[37–42]

A further discussion of waves in materials in which the parameters vary with space and time is beyond the scope of this paper. In the following we consider either arbitrarily inhomogeneous, time-invariant materials, with parameters $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ (Sec. III), or homogeneous, arbitrarily time-variant materials, with parameters $\alpha(t)$ and $\beta(t)$ (Sec. IV).

TABLE I. Specification of the quantities in Eqs. (1)–(4). For 2D and 3D acoustic waves, the quantities are cubic dilatation Θ, mechanical momentum density m, acoustic pressure p, particle velocity v, compressibility κ, mass density ρ, volume-injection rate density q, and external force density f. For 2D SH (horizontally polarized shear) waves in the x–z plane, m, v, ρ, and f are defined the same as for acoustic waves, and the additional quantities are strain e, stress τ, shear modulus μ, and external deformation rate density h. For 2D TE (transverse electric) and 2D TM (transverse magnetic) waves in the x–z plane, the quantities are electric and magnetic flux densities D and B, electric and magnetic field strengths E and H, permittivity ε, permeability μ, and external electric and magnetic current densities $J^{\rm e}$ and $J^{\rm m}$.

	U	V_x	V_y	V_z	P	Q_x	Q_y	Q_z	α	β	a	b_x	b_y	b_z
2D/3D acoustic	$-\Theta$	m_x	m_y	m_z	p	v_x	v_y	v_z	κ	ρ	q	f_x	f_y	f_z
2D SH	m_y	$-2e_{yx}$	—	$-2e_{yz}$	v_y	$-\tau_{yx}$	—	$-\tau_{yz}$	ρ	$1/\mu$	f_y	$2h_{yx}$	—	$2h_{yz}$
2D TE	D_y	B_z	—	$-B_x$	E_y	H_z	—	$-H_x$	ε	μ	$-J_y^{\rm e}$	$-J_z^{\rm m}$	—	$J_x^{\rm m}$
2D TM	B_y	$-D_z$	—	D_x	H_y	$-E_z$	—	E_x	μ	ε	$-J_y^{\rm m}$	$J_z^{\rm e}$	—	$-J_x^{\rm e}$

JASA

https://doi.org/10.1121/10.0044236

III. TIME-REVERSED ACOUSTICS AND GREEN'S FUNCTION RETRIEVAL IN INHOMOGENEOUS, TIME-INVARIANT MATERIALS

In this section, we review classical time-reversed acoustics and Green's function retrieval in inhomogeneous, time-invariant materials, and their mutual relation. In Sec. III A, we discuss the wave equation for an inhomogeneous, time-invariant material, as a special case of the equations in Sec. II. In Sec. III B, we discuss the Green's function of an inhomogeneous, time-invariant material. In Sec. III C, we review a classical representation of the homogeneous Green's function of an inhomogeneous, time-invariant material. In Sec. III D and Sec. III E, we show how this homogeneous Green's function representation leads to time-reversed acoustics and Green's function retrieval, respectively, in inhomogeneous, time-invariant materials. The review in this section serves as an introduction to Sec. IV, where, in an analogous way, we discuss time-reversed acoustics and Green's function retrieval in homogeneous, time-variant materials.

A. Wave equation for an inhomogeneous, time-invariant material

We consider an arbitrarily inhomogeneous, time-invariant material, with parameters $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$. Substituting constitutive relations (3) and (4) into Eqs. (1) and (2), using the fact that $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are time-invariant, we obtain

$$\alpha\partial_t P + \nabla\cdot\mathbf{Q} = a, \tag{5}$$

$$\beta\partial_t\mathbf{Q} + \nabla P = \mathbf{b}. \tag{6}$$

For acoustic waves, these expressions become $\kappa\partial_t p + \nabla\cdot\mathbf{v} = q$ and $\rho\partial_t\mathbf{v} + \nabla p = \mathbf{f}$. We continue with the general notation of Eqs. (5) and (6), so that all that follows also holds for 2D SH, TE, and TM waves.

The parameters $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ in Eqs. (5) and (6) may vary continuously with position. For a material with piecewise continuous parameters, these equations are supplemented with boundary conditions. At a spatial boundary with normal $\mathbf{n}(\mathbf{x})$, where the parameters undergo a finite jump, the boundary conditions state that P and $\mathbf{Q}\cdot\mathbf{n}$ are continuous over that boundary.

By eliminating $\mathbf{Q}$ from Eqs. (5) and (6), we obtain the well-known second order wave equation:

$$\left(\frac{1}{\beta c^2}\partial_t^2 - \nabla\cdot\frac{1}{\beta}\nabla\right)P(\mathbf{x},t) = s(\mathbf{x},t), \tag{7}$$

with $c(\mathbf{x})$ being the space-variant propagation velocity, defined as

$$c = \frac{1}{\sqrt{\alpha\beta}}, \tag{8}$$

and $s(\mathbf{x},t)$ the source distribution, defined as

$$s = \partial_t a - \nabla\cdot\left(\frac{1}{\beta}\mathbf{b}\right). \tag{9}$$

B. Green's function of an inhomogeneous, time-invariant material

We introduce the Green's function $\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,t_A)$ of an inhomogeneous, time-invariant material as the response to an impulsive point source at $\mathbf{x}=\mathbf{x}_A$ at time instant $t=t_A$, observed at position $\mathbf{x}$ and time t. Hence, it obeys the wave equation [Eq. (7)], with the source term $s(\mathbf{x},t)$ on the right-hand side replaced by an impulsive point source, according to

$$\left(\frac{1}{\beta c^2}\partial_t^2 - \nabla\cdot\frac{1}{\beta}\nabla\right)\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,t_A) = \delta(\mathbf{x}-\mathbf{x}_A)\delta(t-t_A), \tag{10}$$

where $\delta(\cdot)$ is the Dirac delta function. The subscript x in $\mathcal{G}_x$ denotes that this is the Green's function of a space-variant material. The Green's function obeys the causality condition

$$\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,t_A) = 0 \quad \text{for} \quad t < t_A. \tag{11}$$

Assuming that the material is homogeneous outside a sphere with arbitrarily large but finite radius, this causality condition implies that $\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,t_A)$ is outward propagating for $|\mathbf{x}-\mathbf{x}_A|\to\infty$.

Since the material is time-invariant, the Green's function is time-shift invariant, hence,

$$\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,t_A) = \mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t-t_A,0). \tag{12}$$

The Green's function obeys the following reciprocity relation:[36,43]

$$\mathcal{G}_x(\mathbf{x}_A,\mathbf{x}_B,t,0) = \mathcal{G}_x(\mathbf{x}_B,\mathbf{x}_A,t,0). \tag{13}$$

Hence, in an inhomogeneous, time-invariant material, the response to a source at $\mathbf{x}_B$, observed at $\mathbf{x}_A$, is equal to the response to a source at $\mathbf{x}_A$, observed at $\mathbf{x}_B$.

Since the wave equation [Eq. (10)] contains only even-order time derivatives, the time-reversed Green's function $\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,-t,0)$ is also a solution of this equation (with $t_A=0$). This acausal Green's function is inward propagating for $|\mathbf{x}-\mathbf{x}_A|\to\infty$. It propagates through the inhomogeneous material, converges to a sink at $\mathbf{x}=\mathbf{x}_A$ and $t=0$, after which it disappears.

We define the homogeneous Green's function of an inhomogeneous, time-invariant material as the difference of the Green's function and its time-reversed version, according to

$$\mathcal{G}_x^h(\mathbf{x},\mathbf{x}_A,t,0) = \mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0) - \mathcal{G}_x(\mathbf{x},\mathbf{x}_A,-t,0). \tag{14}$$

Both terms on the right-hand side obey the same wave equation, with the same singularity $\delta(\mathbf{x}-\mathbf{x}_A)\delta(t)$. The singularities cancel when we subtract the equations for the Green's function and its time-reversed version. Hence, the homogeneous Green's function obeys the following wave equation:

$$\left(\frac{1}{\beta c^2}\partial_t^2 - \nabla\cdot\frac{1}{\beta}\nabla\right)\mathcal{G}_x^h(\mathbf{x},\mathbf{x}_A,t,0) = 0. \tag{15}$$

Note that $\mathcal{G}_x^h(\mathbf{x},\mathbf{x}_A,t,0)$ is called the homogeneous Green's function because it is a solution of this homogeneous differential equation (hence, here "homogeneous" does not refer to the material parameters, which may still be inhomogeneous, but to the absence of the singularity in the wave equation). Unlike the Green's function $\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0)$ and its time-reversed version $\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,-t,0)$, the homogeneous Green's function $\mathcal{G}_x^h(\mathbf{x},\mathbf{x}_A,t,0)$ is not singular for $\mathbf{x}\to\mathbf{x}_A$ and $t\to 0$.

From Eqs. (13) and (14), it follows that the homogeneous Green's function obeys the following reciprocity relation:

$$\mathcal{G}_x^h(\mathbf{x}_A,\mathbf{x}_B,t,0) = \mathcal{G}_x^h(\mathbf{x}_B,\mathbf{x}_A,t,0). \tag{16}$$

C. Homogeneous Green's function representation for an inhomogeneous, time-invariant material

We derive a boundary integral representation for the homogeneous Green's function $\mathcal{G}_x^h(\mathbf{x},\mathbf{x}_A,t,0)$ of an inhomogeneous, time-invariant material. We start by defining a vector function $\mathbf{a}(\mathbf{x}',t)$ as

$$\mathbf{a}(\mathbf{x}',t) = \frac{1}{\beta(\mathbf{x}')}\Big[\{\nabla'\mathcal{G}_x(\mathbf{x}',\mathbf{x}_B,t,0)\}*_t\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,-t,0) - \mathcal{G}_x(\mathbf{x}',\mathbf{x}_B,t,0)*_t\nabla'\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,-t,0)\Big], \tag{17}$$

where $\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,t,0)$ and $\mathcal{G}_x(\mathbf{x}',\mathbf{x}_B,t,0)$ are Green's functions with their sources at $\mathbf{x}_A$ and $\mathbf{x}_B$, respectively. The prime in ∇' denotes that this nabla operator acts on $\mathbf{x}'$, hence $\nabla' = (\partial_{x'},\partial_{y'},\partial_{z'})$ (in 3D) or $\nabla' = (\partial_{x'},\partial_{z'})$ (in 2D). The symbol $*_t$ stands for a temporal convolution, defined as $f(t)*_t g(t) = \int_{-\infty}^{\infty} f(t-t')g(t')\mathrm{d}t'$. When one of the functions is reversed in time, we obtain $f(t)*_t g(-t) = \int_{-\infty}^{\infty} f(t-t')g(-t')\mathrm{d}t' = \int_{-\infty}^{\infty} f(t+t')g(t')\mathrm{d}t'$. Hence, $f(t)*_t g(-t)$ stands for a temporal correlation of the functions $f(t)$ and $g(t)$. Evaluating the divergence $\nabla'\cdot\mathbf{a}(\mathbf{x}',t)$, using the wave equation [Eq. (10), with $\mathbf{x}$ replaced by $\mathbf{x}'$ and $t_A = 0$] and the property $\{\partial_t^2 f(t)\}*_t g(-t) = f(t)*_t\{\partial_t^2 g(-t)\}$, yields

$$\nabla'\cdot\mathbf{a}(\mathbf{x}',t) = \mathcal{G}_x(\mathbf{x}',\mathbf{x}_B,t,0)\delta(\mathbf{x}'-\mathbf{x}_A) - \delta(\mathbf{x}'-\mathbf{x}_B)\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,-t,0). \tag{18}$$

Next, we integrate this over a domain $\mathcal{V}$, enclosed by boundary $\mathcal{S}$, with outward pointing normal vector $\mathbf{n}$ (for the 3D situation, $\mathcal{V}$ is a 3D volume and $\mathcal{S}$ a 2D closed boundary; for the 2D situation, $\mathcal{V}$ is a 2D plane and $\mathcal{S}$ a 1D closed curve in the x–z plane). The Green's sources at $\mathbf{x}_A$ and $\mathbf{x}_B$ are both situated in $\mathcal{V}$. Using the theorem of Gauss, $\int_{\mathcal{V}}\nabla'\cdot\mathbf{a}\mathrm{d}\mathbf{x}' = \oint_{\mathcal{S}}\mathbf{a}\cdot\mathbf{n}\mathrm{d}\mathbf{x}'$, applying the reciprocity relation of Eq. (13) to the Green's functions with the source at $\mathbf{x}_B$, and replacing $\mathbf{x}_B$ by variable $\mathbf{x}$ yields the following representation of the homogeneous Green's function:

$$\mathcal{G}_x^h(\mathbf{x},\mathbf{x}_A,t,0) = \oint_{\mathcal{S}}\frac{1}{\beta(\mathbf{x}')}\Big[\{\nabla'\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)\}*_t\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,-t,0) - \mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)*_t\nabla'\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,-t,0)\Big]\cdot\mathbf{n}\mathrm{d}\mathbf{x}'. \tag{19}$$

This expression is the time-domain equivalent of the frequency-domain homogeneous Green's function representation, introduced for holographic imaging and inverse scattering.[44,45] In the following, we show how the homogeneous Green's function representation of Eq. (19) forms a theoretical basis for time-reversed acoustics and for Green's function retrieval.

D. Time-reversed acoustics in an inhomogeneous, time-invariant material

We start by reviewing the principle of time-reversed acoustics in an inhomogeneous, time-invariant material, in an intuitive way. Consider an impulsive point source at $\mathbf{x}_A$ and $t = 0$ in an arbitrarily inhomogeneous, time-invariant material. The response to this source, $P(\mathbf{x}',t)$, is recorded by receivers at $\mathbf{x}'$ on a closed boundary [see Fig. 1(a)]. In the acoustic situation, P stands for the acoustic pressure. Next, the responses at all $\mathbf{x}'$ are reversed in time, yielding $P(\mathbf{x}',-t)$, and these time-reversed signals are fed to sources at the positions of the receivers [see Fig. 1(b)]. The time-reversed wave field obeys the same wave equation as the original wave field. Hence, the time-reversed field emitted by the sources at the boundary propagates back through the inhomogeneous material, and at $t = 0$ it focuses at the original source position $\mathbf{x}_A$ [see Fig. 1(b)].

In geophysics, this time-reversal principle is used to determine the parameters of earthquake sources[46–48] and to perform seismic imaging.[49–52] Note that in geophysics the time-reversed field is not physically emitted into the earth, but instead the backpropagation is carried out numerically.

Fink and coworkers[16,24,25] developed the field of time-reversed acoustics for ultrasonic measurements, in which piezoelectric transducers are initially used as receivers [as in Fig. 1(a)], and subsequently as sources [as in Fig. 1(b)], emitting the time-reversed responses physically into the material.

Note that once the field has focused at $\mathbf{x}_A$ and $t = 0$, there is no sink to absorb the focused field; hence, the wave field continues its propagation.[32] Hence, the focused field acts as a source which emits waves into the inhomogeneous material [see Fig. 1(c)]. In other words, the focused field at $t = 0$ is a virtual source for the field for $t > 0$.

Let us now review the mathematics behind time-reversed acoustics, using the representation of Eq. (19) as the starting point.[53] We start by assuming that the boundary $\mathcal{S}$ is sufficiently smooth and that the material at and outside $\mathcal{S}$ is homogeneous, with parameters α_0, β_0, and c_0. Then, in

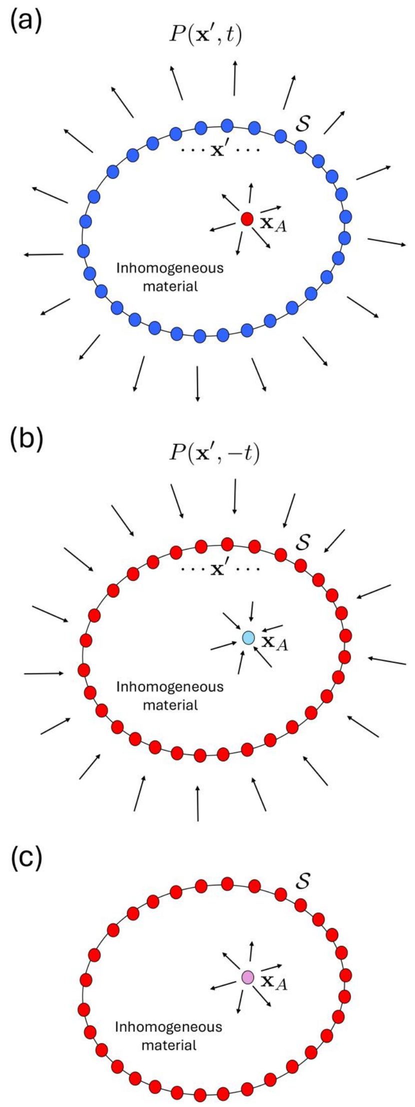

FIG. 1. Principle of time-reversed acoustics in an inhomogeneous, time-invariant material. (a) The response to a source at $\mathbf{x}_A$ and $t = 0$ (indicated by the red dot), recorded by receivers at $\mathbf{x}'$. (b) Time-reversed response, fed to sources at $\mathbf{x}'$. The wave field emitted by these sources focuses at $t = 0$ at the original source position $\mathbf{x}_A$. (c) After having focused, the wave field continues its propagation. The focused field at $t = 0$ is a virtual source (indicated by the pink dot) for the field for $t > 0$.

the high-frequency regime, the contributions of the two terms under the integral in Eq. (19) are identical (but with opposite signs).[33] Hence, we obtain

$$\mathcal{G}_x^h(\mathbf{x},\mathbf{x}_A,t,0) = \frac{2}{\beta_0}\oint_{\mathcal{S}}\{\nabla'\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)\cdot\mathbf{n}\}*_t\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,-t,0)\mathrm{d}\mathbf{x}'. \tag{20}$$

Inside $\mathcal{S}$, we define a source function $s(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{x}_A)s_0(t)$, which is a point source at $\mathbf{x} = \mathbf{x}_A$ with source wavelet $s_0(t)$. From Eqs. (7) and (10) (with $t_A = 0$), it follows that the response to this source, observed at $\mathbf{x}'$ on $\mathcal{S}$, is given by $P(\mathbf{x}',t) = \mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,t,0)*_t s_0(t)$ [see Fig. 1(a)]. Convolving both sides of Eq. (20) with the time-reversal of the source wavelet and substituting Eq. (14) into the left-hand side yields

$$\{\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0)-\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,-t,0)\}*_t s_0(-t) = \frac{2}{\beta_0}\oint_{\mathcal{S}}\underbrace{\{\nabla'\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)\cdot\mathbf{n}\}}_{\text{“propagator”}}*_t\underbrace{P(\mathbf{x}',-t)}_{\text{“source field”}}\,\mathrm{d}\mathbf{x}'. \tag{21}$$

The right-hand side quantifies the propagation of the time-reversed field $P(\mathbf{x}',-t)$ from sources at $\mathbf{x}'$ on $\mathcal{S}$ to any point $\mathbf{x}$ inside $\mathcal{S}$. The second term on the left-hand side, $\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,-t,0)*_t s_0(-t)$, is the field converging to $\mathbf{x}_A$ [Fig. 1(b)]. The first term on the left-hand side, $\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0)*_t s_0(-t)$, is the field diverging from $\mathbf{x}_A$ [Fig. 1(c)].

In most practical situations, the object of investigation cannot be accessed from a closed boundary. Replacing the closed boundary integral in Eq. (21) by an open boundary integral necessarily leads to approximations, which can be partly remedied by iterative methods. A further discussion is beyond the scope of this paper; we refer to the extensive literature on iterative time-reversed acoustics (e.g., Refs. 54–56) and on the Marchenko method (e.g., Refs. 57–59).

E. Green's function retrieval in an inhomogeneous, time-invariant material

It has been shown by many authors that the temporal cross-correlation of passive noise measurements at two receivers in an inhomogeneous, time-invariant material converges approximately to the response at one of these receivers as if there were a source at the position of the other (i.e., the Green's function between the two receivers).[26–29,60–63] We review the mathematics behind this principle of Green's function retrieval, using Eq. (20) as the starting point.[28,33] Using the reciprocity relation of Eq. (13) and substituting Eq. (14) into the left-hand side of Eq. (20) yields

$$\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0)-\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,-t,0) = \frac{2}{\beta_0}\oint_{\mathcal{S}}\{\nabla'\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)\cdot\mathbf{n}\}*_t\mathcal{G}_x(\mathbf{x}_A,\mathbf{x}',-t,0)\mathrm{d}\mathbf{x}'. \tag{22}$$

The Green's function $\mathcal{G}_x(\mathbf{x}_A,\mathbf{x}',t,0)$ on the right-hand side represents the response to an impulsive monopole source at $\mathbf{x}'$ on $\mathcal{S}$ and $t = 0$, recorded by a receiver at $\mathbf{x}_A$. Similarly, $\nabla'\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)\cdot\mathbf{n}$ (with nabla operator ∇' acting on the source position $\mathbf{x}'$) represents the response to an impulsive dipole source at $\mathbf{x}'$ on $\mathcal{S}$ and $t = 0$, recorded by a receiver at $\mathbf{x}$. Both responses are illustrated in Fig. 2(a). The entire right-hand side quantifies the temporal cross-correlation of these responses at $\mathbf{x}_A$ and $\mathbf{x}$, integrated over all sources at $\mathbf{x}'$ on $\mathcal{S}$. The left-hand side consists of the Green's function $\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0)$ (minus its time-reversed version). This is the response to an impulsive virtual source at $\mathbf{x}_A$ and $t = 0$, recorded by a physical receiver at $\mathbf{x}$ [see Fig. 2(b)].

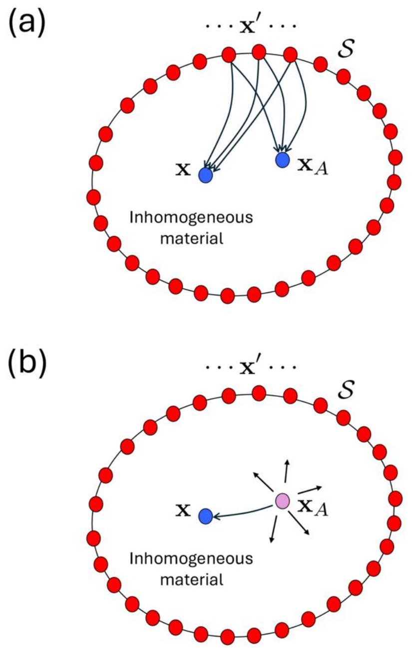

FIG. 2. Principle of Green's function retrieval in an inhomogeneous, time-invariant material. (a) Responses to impulsive sources at $\mathbf{x}'$ and $t=0$, recorded by receivers at $\mathbf{x}_A$ and $\mathbf{x}$. (b) Response to a virtual source at $\mathbf{x}_A$ (indicated by the pink dot), recorded by a receiver at $\mathbf{x}$, obtained from the temporal cross-correlation of the responses in (a), integrated over all sources at $\mathbf{x}'$ on $\mathcal{S}$.

When only monopole sources are available on $\mathcal{S}$, we need to approximate the dipole responses on the right-hand side of Eq. (22) by monopole responses. We already assumed that the material at and outside $\mathcal{S}$ is homogeneous, with parameters α_0, β_0, and c_0. If we assume in addition that $\mathcal{S}$ is a sphere with very large radius, we may use the following approximation on $\mathcal{S}$:[33]

$$\nabla'\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)\cdot\mathbf{n}\approx-\frac{1}{c_0}\partial_t\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0). \tag{23}$$

Substituting this into Eq. (22), using $\{\partial_t f(t)\}*_t g(t)=\partial_t\{f(t)*_t g(t)\}$, yields

$$\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0)-\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,-t,0)\approx-\frac{2}{\beta_0c_0}\partial_t\oint_{\mathcal{S}}\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)*_t\mathcal{G}_x(\mathbf{x}_A,\mathbf{x}',-t,0)\mathrm{d}\mathbf{x}'. \tag{24}$$

Now the right-hand side quantifies the time-derivative of the cross-correlation of monopole responses, integrated over the sources. An alternative form, without the time-derivative, reads[64]

$$G_x(\mathbf{x},\mathbf{x}_A,t,0)+G_x(\mathbf{x},\mathbf{x}_A,-t,0)\approx\frac{2}{\beta_0c_0}\oint_{\mathcal{S}}G_x(\mathbf{x},\mathbf{x}',t,0)*_tG_x(\mathbf{x}_A,\mathbf{x}',-t,0)\mathrm{d}\mathbf{x}', \tag{25}$$

where the Green's function $G_x(\mathbf{x},\mathbf{x}_A,t,0)$ is the response to a source $a(\mathbf{x},t)=\delta(\mathbf{x}-\mathbf{x}_A)\delta(t)$, $\mathbf{b}(\mathbf{x},t)=\mathbf{0}$. Since $s(\mathbf{x},t)=\partial_t a(\mathbf{x},t)$ [Eq. (9)] and the material is time-invariant, we have $G_x(\mathbf{x},\mathbf{x}_A,t,0)=\partial_t\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0)$. Equations (24) and (25) are useful when the responses to all sources at $\mathbf{x}'$ on $\mathcal{S}$ are individually available, such as in controlled-source seismic interferometry.[65–67] Of course, approximations are introduced when the sources are not available on a closed boundary, similar as in time-reversed acoustics, as noted in Sec. III D.

For the derivation of Green's function retrieval from ambient noise, consider the situation in which the sources at all $\mathbf{x}'$ on $\mathcal{S}$ are simultaneously acting noise sources $N(\mathbf{x}',t)$. In that case, the fields at $\mathbf{x}_A$ and $\mathbf{x}$ are given by $P(\mathbf{x}_A,t)=\oint_{\mathcal{S}}\mathcal{G}_x(\mathbf{x}_A,\mathbf{x}',t,0)*_tN(\mathbf{x}',t)\mathrm{d}\mathbf{x}'$ and $P(\mathbf{x},t)=\oint_{\mathcal{S}}\mathcal{G}_x(\mathbf{x},\mathbf{x}'',t,0)*_tN(\mathbf{x}'',t)\mathrm{d}\mathbf{x}''$, respectively. We assume that the noise sources are mutually uncorrelated, according to $\langle N(\mathbf{x}'',t)*_tN(\mathbf{x}',-t)\rangle=\delta(\mathbf{x}'-\mathbf{x}'')C(t)$. Here, $C(t)$ is the temporal autocorrelation of the noise (which is assumed to be the same for all sources) and $\langle\cdot\rangle$ stands for ensemble averaging, which in practice is replaced by averaging over very long time. Further, $\delta(\mathbf{x}'-\mathbf{x}'')$ is a delta function defined in $\mathcal{S}$. The cross-correlation of the fields at $\mathbf{x}_A$ and $\mathbf{x}$ thus gives[64]

$$\langle P(\mathbf{x},t)*_tP(\mathbf{x}_A,-t)\rangle=\oint_{\mathcal{S}}\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)*_t\mathcal{G}_x(\mathbf{x}_A,\mathbf{x}',-t,0)*_tC(t)\mathrm{d}\mathbf{x}'. \tag{26}$$

Convolving both sides of Eq. (24) with the autocorrelation $C(t)$, using $\{\partial_t f(t)\}*_t g(t)=\partial_t\{f(t)*_t g(t)\}$ and Eq. (26) to simplify the right-hand side, we obtain

$$\{\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,t,0)-\mathcal{G}_x(\mathbf{x},\mathbf{x}_A,-t,0)\}*_tC(t)\approx-\frac{2}{\beta_0c_0}\partial_t\langle P(\mathbf{x},t)*_tP(\mathbf{x}_A,-t)\rangle. \tag{27}$$

This is the basic expression for Green's function retrieval from noise, also known as ambient-noise interferometry.[26–29,32,64] It states that the time-derivative of the cross-correlation of two noise recordings at receiver positions $\mathbf{x}_A$ and $\mathbf{x}$, averaged over long enough time, converges to the Green's function between $\mathbf{x}_A$ and $\mathbf{x}$ (minus its time-reversed version), convolved with the temporal autocorrelation of the noise. Hence, the receiver at $\mathbf{x}_A$ is turned into a virtual source.

In most practical situations, the sources are not distributed along a closed boundary, the noise sources may be mutually correlated, the noise spectra may not be the same for all sources, the wave fields may not obey a simple scalar wave equation, etc. Hence, the expressions derived in this section are at best approximations for real situations. Much research has been done to overcome these limitations, of which we mention iterative correlation,[68] multidimensional deconvolution,[67,69] directional balancing,[70] and generalized interferometry.[71] A further discussion is beyond the scope of this paper.

IV. TIME-REVERSED ACOUSTICS AND GREEN'S FUNCTION RETRIEVAL IN HOMOGENEOUS, TIME-VARIANT MATERIALS

We discuss time-reversed acoustics and Green's function retrieval in homogeneous, time-variant materials, and their

https://doi.org/10.1121/10.0044236

mutual relation. The setup is analogous to that in Sec. III for inhomogeneous, time-invariant materials, which facilitates the comparison of the methodologies for both classes of material. In Sec. IV A, we discuss the wave equation for a homogeneous, time-variant material, as a special case of the equations in Sec. II. In Sec. IV B, we discuss the Green's function of a homogeneous, time-variant material. In Sec. IV C, we discuss the propagator matrix for a homogeneous, time-variant material. In Sec. IV D, we use this propagator matrix as the basis for deriving a representation of the homogeneous Green's function of a homogeneous, time-variant material. In Sec. IV E and Sec. IV F, we show how this homogeneous Green's function representation leads to time-reversed acoustics and Green's function retrieval, respectively, in homogeneous, time-variant materials.

A. Wave equation for a homogeneous, time-variant material

We consider a homogeneous, arbitrarily time-variant material, with parameters $\alpha(t)$ and $\beta(t)$. Substituting the constitutive relations [Eqs. (3) and (4)] into Eqs. (1) and (2), using the fact that $\alpha(t)$ and $\beta(t)$ are space-invariant, we obtain

$$\partial_t U + \frac{1}{\beta}\nabla\cdot\mathbf{V} = a, \tag{28}$$

$$\partial_t \mathbf{V} + \frac{1}{\alpha}\nabla U = \mathbf{b}. \tag{29}$$

For acoustic waves, these expressions become $-\partial_t\Theta + (1/\rho)\nabla\cdot\mathbf{m} = q$ and $\partial_t\mathbf{m} - (1/\kappa)\nabla\Theta = \mathbf{f}$. We continue with the general notation of Eqs. (28) and (29), so that all that follows also holds for 2D SH, TE, and TM waves.

The parameters $\alpha(t)$ and $\beta(t)$ in Eqs. (28) and (29) may vary continuously with time. For a material with piecewise continuous parameters, these equations are supplemented with boundary conditions. At a time boundary, where the parameters undergo a finite jump, the boundary conditions state that U and $\mathbf{V}$ are continuous over that boundary.

By eliminating $\mathbf{V}$ from Eqs. (28) and (29), we obtain the second-order wave equation

$$\left(\partial_t\beta\partial_t - \beta c^2\nabla^2\right)U(\mathbf{x},t) = s(\mathbf{x},t), \tag{30}$$

with $c(t)$ being the time-variant propagation velocity, defined again as $c = 1/\sqrt{\alpha\beta}$, and $s(\mathbf{x},t)$ the source distribution, defined as

$$s = \partial_t(\beta a) - \nabla\cdot\mathbf{b}. \tag{31}$$

B. Green's function of a homogeneous, time-variant material

We introduce the Green's function $\mathcal{G}_t(\mathbf{x},\mathbf{x}_A,t,t_A)$ of a homogeneous, time-variant material as the response to an impulsive point source at $\mathbf{x}=\mathbf{x}_A$ at time instant $t=t_A$, observed at position $\mathbf{x}$ and time t. Hence, it obeys the wave equation [Eq. (30)], with the source term $s(\mathbf{x},t)$ on the right-hand side replaced by an impulsive point source, according to

$$\left(\partial_t\beta\partial_t - \beta c^2\nabla^2\right)\mathcal{G}_t(\mathbf{x},\mathbf{x}_A,t,t_A) = \delta(\mathbf{x}-\mathbf{x}_A)\delta(t-t_A). \tag{32}$$

The subscript t in $\mathcal{G}_t$ denotes that this is the Green's function of a time-variant material. The Green's function obeys the causality condition

$$\mathcal{G}_t(\mathbf{x},\mathbf{x}_A,t,t_A) = 0 \quad \text{for} \quad t < t_A. \tag{33}$$

Assuming that the material is time-invariant after an arbitrarily large but finite time, this causality condition implies that $\mathcal{G}_t(\mathbf{x},\mathbf{x}_A,t,t_A)$ is outward propagating for $|\mathbf{x}-\mathbf{x}_A|\to\infty$. Analytical expressions of $\mathcal{G}_t(\mathbf{x},\mathbf{x}_A,t,t_A)$ for the special case of a homogeneous, time-invariant material are given in the Appendix. Here, we continue with expressions for a time-variant material.

Since the material is homogeneous, the Green's function is space-shift invariant, hence,

$$\mathcal{G}_t(\mathbf{x},\mathbf{x}_A,t,t_A) = \mathcal{G}_t(\mathbf{x}-\mathbf{x}_A,\mathbf{0},t,t_A). \tag{34}$$

We introduce an acausal Green's function $\mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,t_A)$, obeying the same wave equation as the causal Green's function [Eq. (32), with $\mathbf{x}_A=\mathbf{0}$], but with the acausality condition

$$\mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,t_A) = 0 \quad \text{for} \quad t > t_A. \tag{35}$$

Assuming that the material is time-invariant before an arbitrarily large but finite negative time, this acausality condition implies that $\mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,t_A)$ is inward propagating for $|\mathbf{x}|\to\infty$. Subsequently, it propagates through the time-variant material, converges to a sink at $\mathbf{x}=\mathbf{0}$ and $t=t_A$, after which it disappears. Note, however, that since the material is time-variant, in general this acausal Green's function is not the time-reversal of the causal Green's function, i.e., $\mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,0) \neq \mathcal{G}_t(\mathbf{x},\mathbf{0},-t,0)$.

The causal and acausal Green's functions are mutually related via the reciprocity relation[34]

$$\mathcal{G}_t(\mathbf{x},\mathbf{0},t_A,t_B) = \mathcal{G}_t^a(\mathbf{x},\mathbf{0},t_B,t_A). \tag{36}$$

Hence, in a homogeneous, time-variant material, the causal response to a source at t_B, observed at t_A, is equal to the acausal response observed at t_B, absorbed by a sink at t_A. This is a meaningful relation for $t_A > t_B$. For $t_A < t_B$, Eq. (36) reduces to the trivial relation $0=0$, on account of Eqs. (33) and (35).

We define the homogeneous Green's function of a homogeneous, time-variant material, as the difference of the causal and acausal Green's functions, according to

$$\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A) = \mathcal{G}_t(\mathbf{x},\mathbf{0},t,t_A) - \mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,t_A). \tag{37}$$

Both terms on the right-hand side obey the same wave equation, with the same singularity $\delta(\mathbf{x})\delta(t-t_A)$. The singularities cancel when we subtract the equations for the causal and acausal Green's functions. Hence, the homogeneous Green's function obeys the following equation:

$$\left(\partial_t\beta\partial_t - \beta c^2\nabla^2\right)\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A) = 0. \tag{38}$$

Unlike the Green's function $\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t_A)$ and its acausal counterpart $\mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,t_A)$, the homogeneous Green's function $\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A)$ is not singular for $\mathbf{x}\to\mathbf{0}$ and $t\to t_A$.

From Eqs. (36) and (37), it follows that the homogeneous Green's function obeys the following reciprocity relation:

$$\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t_A,t_B) = -\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t_B,t_A). \tag{39}$$

In Sec. IV C, we discuss the propagator matrix for a homogeneous, time-variant material. The elements of this matrix can be expressed in terms of the homogeneous Green's function $\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A)$. The propagator matrix formalism will be used in Sec. IV D as a basis for deriving a representation of the homogeneous Green's function for a homogeneous, time-variant material [as the counterpart of the representation of Eq. (19) for an inhomogeneous, time-invariant material].

C. Propagator matrix of a homogeneous, time-variant material

We reorganize Eqs. (28) and (29) into a matrix-vector wave equation for a homogeneous, time-variant material, as the counterpart of a matrix-vector wave equation for an inhomogeneous, time-invariant material.[72–77] This equation will form the basis for discussing the propagator matrix of a homogeneous, time-variant material.

We reduce the number of field quantities in Eqs. (28) and (29) by defining a scalar wave field $V_{\rm d}(\mathbf{x},t)$ as

$$V_{\rm d} = \nabla\cdot\mathbf{V}, \tag{40}$$

where subscript d refers to "divergence." Using this in Eqs. (28) and (29), we obtain

$$\partial_t U + \frac{1}{\beta}V_{\rm d} = a, \tag{41}$$

$$\partial_t V_{\rm d} + \frac{1}{\alpha}\nabla^2 U = b_{\rm d}, \tag{42}$$

with

$$b_{\rm d} = \nabla\cdot\mathbf{b}. \tag{43}$$

Equations (41) and (42) can be reorganized into a matrix-vector wave equation, according to

$$\partial_t\mathbf{q} = \mathbf{A}\mathbf{q} + \mathbf{d}, \tag{44}$$

with

$$\mathbf{q} = \begin{pmatrix} U \\ V_{\rm d}\end{pmatrix},\quad \mathbf{A} = \begin{pmatrix} 0 & -\dfrac{1}{\beta} \\ -\dfrac{1}{\alpha}\nabla^2 & 0\end{pmatrix},\quad \mathbf{d} = \begin{pmatrix} a \\ b_{\rm d}\end{pmatrix}, \tag{45}$$

where $\mathbf{q}(\mathbf{x},t)$ is a wave field vector, $\mathbf{A}(t)$ an operator matrix, and $\mathbf{d}(\mathbf{x},t)$ a source vector. The parameters $\alpha(t)$ and $\beta(t)$ in operator matrix $\mathbf{A}(t)$ may vary continuously with time. For a material with piecewise continuous parameters, Eq. (44) is supplemented with boundary conditions: at each time boundary, the wave field vector $\mathbf{q}(\mathbf{x},t)$ is continuous over that boundary.

We introduce the 2×2 propagator matrix $\mathbf{W}(\mathbf{x},t,t_A)$ as the solution of Eq. (44) for the source-free situation,[19,20,34] hence,

$$\partial_t\mathbf{W} = \mathbf{A}\mathbf{W}. \tag{46}$$

For $t=t_A$, it obeys the initial condition

$$\mathbf{W}(\mathbf{x},t_A,t_A) = \mathbf{I}\delta(\mathbf{x}), \tag{47}$$

where $\mathbf{I}$ is the identity matrix. The propagator matrix $\mathbf{W}(\mathbf{x},t,t_A)$ forms the counterpart of the propagator matrix for an inhomogeneous, time-invariant material.[78–82] In the literature on time-variant materials, this matrix is also known as the transfer matrix.[19,20,83] From Eqs. (44), (46), and (47), assuming $\alpha(t)$ and $\beta(t)$ vary continuously and there are no sources between t_A and t, it follows that $\mathbf{q}(\mathbf{x},t)$ can be expressed as

$$\mathbf{q}(\mathbf{x},t) = \mathbf{W}(\mathbf{x},t,t_A)*_x\mathbf{q}(\mathbf{x},t_A). \tag{48}$$

Here, t may be larger or smaller than, or equal to t_A. The symbol $*_x$ stands for a 3D or 2D spatial convolution, according to $f(\mathbf{x})*_x g(\mathbf{x}) = \int_{\mathbb{R}^m} f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}'){\rm d}\mathbf{x}'$, with $m=3$ for the 3D situation and $m=2$ for the 2D situation; $\mathbb{R}$ denotes the set of real numbers.

Let $t_1\ldots t_n\ldots t_{N-1}$ be time instants where the parameters $\alpha(t)$ and $\beta(t)$ may be discontinuous. Then, by applying Eq. (48) recursively (starting with $t_A = t_0$), we obtain

$$\mathbf{W}(\mathbf{x},t_N,t_0) = \mathbf{W}(\mathbf{x},t_N,t_{N-1})*_x\cdots *_x\mathbf{W}(\mathbf{x},t_n,t_{n-1})*_x\cdots *_x\mathbf{W}(\mathbf{x},t_1,t_0). \tag{49}$$

This expression is useful for numerical modeling, in which case the t_n are ordered from small to large (an example is given at the end of this section). However, this expression remains valid for arbitrarily ordered t_n.

We partition the 2×2 matrix $\mathbf{W}(\mathbf{x},t,t_A)$ as follows:

$$\mathbf{W}(\mathbf{x},t,t_A) = \begin{pmatrix} W^{U,U} & W^{U,V} \\ W^{V,U} & W^{V,V}\end{pmatrix}(\mathbf{x},t,t_A). \tag{50}$$

According to Ref. 34 [their Eq. (11.27)], $W^{U,V}(\mathbf{x},t,t_A)$ is related to the causal and acausal Green's functions via

JASA
https://doi.org/10.1121/10.0044236

$$W^{U,V}(\mathbf{x},t,t_A) = \mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,t_A) - \mathcal{G}_t(\mathbf{x},\mathbf{0},t,t_A) = -\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A). \tag{51}$$

According to Ref. 34 [their Eqs. (11.18), (11.20), and (11.21)], the other components of the propagator matrix can be expressed in terms of $W^{U,V}(\mathbf{x},t,t_A)$. Combining those relations with Eq. (51), we obtain

$$W^{V,V}(\mathbf{x},t,t_A) = \beta(t)\partial_t\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A), \tag{52}$$

$$W^{U,U}(\mathbf{x},t,t_A) = -\beta(t_A)\partial_{t_A}\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A), \tag{53}$$

$$W^{V,U}(\mathbf{x},t,t_A) = \beta(t)\beta(t_A)\partial_t\partial_{t_A}\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A). \tag{54}$$

Equations (50)–(54) show that the propagator matrix is expressed in terms of the Green's function and its derivatives. This property is restricted to homogeneous, time-variant materials (i.e., it does not hold for the analogous case of inhomogeneous, time-invariant materials). This is explained as follows. For inhomogeneous, time-invariant materials, the initial condition of the propagator matrix [Eq. (47)] is replaced by a boundary condition, but the causality condition of the Green's function [Eq. (33)] remains valid (i.e., it is not replaced by a boundary condition). Therefore, since time and space are interchangeable for the propagator matrix but not for the Green's function, the relations between the propagator matrix and the Green's function, as expressed by Eqs. (51)–(54), have no counterpart for inhomogeneous, time-invariant materials. In Sec. IV D, we make use of the simple relations between the propagator matrix and the Green's function to derive a representation of the homogeneous Green's function from a representation of the propagator matrix.

We conclude this section with a 2D numerical example of a Green's function in a homogeneous, time-variant material. We consider a piecewise constant material consisting of five time-invariant slabs, with propagation velocities of 1500, 2500, 2000, 1400, and 1400 m s^{-1}. The parameter β is taken constant throughout. The duration of each time-invariant slab is 20 μs. We use Eq. (49), starting with $t_0 = 0\,\mu$s, stepping through the material with steps of 2.5 μs. For $\mathbf{W}(\mathbf{x},t_n,t_{n-1})$, we use the analytical expression given in the Appendix. After each time step, we extract the upper-right element, multiplied by -1, i.e., $-W^{U,V}(\mathbf{x},t_n,t_0) = \mathcal{G}_t(\mathbf{x},\mathbf{0},t_n,t_0)$ (the acausal Green's function is zero since $t_n > t_0$ for all n). The result (convolved with a spatial wavelet with a central wavenumber $k_0/2\pi = 100$ m^{-1}) is shown in Fig. 3 as a function of $\mathbf{x} = (x,z)$ and t (for later convenience, we have replaced t_0 by $t' = 0$). Snapshots for constant t are shown in Fig. 4 and a cross section as a function of (x,t) for $z = 0$ in Fig. 5. These figures clearly show that, when a wave encounters a time boundary (indicated by the blue dashed lines in Fig. 5), it is partly transmitted and partly reflected.

D. Homogeneous Green's function representation for a homogeneous, time-variant material

As a counterpart of Eq. (19), which is a representation of the homogeneous Green's function $\mathcal{G}_x^h(\mathbf{x},\mathbf{x}_A,t,0)$ in an inhomogeneous, time-invariant material, here we seek for a representation of the homogeneous Green's function $\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A)$ in a homogeneous, time-variant material. To this end, we first formulate a representation of the propagator matrix $\mathbf{W}(\mathbf{x},t,t_A)$. As a special case of Eq. (49), we write

$$\mathbf{W}(\mathbf{x},t,t_A) = \mathbf{W}(\mathbf{x},t,t') *_x \mathbf{W}(\mathbf{x},t',t_A). \tag{55}$$

In general, the order of t, t' and t_A is arbitrary; but in the following, we assume that t' is smaller than both t_A and t. Hence, on the right-hand side, the elements of matrix $\mathbf{W}(\mathbf{x},t',t_A)$ (with $t' < t_A$) consist only of (derivatives of) the

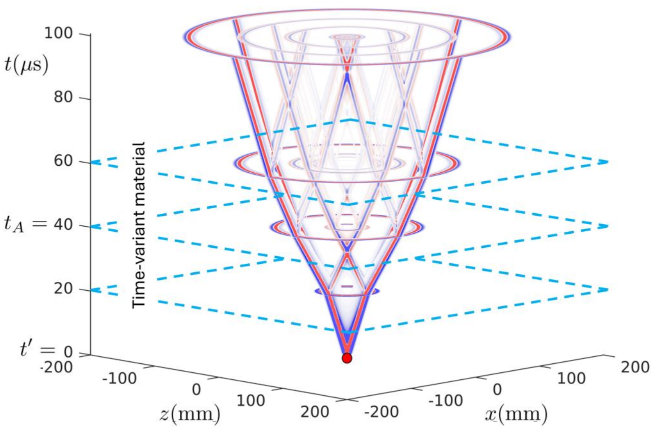

FIG. 3. 2D Green's function $\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t'=0)$ (convolved with a spatial wavelet) in a piecewise constant, time-variant material. The red dot indicates the source at $\mathbf{x} = \mathbf{0}$ and $t' = 0$. The dashed blue planes indicate the time boundaries. Movie available at https://www.keeswapenaar.nl/TimeMaterial/Green5.mp4.

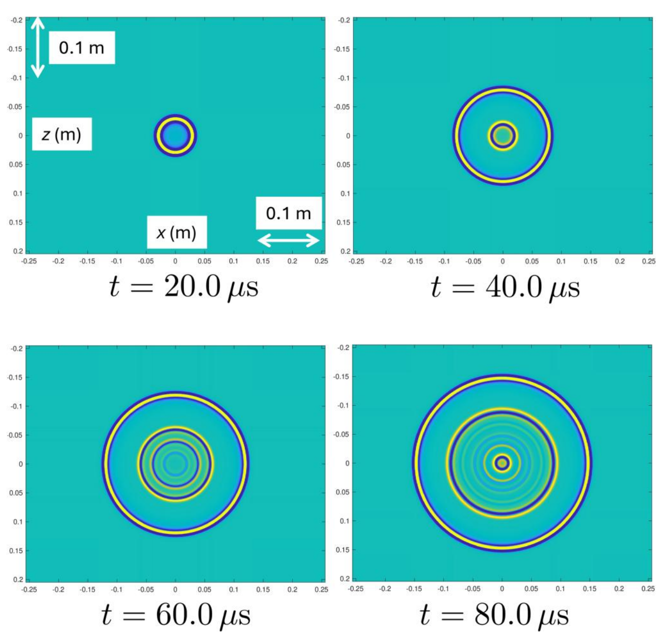

FIG. 4. Snapshots for constant t of the Green's function of Fig. 3 as a function of $\mathbf{x} = (x,z)$.

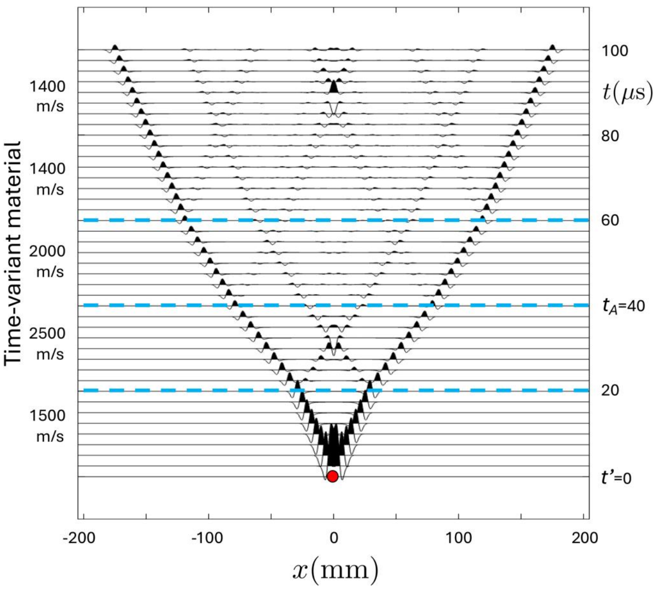

FIG. 5. Cross-section of the Green's function of Fig. 3 for $z = 0$ as a function of (x, t). The red dot indicates the source at $\mathbf{x} = \mathbf{0}$ and $t' = 0$. The dashed blue lines indicate the time boundaries.

acausal part of the homogeneous Green's functions. On the other hand, the elements of matrix $\mathbf{W}(\mathbf{x}, t, t')$ (with $t' < t$) consist only of (derivatives of) the causal part of the homogeneous Green's functions. Using the partitioning defined in Eq. (50), we thus obtain for the upper-right element of $\mathbf{W}(\mathbf{x}, t, t_A)$:

$$W^{U,V}(\mathbf{x}, t, t_A) = W^{U,U}(\mathbf{x}, t, t') *_x W^{U,V}(\mathbf{x}, t', t_A) + W^{U,V}(\mathbf{x}, t, t') *_x W^{V,V}(\mathbf{x}, t', t_A), \quad (56)$$

with $t' < t_A$ and $t' < t$, or, substituting Eqs. (51)–(53),

$$\mathcal{G}_t^h(\mathbf{x}, \mathbf{0}, t, t_A) = \beta(t') \Big[\{ \partial_{t'} \mathcal{G}_t(\mathbf{x}, \mathbf{0}, t, t') \} *_x \mathcal{G}_t^a(\mathbf{x}, \mathbf{0}, t', t_A) - \mathcal{G}_t(\mathbf{x}, \mathbf{0}, t, t') *_x \partial_{t'} \mathcal{G}_t^a(\mathbf{x}, \mathbf{0}, t', t_A) \Big]. \quad (57)$$

This is the homogeneous Green's function representation for a homogeneous, time-variant material. It is the counterpart of Eq. (19), with coordinates $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{x}_A$, and t replaced by t, t', t_A, and $\mathbf{x}$. Furthermore, ∇' is replaced by $\partial_{t'}$ and $1/\beta(\mathbf{x}')$ by $\beta(t')$. Whereas the temporal convolution in Eq. (19) is a 1D integral, and the integral over $\mathbf{x}'$ is a 2D surface integral (for the 3D situation) or a 1D line integral (for the 2D situation), the spatial convolution in Eq. (57) is a 3D or 2D integral, respectively, and there is no integral over t' [t' has only a single value in Eq. (57)]. Hence, the total number of dimensions of the integrals is the same in both representations. Last, but not least, the time-reversed Green's function $\mathcal{G}_x(\mathbf{x}', \mathbf{x}_A, -t, 0)$ and its derivative in Eq. (19) are replaced by the acausal Green's function $\mathcal{G}_t^a(\mathbf{x}, \mathbf{0}, t', t_A)$ and its derivative in Eq. (57).

Whereas time-reversal can be applied to recordings of physical measurements, acausal responses cannot be obtained from physical measurements. Therefore, we need to modify Eq. (57), or more generally, Eq. (55), before we can use it in physical situations. In particular, we need to use symmetry properties of the Green's functions to transform Eq. (55) into a form that complies with physics. In Sec. IV E, we transform the acausal matrix $\mathbf{W}(\mathbf{x}, t', t_A)$, propagating from t_A to t' (with $t' < t_A$), into a causal matrix, propagating from $-t_A$ to $-t'$ (with $-t' > -t_A$) in a time-reversed material. In this case, Eq. (55) will form the theoretical basis for time-reversed acoustics. In Sec. IV F, we transform the acausal matrix $\mathbf{W}(\mathbf{x}, t', t_A)$ into a causal matrix, propagating from t' to t_A (with $t_A > t'$). In that case, Eq. (55) will form the theoretical basis for Green's function retrieval by spatial correlation.

E. Time-reversed acoustics in a homogeneous, time-variant material

We start by introducing the principle of idealized time-reversed acoustics in a homogeneous, time-variant material in an intuitive way. We seek for a counterpart of time-reversed acoustics in an inhomogeneous, time-invariant material, as discussed in Sec. III D. Consider again Figs. 1(b) and 1(c), which illustrate a wave field, emitted by sources at $\mathbf{x}'$ on a closed boundary, propagating through the inhomogeneous material, and focusing at $\mathbf{x}_A$ and $t = 0$ [Fig. 1(b)]. The focus at $\mathbf{x}_A$ is a virtual source, radiating waves into the inhomogeneous material [Fig. 1(c)]. The counterpart of these figures is shown in the upper frame of Fig. 6. Here, we have sources at a single time-instant $t' = 0$ μs, emitting waves into a homogeneous, time-variant material (the same material as used in the numerical example in Sec. IV C). The field focuses at $t_A = 40$ μs and $\mathbf{x} = \mathbf{0}$. The focus at t_A is a virtual source, radiating waves into the time-variant material beyond t_A. The question is: how do we obtain the wave field at t', such that it focuses at t_A? Unlike in Sec. III D, where the field emitted into the inhomogeneous material was simply the time-reversal of the causal field, radiated by a source at $\mathbf{x}_A$ and $t = 0$, and observed at $\mathbf{x}'$ [Fig. 1(a)], in the present situation we cannot measure a field at t', radiated by a source at $t_A > t'$. Instead, we consider waves in a time-reversed version of the material between t' and t_A. The lower frame of Fig. 6 shows a wave field in the time-reversed material between $-t_A$ and $-t'$. A source at $-t_A$ and $\mathbf{x} = \mathbf{0}$ (indicated by the red dot) emits a wave field into this time-reversed material. Note that the wave paths are opposite to those in the lower part of the upper frame. The wave field is observed by receivers at $-t' > -t_A$ (indicated by the blue dots). These receivers measure the wave field $U_{\rm TR}(\mathbf{x}', -t')$ and its time-derivative $\partial_{t'} U_{\rm TR}(\mathbf{x}', -t')$ (with subscripts TR denoting that these fields are measured in the time-reversed material; for simplicity, Fig. 6 only shows the response $U_{\rm TR}$). To let the waves change their propagation direction, we need to change the sign of $\partial_{t'} U_{\rm TR}(\mathbf{x}', -t')$.[11,12] Hence, $U_{\rm TR}(\mathbf{x}', -t')$ and $-\partial_{t'} U_{\rm TR}(\mathbf{x}', -t')$ form the field that needs to be fed to the sources at all $\mathbf{x}'$ at t' and emitted into the actual time-variant material, to focus at t_A and $\mathbf{x} = \mathbf{0}$. A 3D

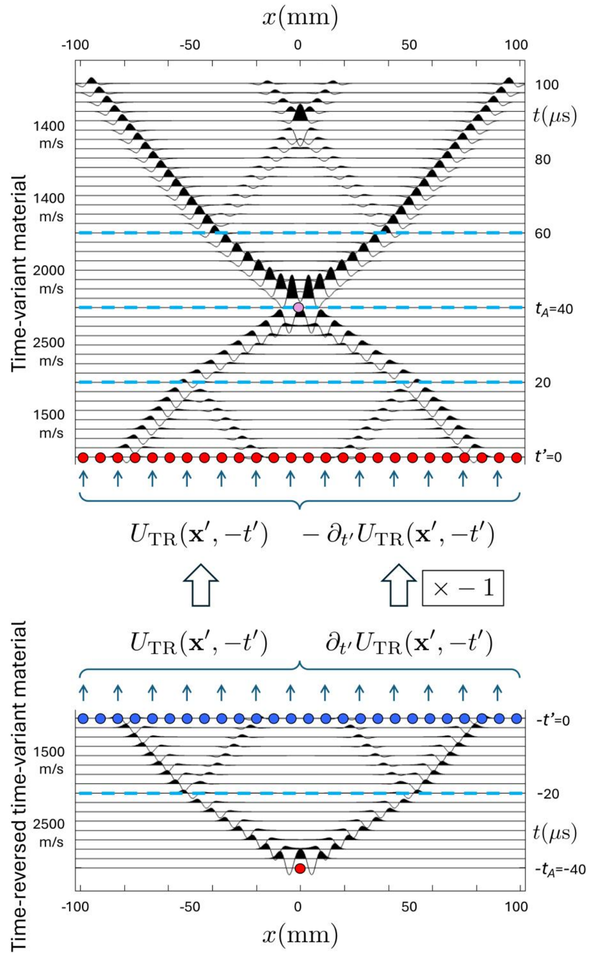

FIG. 6. Illustration of the principle of time-reversed acoustics in a homogeneous, time-variant material. The dashed blue lines indicate the time boundaries. Bottom: The response $U_{\rm TR}(\mathbf{x}',-t')$ [and its time-derivative $\partial_{t'}U_{\rm TR}(\mathbf{x}',-t')$] to a source at $-t_A=-40\,\mu$s and $\mathbf{x}=\mathbf{0}$ (indicated by the red dot) in the time-reversed version of the actual material, is recorded by receivers at $-t'=0\,\mu$s (only the response $U_{\rm TR}$ is shown). Top: After changing the sign of the time-derivative $\partial_{t'}U_{\rm TR}(\mathbf{x}',-t')$, the recorded fields are fed to sources at all $\mathbf{x}'$ at $t'=0\,\mu$s. The wave field emitted by these sources into the actual time-variant material focuses at $t_A=40\,\mu$s and $\mathbf{x}=\mathbf{0}$. After having focused, the wave field continues its propagation. The focused field at $t_A=40\,\mu$s and $\mathbf{x}=\mathbf{0}$ is a virtual source (indicated by the pink dot) for the field for $t>t_A$. In Sec. IV F, we show that the top frame can alternatively be obtained by the principle of Green's function retrieval in a homogeneous, time-variant material.

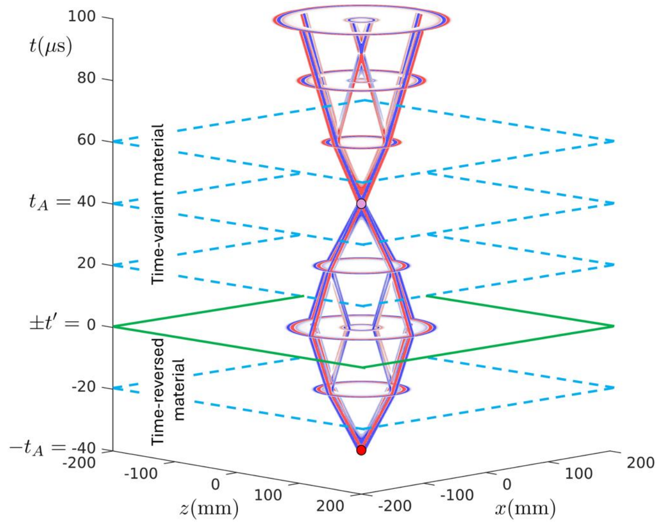

FIG. 7. Wave field of Fig. 6, shown here as a function of $\mathbf{x}=(x,z)$ and t. The dashed blue planes indicate the time boundaries. The green plane indicates the interface between the actual material and its time-reversed version. The pink dot at $t_A=40\,\mu$s and $\mathbf{x}=\mathbf{0}$ indicates the virtual source. Movie available at https://www.keeswapenaar.nl/TimeMaterial/TimeMirror.mp4.

presentation of the wave fields in Fig. 6 can be seen in Fig. 7.

Next, we formalize this principle of idealized time-reversed acoustics in a homogeneous, time-variant material, using the representation of Eq. (55) as starting point. The acausal Green's functions in matrix $\mathbf{W}(\mathbf{x},t',t_A)$, with $t'<t_A$, are not associated with a physical situation. We use symmetry properties of the Green's functions to transform $\mathbf{W}(\mathbf{x},t',t_A)$ in Eq. (55) into a form that complies with physics.

Consider the Green's function $\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t_A)$, obeying wave Eq. (32), with the causality condition of Eq. (33) (for $\mathbf{x}_A=\mathbf{0}$). We introduce $\mathcal{G}_{t,\rm TR}(\mathbf{x},\mathbf{0},t,-t_A)$ as the response to a source at $-t_A$ in the time-reversed version of the actual material. Hence, it obeys the following equation:

$$\big(\partial_t\beta(-t)\partial_t-\beta(-t)c^2(-t)\nabla^2\big)\mathcal{G}_{t,\rm TR}(\mathbf{x},\mathbf{0},t,-t_A)$$
$$=\delta(\mathbf{x})\delta(t+t_A), \tag{58}$$

with causality condition

$$\mathcal{G}_{t,\rm TR}(\mathbf{x},\mathbf{0},t,-t_A)=0 \quad \text{for} \quad t<-t_A. \tag{59}$$

Note that the subscript TR denotes that the material is time-reversed (hence, TR does not refer to time-reversal of the wave field). Replacing t by $-t$ in Eq. (58) gives

$$\big(\partial_t\beta(t)\partial_t-\beta(t)c^2(t)\nabla^2\big)\mathcal{G}_{t,\rm TR}(\mathbf{x},\mathbf{0},-t,-t_A)$$
$$=\delta(\mathbf{x})\delta(-t+t_A), \tag{60}$$

with causality condition

$$\mathcal{G}_{t,\rm TR}(\mathbf{x},\mathbf{0},-t,-t_A)=0 \quad \text{for} \quad -t<-t_A. \tag{61}$$

Note that $\delta(-t+t_A)$ in Eq. (60) is equal to $\delta(t-t_A)$, and the condition $-t<-t_A$ in Eq. (61) is equivalent with the condition $t>t_A$. Hence, $\mathcal{G}_{t,\rm TR}(\mathbf{x},\mathbf{0},-t,-t_A)$ obeys the same wave equation [Eq. (32), with $\mathbf{x}_A=\mathbf{0}$] and the same causality condition [Eq. (35)] as $\mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,t_A)$. Hence,

$$\mathcal{G}_t^a(\mathbf{x},\mathbf{0},t,t_A)=\mathcal{G}_{t,\rm TR}(\mathbf{x},\mathbf{0},-t,-t_A). \tag{62}$$

Similarly,

$$\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t_A)=\mathcal{G}_{t,\rm TR}^a(\mathbf{x},\mathbf{0},-t,-t_A), \tag{63}$$

$$\mathcal{G}_t^h(\mathbf{x},\mathbf{0},t,t_A)=-\mathcal{G}_{t,\rm TR}^h(\mathbf{x},\mathbf{0},-t,-t_A), \tag{64}$$

where $\mathcal{G}_{t,\rm TR}^a$ and $\mathcal{G}_{t,\rm TR}^h$ are the acausal and homogeneous Green's functions, respectively, in the time-reversed material. Using Eq. (64) in Eqs. (51)–(54), we find

$$W^{U,V}(\mathbf{x},t,t_A) = -W^{U,V}_{\mathrm{TR}}(\mathbf{x},-t,-t_A), \tag{65}$$

$$W^{V,V}(\mathbf{x},t,t_A) = W^{V,V}_{\mathrm{TR}}(\mathbf{x},-t,-t_A), \tag{66}$$

$$W^{U,U}(\mathbf{x},t,t_A) = W^{U,U}_{\mathrm{TR}}(\mathbf{x},-t,-t_A), \tag{67}$$

$$W^{V,U}(\mathbf{x},t,t_A) = -W^{V,U}_{\mathrm{TR}}(\mathbf{x},-t,-t_A). \tag{68}$$

Hence,

$$\mathbf{W}(\mathbf{x},t,t_A) = \mathbf{J}\mathbf{W}_{\mathrm{TR}}(\mathbf{x},-t,-t_A)\mathbf{J}^{-1}, \tag{69}$$

where $\mathbf{W}_{\mathrm{TR}}(\mathbf{x},-t,-t_A)$ is the propagator matrix for the time-reversed material, and

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{70}$$

Substituting Eq. (69) into Eq. (55) and post-multiplying both sides by $\mathbf{J}$ yields

$$\mathbf{W}(\mathbf{x},t,t_A)\mathbf{J} = \mathbf{W}(\mathbf{x},t,t')*_x\mathbf{J}\mathbf{W}_{\mathrm{TR}}(\mathbf{x},-t',-t_A). \tag{71}$$

In Eq. (71), $\mathbf{W}_{\mathrm{TR}}(\mathbf{x},-t',-t_A)$ propagates from $-t_A$ to $-t'$ through the time-reversed version of the actual material between t' and t_A. Hence, the elements of both propagator matrices on the right-hand side of Eq. (71) consist only of (derivatives of) the causal part of the homogeneous Green's functions. Let us compare Eq. (71) with its counterpart of Eq. (20). To this end, for the upper-right element of $\mathbf{W}(\mathbf{x},t,t_A)\mathbf{J}$, we write

$$-W^{U,V}(\mathbf{x},t,t_A) = W^{U,U}(\mathbf{x},t,t')*_xW^{U,V}_{\mathrm{TR}}(\mathbf{x},-t',-t_A) - W^{U,V}(\mathbf{x},t,t')*_xW^{V,V}_{\mathrm{TR}}(\mathbf{x},-t',-t_A). \tag{72}$$

Substituting Eqs. (51)–(53), using $\beta_{\mathrm{TR}}(-t') = \beta(t')$, yields

$$\mathcal{G}^h_t(\mathbf{x},\mathbf{0},t,t_A) = \beta(t')\Big[\{\partial_{t'}\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t')\}*_x\mathcal{G}_{t,\mathrm{TR}}(\mathbf{x},\mathbf{0},-t',-t_A) - \mathcal{G}_t(\mathbf{x},\mathbf{0},t,t')*_x\partial_{t'}\mathcal{G}_{t,\mathrm{TR}}(\mathbf{x},\mathbf{0},-t',-t_A)\Big]. \tag{73}$$

This expression is the counterpart of Eq. (20). Unlike in Eq. (20), where we captured the two terms under the integral in Eq. (19) by a single term, no such simplification is possible for Eq. (73).

We continue with the matrix representation of Eq. (71) and use this as the basis for time-reversed acoustics in a homogeneous, time-variant material. To this end, we define a source vector $\mathbf{d}(\mathbf{x},t)$ at $t=-t_A$ in the time-reversed material, according to $\mathbf{d}(\mathbf{x},t) = \mathbf{d}_0(\mathbf{x})\delta(t+t_A)$. Then the causal response to this source in the time-reversed material is given by

$$\mathbf{q}_{\mathrm{TR}}(\mathbf{x},t) = \mathbf{W}_{\mathrm{TR}}(\mathbf{x},t,-t_A)*_x\mathbf{d}_0(\mathbf{x}). \tag{74}$$

Convolving both sides of Eq. (71) with the source distribution $\mathbf{d}_0(\mathbf{x})$, using Eq. (74) for $t=-t'$, yields

$$\mathbf{W}(\mathbf{x},t,t_A)*_x\mathbf{J}\mathbf{d}_0(\mathbf{x}) = \mathbf{W}(\mathbf{x},t,t')*_x\mathbf{J}\mathbf{q}_{\mathrm{TR}}(\mathbf{x},-t'). \tag{75}$$

We discuss this expression from right to left. For this discussion, recall that t_A and t' are fixed, whereas t is a variable; moreover, the order of t_A and t is arbitrary and t' is smaller than both t_A and t. The term

$$\mathbf{q}_{\mathrm{TR}}(\mathbf{x},-t') = \begin{pmatrix} U_{\mathrm{TR}}(\mathbf{x},-t') \\ V_{\mathrm{d,TR}}(\mathbf{x},-t') \end{pmatrix} \tag{76}$$

is, according to Eq. (74), the causal response to the source at $-t_A$ in the time-reversed material, observed at $-t'$ (see the lower frame of Fig. 6; recall that for simplicity Fig. 6 only shows the component U_{TR}). According to Eq. (41) (with $a=0$), we have $V_{\mathrm{d,TR}}(\mathbf{x},-t') = \beta(t')\partial_{t'}U_{\mathrm{TR}}(\mathbf{x},-t')$. The matrix $\mathbf{J}$ in the right-hand side of Eq. (75) changes the sign of the lower component of $\mathbf{q}_{\mathrm{TR}}(\mathbf{x},-t')$, hence,

$$\mathbf{J}\mathbf{q}_{\mathrm{TR}}(\mathbf{x},-t') = \begin{pmatrix} U_{\mathrm{TR}}(\mathbf{x},-t') \\ -\beta(t')\partial_{t'}U_{\mathrm{TR}}(\mathbf{x},-t') \end{pmatrix}. \tag{77}$$

This sign change is indicated between the two frames in Fig. 6. Next, matrix $\mathbf{W}(\mathbf{x},t,t')$ in Eq. (75) propagates this field from t' through the actual time-variant material to t (see the upper frame of Fig. 6). For $t<t_A$, the left-hand side of Eq. (75) is interpreted as an acausal field observed at t, propagating to a sink at t_A. For $t>t_A$, it is interpreted as a causal field observed at t, radiated by a source at t_A of strength $\mathbf{J}\mathbf{d}_0(\mathbf{x})$. Since there is no real sink nor a real source at t_A, the sink and the source at t_A in the interpretation of the left-hand side of Eq. (75) are both virtual. In the numerical example in Figs. 6 and 7, the virtual source at $t_A = 40\mu$s coincides with a time boundary, but note that the theory described here holds for any choice of t_A.

In summary, Eq. (75) formalizes the principle of idealized time-reversed acoustics in a homogeneous, time-variant material. To compare this expression with Eq. (21), we choose for convenience

$$\mathbf{d}_0(\mathbf{x}) = \begin{pmatrix} 0 \\ b_{\mathrm{d},0}(\mathbf{x}) \end{pmatrix}. \tag{78}$$

For the upper element of Eq. (75), we thus obtain [using Eq. (77)]

$$-W^{U,V}(\mathbf{x},t,t_A)*_xb_{\mathrm{d},0}(\mathbf{x}) = W^{U,U}(\mathbf{x},t,t')*_xU_{\mathrm{TR}}(\mathbf{x},-t') + W^{U,V}(\mathbf{x},t,t')*_x\{-\beta(t')\partial_{t'}U_{\mathrm{TR}}(\mathbf{x},-t')\}, \tag{79}$$

or, using Eqs. (51) and (53) and the (a)causality conditions of Eqs. (33) and (35),

$$\{\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t_A) - \mathcal{G}^a_t(\mathbf{x},\mathbf{0},t,t_A)\}*_xb_{\mathrm{d},0}(\mathbf{x}) = -\beta(t')\Big[\{\partial_{t'}\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t')\}*_xU_{\mathrm{TR}}(\mathbf{x},-t') + \mathcal{G}_t(\mathbf{x},\mathbf{0},t,t')*_x\{-\partial_{t'}U_{\mathrm{TR}}(\mathbf{x},-t')\}\Big]. \tag{80}$$

Using the integral notation for the spatial convolutions at the right-hand side and the space-shift invariance of the Green's function [Eq. (34)], this becomes

$$\int_{\mathbb{R}^m}\{\mathcal{G}_t(\mathbf{x},\mathbf{x}',t,t_A)-\mathcal{G}_t^a(\mathbf{x},\mathbf{x}',t,t_A)\}b_{\mathrm{d},0}(\mathbf{x}')\mathrm{d}\mathbf{x}'$$
$$=-\beta(t')\int_{\mathbb{R}^m}\Bigg[\underbrace{\{\partial_{t'}\mathcal{G}_t(\mathbf{x},\mathbf{x}',t,t')\}}_{\text{“propagator”}}\underbrace{U_{\mathrm{TR}}(\mathbf{x}',-t')}_{\text{“source field”}}$$
$$+\underbrace{\mathcal{G}_t(\mathbf{x},\mathbf{x}',t,t')}_{\text{“propagator”}}\underbrace{\{-\partial_{t'}U_{\mathrm{TR}}(\mathbf{x}',-t')\}}_{\text{“source field”}}\Bigg]\mathrm{d}\mathbf{x}'. \tag{81}$$

This expression is the counterpart of Eq. (21). The time-reversed field $P(\mathbf{x}',-t)$ in Eq. (21) is replaced by the fields $U_{\mathrm{TR}}(\mathbf{x}',-t')$ and $-\partial_{t'}U_{\mathrm{TR}}(\mathbf{x}',-t')$ in a time-reversed version of the actual material. The right-hand side quantifies the propagation of $U_{\mathrm{TR}}(\mathbf{x}',-t')$ and $-\partial_{t'}U_{\mathrm{TR}}(\mathbf{x}',-t')$ from sources at all $\mathbf{x}'$ at a single time instant t' to any t larger than t'. Unlike in Eq. (21), the two terms on the right-hand side of Eq. (81) cannot be recast into a single term. The second term on the left-hand side, $\int_{\mathbb{R}^m}\mathcal{G}_t^a(\mathbf{x},\mathbf{x}',t,t_A)b_{\mathrm{d},0}(\mathbf{x}')\mathrm{d}\mathbf{x}'$, is the field that focuses at t_A, and the first term on the left-hand side, $\int_{\mathbb{R}^m}\mathcal{G}_t(\mathbf{x},\mathbf{x}',t,t_A)b_{\mathrm{d},0}(\mathbf{x}')\mathrm{d}\mathbf{x}'$, is the field emitted by the virtual source distribution $b_{\mathrm{d},0}(\mathbf{x})$ at t_A (see the upper frame of Fig. 6).

What we have discussed in this section is the ideal version of time-reversed acoustics in a homogeneous, time-variant material. It requires a wave field in a time-reversed version of the actual material, measurements of this field and its time derivative at all $\mathbf{x}'$ at a single time-instant $-t'$, and emission of this field and its time derivative (the latter with reversed sign) into the actual material by sources at all $\mathbf{x}'$ at time-instant t'. Experiments described by Refs. 11, 12, 14, and 84 make use of several shortcuts. These experiments are carried out in a material that is time-invariant, except for an impulsive temporal disruption at $t'=0$. Hence, the time-reversed material is the same as the actual material. Furthermore, instead of measuring the field and its time derivative and re-emitting this (with a sign change) into the material, the impulsive temporal disruption of the material at $t'=0$ acts as a time boundary, which partly reflects and partly transmits the field into the material. The reflected field approximates the re-emitted field and focuses at the position of the original source(s), whereas the transmitted field, which continues propagating forward, is ignored. Despite the approximations, these experiments are fascinating and triggered the analysis described in this paper.

F. Green's function retrieval in a homogeneous, time-variant material

We start by introducing the principle of Green's function retrieval in a homogeneous, time-variant material in an intuitive way. We seek for a counterpart of Green's function retrieval in an inhomogeneous, time-invariant material, as discussed in Sec. III E. Consider again Fig. 2(a), which illustrates Green's functions in an inhomogeneous material, with impulsive sources at $t=0$ and $\mathbf{x}'$ on a closed boundary, and receivers at $\mathbf{x}_A$ and $\mathbf{x}$ inside the closed boundary. The temporal cross-correlation of the Green's functions, observed at $\mathbf{x}_A$ and $\mathbf{x}$, integrated over all sources at $\mathbf{x}'$, yields the Green's function (minus its time-reversed version) between $\mathbf{x}_A$ and $\mathbf{x}$. Hence, the receiver at $\mathbf{x}_A$ has turned into a virtual source, and the response to this source is observed at $\mathbf{x}$ [see Fig. 2(b)]. The counterpart of Fig. 2(a) is shown in the left frame of Fig. 8. It shows the Green's function $\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t'=0)$ (convolved with a spatial wavelet) in a time-variant material, with its source at $\mathbf{x}=\mathbf{0}$ and $t'=0$ (indicated by the red dot), and receivers at $t_A=40\,\mu$s (indicated by the dashed blue line) and variable t (two of them indicated by the dotted

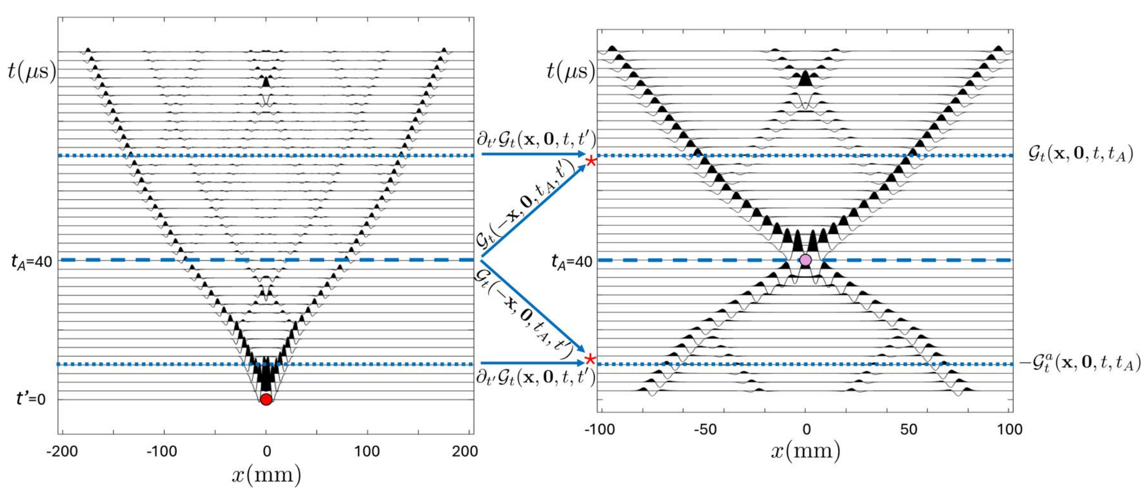

FIG. 8. Illustration of the principle of Green's function retrieval in a homogeneous, time-variant material. The left frame shows the Green's function $\mathcal{G}_t(\mathbf{x},\mathbf{0},t,t'=0)$ (the same as in Fig. 5), with its source indicated by the red dot. The spatial cross-correlation of the response at $t_A=40\,\mu$s with that at any other t (explained in more detail in the text) yields the response to a virtual source (sink), indicated by the pink dot in the right frame (note the different scales of the space axes). The response in the right frame is identical to that in the upper frame of Fig. 6, which was obtained by the principle of time-reversed acoustics, discussed in Sec. IV E.

blue lines). We expect that the spatial cross-correlation of the Green's function observed at $t_A = 40\,\mu$s (the dashed blue line), with that at any other t (the dotted blue lines), yields the Green's function (minus its acausal version) between t_A and t, as shown in the right frame of Fig. 8. This frame can be interpreted as the counterpart of Fig. 2(b). It shows the response to a virtual source at $\mathbf{x} = \mathbf{0}$ and $t_A = 40\,\mu$s, indicated by the pink dot, observed by receivers at variable $\mathbf{x}$ and t (the dotted blue lines). In the following, we show mathematically that this Green's function (minus its acausal version) indeed follows from spatial cross-correlations of Green's functions, albeit in a somewhat more complex way than described above.

Our starting point is again the representation of Eq. (55). The acausal Green's functions in matrix $\mathbf{W}(\mathbf{x}, t', t_A)$, with $t' < t_A$, are not associated with a physical situation. We use symmetry properties of the Green's functions to transform this matrix into a form that complies with physics. Using the reciprocity relation of Eq. (39) in Eqs. (51)–(54), we find

$$W^{U,V}(\mathbf{x}, t, t_A) = -W^{U,V}(\mathbf{x}, t_A, t), \tag{82}$$

$$W^{V,V}(\mathbf{x}, t, t_A) = W^{U,U}(\mathbf{x}, t_A, t), \tag{83}$$

$$W^{U,U}(\mathbf{x}, t, t_A) = W^{V,V}(\mathbf{x}, t_A, t), \tag{84}$$

$$W^{V,U}(\mathbf{x}, t, t_A) = -W^{V,U}(\mathbf{x}, t_A, t). \tag{85}$$

Hence,

$$\mathbf{W}(\mathbf{x}, t, t_A) = \mathbf{N}\mathbf{W}^t(\mathbf{x}, t_A, t)\mathbf{N}^{-1}, \tag{86}$$

where superscript t denotes transposition and

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{87}$$

Substituting Eq. (86) into Eq. (55) and post-multiplying both sides by $\mathbf{N}$, we obtain

$$\mathbf{W}(\mathbf{x}, t, t_A)\mathbf{N} = \mathbf{W}(\mathbf{x}, t, t')*_x\mathbf{N}\mathbf{W}^t(\mathbf{x}, t_A, t'). \tag{88}$$

The right-hand side represents a spatial convolution. From Eqs. (45)–(47), it follows that the propagator matrices are symmetric in $\mathbf{x}$. Hence, we may replace $\mathbf{x}$ in the rightmost matrix in Eq. (88) by $-\mathbf{x}$, so that the right-hand side becomes a spatial cross-correlation. Then, for the upper-left element, we obtain

$$-W^{U,V}(\mathbf{x},t,t_A) = W^{U,U}(\mathbf{x},t,t')*_x W^{U,V}(-\mathbf{x},t_A,t') - W^{U,V}(\mathbf{x},t,t')*_x W^{U,U}(-\mathbf{x},t_A,t'). \tag{89}$$

We assume again that the order of t_A and t is arbitrary and that t' is smaller than both t_A and t. Using Eqs. (51) and (53) and the (a)causality conditions of Eqs. (33) and (35), we thus obtain

$$\begin{aligned}\mathcal{G}_t(\mathbf{x}, \mathbf{0}, t, t_A) - \mathcal{G}_t^a(\mathbf{x}, \mathbf{0}, t, t_A) &= \beta(t')\Big[\big\{\partial_{t'}\mathcal{G}_t(\mathbf{x}, \mathbf{0}, t, t')\big\}*_x\mathcal{G}_t(-\mathbf{x}, \mathbf{0}, t_A, t') \\ &\quad - \mathcal{G}_t(\mathbf{x}, \mathbf{0}, t, t')*_x\partial_{t'}\mathcal{G}_t(-\mathbf{x}, \mathbf{0}, t_A, t')\Big].\end{aligned} \tag{90}$$

This expression is the counterpart of Eq. (22). The right-hand side quantifies the spatial cross-correlation of specific combinations of Green's functions and their derivatives, observed at all $\mathbf{x}$ at t_A and t, in response to sources at a single time instant t'. Unlike in Eq. (24), the two terms on the right-hand side of Eq. (90) cannot be recast into a single term with a single source type, meaning that we require responses to two types of sources at t'. The left-hand side is the retrieved Green's function between t_A and t (minus its acausal version) (see the right frame of Fig. 8, which is identical to the upper frame of Fig. 6 in Sec. IV E). Whereas in Sec. IV E this was obtained by emitting the response of a time-reversed material into the actual time-variant material [Eq. (81) and Fig. 6], here it is explained as the result of cross-correlating responses at two time instances in one-and-the-same material [Eq. (90) and Fig. 8]. Both methods were derived from Eq. (55) and a specific reorganization of the matrix $\mathbf{W}(\mathbf{x}, t', t_A)$. This confirms the link between time-reversal acoustics and Green's function retrieval by spatial cross-correlation in a homogeneous, time-variant material.

The modification of Eq. (90) for Green's function retrieval by spatial cross-correlation of ambient noise observations at t_A and t is more complicated than in Sec. III E because the two terms on the right-hand side of Eq. (90) cannot be recast into a single term with a single source type. In a companion paper,[35] the author discusses a compromise for the special case of a time-invariant material, and discusses its relation with the SPAC (spatial autocorrelation) method.[85–87] Extending this approach to time-variant materials leads to artifacts in the retrieved Green's function that increase with increasing complexity of the time-variant material. A further discussion of Green's function retrieval by spatial cross-correlation of ambient noise is beyond the scope of this paper.

V. CONCLUSION

We started this paper with a review of the principles of classical time-reversed acoustics and Green's function retrieval for inhomogeneous, time-invariant materials. Along the same lines, we discussed the counterparts of these principles for homogeneous, time-variant materials, and compared those with the classical principles. One observation from this comparison is that the space and time coordinates are interchanged between the methods for the two classes of material. As a consequence, temporal convolutions and correlations in the classical approaches for inhomogeneous, time-invariant materials are replaced by spatial convolutions and correlations in their counterparts for homogeneous, time-variant materials. However, the interchangement of space and time does not apply to causality conditions, which are always related to the time coordinate.

This complicates the translation of methods from one class of material to the other and *vice versa*. Another complicating factor is that the number of time dimensions (one) is not the same as the number of space dimensions (two or three).

For time-reversed acoustics and for Green's function retrieval, we separately review the relations between applications in the two classes of material.

The principle of classical time-reversed acoustics in an inhomogeneous, time-invariant material involves in essence:

1. Emission of a wave field by a source at $\mathbf{x}_A$ into the inhomogeneous material.
2. Recording of this wave field by receivers on a closed boundary.
3. Emission of the time-reversal of the recorded field into the inhomogeneous material by sources at the positions of the receivers.
4. The emitted wave field focuses at $\mathbf{x}_A$ and the focused field is a virtual source, emitting waves into the inhomogeneous material.

The counterpart of the principle of classical time-reversed acoustics for a homogeneous, time-variant material involves:

1. Emission of a wave field by a source at $-t_A$ into a time-reversed version of the actual time-variant material.
2. Recording of this wave field and its time-derivative by receivers at a single time instant.
3. Emission of the recorded field and its time-derivative (the latter with reversed sign) into the actual time-variant material by sources at the positions of the receivers.
4. The emitted wave field focuses at t_A and the focused field is a virtual source, emitting waves into the time-variant material.

This summarizes ideal time-reversed acoustics in a homogeneous, time-variant material. In actual experiments,[11,12,14,84] the material is time-invariant except for an impulsive temporal disruption. This disruption acts as a time boundary, which partly reflects the field into the material and as such avoids recording and re-emission of the wave field.

The principle of classical Green's function retrieval in an inhomogeneous, time-invariant material involves in essence:

1. Recording of passive wave fields at $\mathbf{x}_A$ and $\mathbf{x}$ in response to sources at $\mathbf{x}'$ on a closed boundary.
2. Temporal cross-correlating the recorded fields at $\mathbf{x}_A$ and $\mathbf{x}$ per source, followed by a summation over all sources on the closed boundary.
3. The time derivative of the result is the Green's function (minus its time-reversed version) between a virtual source (sink) at $\mathbf{x}_A$ and a physical receiver at $\mathbf{x}$.
4. In the case that the sources are simultaneously acting uncorrelated noise sources, step 2 is replaced by a single cross-correlation of the ambient noise recordings at $\mathbf{x}_A$ and $\mathbf{x}$.

The counterpart of the principle of classical Green's function retrieval for a homogeneous, time-variant material involves:

1. Recording of passive wave fields at t_A and t in response to two types of sources at a single time instant t'.
2. Spatial cross-correlating two specific combinations of the recorded fields at t_A and t per source type, followed by a summation of the two terms.
3. The result is the Green's function (minus its acausal version) between a virtual source (sink) at t_A and a physical receiver at t.
4. A modification of Green's function retrieval by spatial cross-correlation in the case of simultaneously acting uncorrelated noise sources has thus far only been derived for a time-invariant material.[35]

Finally, we review the relations between time-reversed acoustics and Green's function retrieval, first for an inhomogeneous, time-invariant material, and after that for a homogeneous, time-variant material.

For an inhomogeneous, time-invariant material, the principles of time-reversed acoustics and Green's function retrieval are variations of the homogeneous Green's function representation of Eq. (20) for this class of material. In particular, consider the temporal convolutional product under the integral in Eq. (20):

$$\left\{\nabla'\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)\cdot\mathbf{n}\right\}*_t\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,-t,0). \tag{91}$$

- In time-reversed acoustics, the second Green's function (after a temporal convolution with a source wavelet) stands for the time-reversed field $P(\mathbf{x}',-t)$. This field is fed to sources at $\mathbf{x}'$ on the closed boundary $\mathcal{S}$ and propagated by the first Green's function from $\mathbf{x}'$ to any point $\mathbf{x}$ inside $\mathcal{S}$, as illustrated in Fig. 1(b). The result, after integration over $\mathcal{S}$, is the homogeneous Green's function $\mathcal{G}^h_x(\mathbf{x},\mathbf{x}_A,t,0)$ (temporally convolved with the time-reversal of the source wavelet). This is interpreted as a field focusing at $\mathbf{x}_A$ [Fig. 1(b)] and, after having focused, propagating away from $\mathbf{x}_A$ [Fig. 1(c)].
- In Green's function retrieval, the second Green's function in Eq. (91) is, on basis of the reciprocity relation of Eq. (13), replaced by $\mathcal{G}_x(\mathbf{x}_A,\mathbf{x}',-t,0)$. Now the temporal convolutional product can be interpreted as a temporal cross-correlation (due to the time-reversal of the second Green's function) of observations by receivers at $\mathbf{x}$ and $\mathbf{x}_A$, in response to sources at $\mathbf{x}'$ on $\mathcal{S}$, as illustrated in Fig. 2(a). The result, after integration over $\mathcal{S}$, is the homogeneous Green's function $\mathcal{G}^h_x(\mathbf{x},\mathbf{x}_A,t,0)$, which (for $t>0$) is interpreted as the response to a virtual source at one of the receiver positions ($\mathbf{x}_A$), observed by the receiver at $\mathbf{x}$ [Fig. 2(b)].

In summary, whereas in time-reversed acoustics, $\mathcal{G}_x(\mathbf{x}',\mathbf{x}_A,-t,0)$ in Eq. (91) stands for the time-reversed field that is fed to sources at $\mathbf{x}'$ on $\mathcal{S}$ and propagated by $\nabla'\mathcal{G}_x(\mathbf{x},\mathbf{x}',t,0)\cdot\mathbf{n}$ to any $\mathbf{x}$ inside $\mathcal{S}$ (Fig. 1), in Green's function retrieval its reciprocal version $\mathcal{G}_x(\mathbf{x}_A, \mathbf{x}', t, 0)$ is temporally cross-correlated with $\nabla' \mathcal{G}_x(\mathbf{x}, \mathbf{x}', t, 0) \cdot \mathbf{n}$ and integrated over all $\mathbf{x}'$ on $\mathcal{S}$ (Fig. 2). Both approaches lead to the same homogeneous Green's function $\mathcal{G}_x^h(\mathbf{x}, \mathbf{x}_A, t, 0)$ between $\mathbf{x}_A$ and $\mathbf{x}$.

For a homogeneous, time-variant material, the principles of time-reversed acoustics and Green's function retrieval are variations of the homogeneous Green's function representation of Eq. (57) for this class of material. In particular, consider the spatial convolutional product in Eq. (57) (the analysis of the other convolutional product follows the same arguments):

$$\{\partial_{t'} \mathcal{G}_t(\mathbf{x}, \mathbf{0}, t, t')\} *_x \mathcal{G}_t^a(\mathbf{x}, \mathbf{0}, t', t_A). \tag{92}$$

- In time-reversed acoustics, the second Green's function (after applying the reciprocity relation $\mathcal{G}_t^a(\mathbf{x}, \mathbf{0}, t', t_A) = \mathcal{G}_{t,\mathrm{TR}}(\mathbf{x}, \mathbf{0}, -t', -t_A)$ and a spatial convolution with a source function) stands for the field $U_{\mathrm{TR}}(\mathbf{x}, -t')$ in a time-reversed version of the actual material. This field (together with its sign-reversed time-derivative) is fed to sources at time instance t' and propagated by the first Green's function to any time t beyond t', as illustrated in Fig. 6. The result is the homogeneous Green's function $\mathcal{G}_t^h(\mathbf{x}, \mathbf{0}, t, t_A)$ (spatially convolved with the source function). This is interpreted as a field focusing at t_A and, after having focused, propagating away from t_A (Fig. 6).
- In Green's function retrieval, the second Green's function in Eq. (92) is, on basis of the reciprocity relation of Eq. (36) and its symmetry in $\mathbf{x}$, replaced by $\mathcal{G}_t(-\mathbf{x}, \mathbf{0}, t_A, t')$. Now the spatial convolutional product can be interpreted as a spatial cross-correlation (due to the space-reversal of the second Green's function) of observations by receivers at t and t_A, in response to a source at t', as illustrated in the left frame of Fig. 8. The result, after adding the other convolutional product of Eq. (57), is the homogeneous Green's function $\mathcal{G}_t^h(\mathbf{x}, \mathbf{0}, t, t_A)$, which (for $t > t_A$) is interpreted as the response to a virtual source at one of the receiver times (t_A), observed by receivers at t (Fig. 8, right).

In summary, whereas in time-reversed acoustics, $\mathcal{G}_t^a(\mathbf{x}, \mathbf{0}, t', t_A) = \mathcal{G}_{t,\mathrm{TR}}(\mathbf{x}, \mathbf{0}, -t', -t_A)$ in Eq. (92) stands for the field in the time-reversed material that is fed to sources at t' and propagated by $\partial_{t'} \mathcal{G}_t(\mathbf{x}, \mathbf{0}, t, t')$ to any t beyond t' (Fig. 6), in Green's function retrieval its reciprocal version $\mathcal{G}_t(\mathbf{x}, \mathbf{0}, t_A, t')$ is spatially cross-correlated with $\partial_{t'} \mathcal{G}_t(\mathbf{x}, \mathbf{0}, t, t')$ (Fig. 8). Both approaches lead to the same homogeneous Green's function $\mathcal{G}_t^h(\mathbf{x}, \mathbf{0}, t, t_A)$ between t_A and t.

ACKNOWLEDGMENTS

J.A. and D.-J.v.M. were supported by Swiss National Science Foundation Grant No. 197182.

AUTHOR DECLARATIONS

Conflict of interest

The authors have no conflicts of interests to disclose.

DATA AVAILABILITY

No data have been used for this study. Animations related to Figs. 3 and 7 can be accessed via the following URL https://gitlab.com/geophysicsdelft/OpenSource in the directory .../Green-time-variant.

APPENDIX: ANALYTICAL EXPRESSIONS FOR A HOMOGENEOUS, TIME-INVARIANT MATERIAL

1. Green's function of a homogeneous, time-invariant material

For the special case of a homogeneous, time-invariant material, the analytical solution of Eq. (32), with causality condition Eq. (33), reads

$$\mathcal{G}_t(\mathbf{x}, \mathbf{x}_A, t, t_A) = \begin{cases} \dfrac{1}{4\pi\beta c^2} \dfrac{\delta(t - t_A - |\mathbf{x} - \mathbf{x}_A|/c)}{|\mathbf{x} - \mathbf{x}_A|}, & \text{3D}, \\[2ex] \dfrac{1}{2\pi\beta c^2} \dfrac{H(t - t_A - |\mathbf{x} - \mathbf{x}_A|/c)}{\sqrt{(t - t_A)^2 - |\mathbf{x} - \mathbf{x}_A|^2/c^2}}, & \text{2D}, \end{cases} \tag{A1}$$

where $H(t)$ is the Heaviside step function. We define the spatial Fourier transformation as

$$\check{U}(\mathbf{k}) = \int_{\mathbb{R}^m} U(\mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}) \mathrm{d}\mathbf{x}, \tag{A2}$$

with $m = 3$, $\mathbf{x} = (x, y, z)$, $\mathbf{k} = (k_x, k_y, k_z)$ for the 3D situation, and $m = 2$, $\mathbf{x} = (x, z)$, $\mathbf{k} = (k_x, k_z)$ for the 2D situation; $\mathbb{R}$ denotes the set of real numbers. The inverse spatial Fourier transformation is defined as

$$U(\mathbf{x}) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \check{U}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \mathrm{d}\mathbf{k}. \tag{A3}$$

In the spatial Fourier domain, the Green's function reads

$$\check{\mathcal{G}}_t(\mathbf{k}, \mathbf{x}_A, t, t_A) = \exp(-i\mathbf{k} \cdot \mathbf{x}_A) H(t - t_A) \frac{\sin(|\mathbf{k}| c (t - t_A))}{\beta c |\mathbf{k}|}. \tag{A4}$$

This expression holds for the 3D as well as for the 2D situation. Analogous to Eq. (36), we have for the acausal Green's function in the spatial Fourier domain

$$\check{\mathcal{G}}_t^a(\mathbf{k}, \mathbf{x}_A, t, t_A) = \check{\mathcal{G}}_t(\mathbf{k}, \mathbf{x}_A, t_A, t). \tag{A5}$$

2. Propagator matrix for a homogeneous, time-invariant slab

We consider the propagator matrix $\mathbf{W}(\mathbf{x}, t_n, t_{n-1})$ for a homogeneous, time-invariant slab $t_{n-1} < t < t_n$, with parameters α_n, β_n, and c_n. According to Eqs. (36), (51), and (A1), the upper-right element of this matrix is given by

$$W^{U,V}(\mathbf{x},t_n,t_{n-1}) = \mathcal{G}_t^a(\mathbf{x},\mathbf{0},t_n,t_{n-1}) - \mathcal{G}_t(\mathbf{x},\mathbf{0},t_n,t_{n-1}) = \begin{cases} \dfrac{1}{4\pi\beta_n c_n^2}\dfrac{\delta(\Delta t_n + |\mathbf{x}|/c_n) - \delta(\Delta t_n - |\mathbf{x}|/c_n)}{|\mathbf{x}|}, & \text{3D}, \\ \dfrac{1}{2\pi\beta_n c_n^2}\dfrac{H(-\Delta t_n - |\mathbf{x}|/c_n) - H(\Delta t_n - |\mathbf{x}|/c_n)}{\sqrt{\Delta t_n^2 - |\mathbf{x}|^2/c_n^2}}, & \text{2D}, \end{cases} \quad \text{(A6)}$$

with $\Delta t_n = t_n - t_{n-1}$. The other elements of $\mathbf{W}(\mathbf{x},t_n,t_{n-1})$ follow from Eqs. (52)–(54) and (A6).

Using Eqs. (A4) and (A5), we obtain for the upper-right element of the propagator matrix $\check{\mathbf{W}}(\mathbf{k},t_n,t_{n-1})$ in the spatial Fourier domain

$$\begin{aligned}\check{W}^{U,V}(\mathbf{k},t_n,t_{n-1}) &= \check{\mathcal{G}}_t^a(\mathbf{k},\mathbf{0},t_n,t_{n-1}) - \check{\mathcal{G}}_t(\mathbf{k},\mathbf{0},t_n,t_{n-1}) \\ &= -\frac{1}{\beta_n c_n|\mathbf{k}|}\sin(|\mathbf{k}|c_n\Delta t_n). \end{aligned} \quad \text{(A7)}$$

This expression holds for the 3D as well as for the 2D situation. The other elements of $\check{\mathbf{W}}(\mathbf{k},t_n,t_{n-1})$ follow from the Fourier transforms of Eqs. (52)–(54) and Eq. (A7). Hence,

$$\check{\mathbf{W}}(\mathbf{k},t_n,t_{n-1}) = \begin{pmatrix} \cos(|\mathbf{k}|c_n\Delta t_n) & -\dfrac{1}{\beta_n c_n|\mathbf{k}|}\sin(|\mathbf{k}|c_n\Delta t_n) \\ \beta_n c_n|\mathbf{k}|\sin(|\mathbf{k}|c_n\Delta t_n) & \cos(|\mathbf{k}|c_n\Delta t_n) \end{pmatrix}. \quad \text{(A8)}$$

[1]F. R. Morgenthaler, "Velocity modulation of electromagnetic waves," IRE Trans. Microwave Theory Tech. **6**, 167–172 (1958).

[2]C. L. Jiang, "Wave propagation and dipole radiation in a suddenly created plasma," IEEE Trans. Antennas Propag. **23**, 83–90 (1975).

[3]J. T. Mendonça and P. K. Shukla, "Time refraction and time reflection: Two basic concepts," Phys. Scr. **65**, 160–163 (2002).

[4]T. T. Koutserimpas and R. Fleury, "Electromagnetic waves in a time periodic medium with step-varying refractive index," IEEE Trans. Antennas Propag. **66**, 5300–5307 (2018).

[5]C. Caloz and Z. L. Deck-Léger, "Spacetime metamaterials: I. General concepts," IEEE Trans. Antennas Propag. **68**, 1569–1582 (2020).

[6]D. Ramaccia, A. Toscano, and F. Bilotti, "Light propagation through metamaterial temporal slabs: Reflection, refraction, and special cases," Opt. Lett. **45**, 5836–5839 (2020).

[7]B. Apffel and E. Fort, "Frequency conversion cascade by crossing multiple space and time interfaces," Phys. Rev. Lett. **128**, 064501 (2022).

[8]H. Moussa, G. Xu, S. Yin, E. Galiffi, Y. Ra'di, and A. Alù, "Observation of temporal reflection and broadband frequency translation at photonic time interfaces," Nat. Phys. **19**, 863–868 (2023).

[9]V. Pacheco-Peña, M. Fink, and N. Engheta, "Temporal chirp, temporal lensing, and temporal routing via space-time interfaces," Phys. Rev. B **111**, L100306 (2025).

[10]L. Stefanini, E. Galiffi, S. Yin, S. Singh, D. M. Solís, N. Engheta, A. Toscano, D. Ramaccia, F. Bilotti, and A. Alù, "Theory and experimental observation of scattering by a space-time corner," Phys. Rev. Lett. **135**, 113802 (2025).

[11]V. Bacot, M. Labousse, A. Eddi, M. Fink, and E. Fort, "Time reversal and holography with spacetime transformations," Nat. Phys. **12**, 972–977 (2016).

[12]M. Fink and E. Fort, "From the time-reversal mirror to the instantaneous time mirror," Eur. Phys. J. Spec. Top. **226**, 1477–1486 (2017).

[13]G. Bal, M. Fink, and O. Pinaud, "Time-reversal by time-dependent perturbations," SIAM J. Appl. Math. **79**, 754–780 (2019).

[14]S. Hidalgo-Caballero, S. K. Sreenivas, V. Bacot, S. Wildeman, M. Harazi, X. Jia, A. Tourin, M. Fink, A. Cassinelli, M. Labousse, and E. Fort, "Damping-driven time reversal for waves," Phys. Rev. Lett. **130**, 087201 (2023).

[15]A. Delory, C. Prada, M. Lanoy, A. Eddi, M. Fink, and F. Lemoult, "Elastic wave packets crossing a space-time interface," Phys. Rev. Lett. **133**, 267201 (2025).

[16]M. Fink, "Time-reversal of ultrasonic fields: I. Basic principles," IEEE Trans. Ultras. Ferroelectr. Freq. Contr. **39**, 555–566 (1992).

[17]Y. Xiao, D. N. Maywar, and G. P. Agrawal, "Reflection and transmission of electromagnetic waves at a temporal boundary," Opt. Lett. **39**, 574–577 (2014).

[18]A. T. de Hoop and I. E. Lager, "Closed-form analytic expressions for the pulsed-source radiated electromagnetic field in a class of media with time-varying wave speed," Wave Motion **51**, 418–424 (2014).

[19]M. A. Salem and C. Caloz, "Space-time cross-mapping and application to wave scattering," arXiv:1504.02012 (2015).

[20]D. Torrent, W. J. Parnell, and A. N. Norris, "Loss compensation in time-dependent elastic metamaterials," Phys. Rev. B **97**, 014105 (2018).

[21]C. Caloz and Z. L. Deck-Léger, "Spacetime metamaterials: II. Theory and applications," IEEE Trans. Antennas Propag. **68**, 1583–1598 (2020).

[22]D.-J. Van Manen, J. Aichele, J. Müller, M. Serra-Garcia, and K. Wapenaar, "On acoustic space-time media that compute their own inverse," arXiv:2406.15203 (2024).

[23]K. Wapenaar, J. Aichele, and D.-J. van Manen, "Waves in space-dependent and time-dependent materials: A systematic comparison," Wave Motion **130**, 103374 (2024).

[24]C. Draeger and M. Fink, "One-channel time-reversal in chaotic cavities: Theoretical limits," J. Acoust. Soc. Am. **105**, 611–617 (1999).

[25]J. de Rosny and M. Fink, "Overcoming the diffraction limit in wave physics using a time-reversal mirror and a novel acoustic sink," Phys. Rev. Lett. **89**, 124301 (2002).

[26]R. L. Weaver and O. I. Lobkis, "Ultrasonics without a source: Thermal fluctuation correlations at MHz frequencies," Phys. Rev. Lett. **87**, 134301 (2001).

[27]M. Campillo and A. Paul, "Long-range correlations in the diffuse seismic coda," Science **299**, 547–549 (2003).

[28]K. Wapenaar, "Synthesis of an inhomogeneous medium from its acoustic transmission response," Geophysics **68**, 1756–1759 (2003).

[29]R. Snieder, "Extracting the Green's function from the correlation of coda waves: A derivation based on stationary phase," Phys. Rev. E **69**, 046610 (2004).
[30]G. T. Schuster, J. Yu, J. Sheng, and J. Rickett, "Interferometric/daylight seismic imaging," Geophys. J. Int. **157**, 838–852 (2004).
[31]D.-J. Van Manen, J. O. A. Robertsson, and A. Curtis, "Modeling of wave propagation in inhomogeneous media," Phys. Rev. Lett. **94**, 164301 (2005).
[32]A. Derode, E. Larose, M. Tanter, J. de Rosny, A. Tourin, M. Campillo, and M. Fink, "Recovering the Green's function from field-field correlations in an open scattering medium (L)," J. Acoust. Soc. Am. **113**, 2973–2976 (2003).
[33]K. Wapenaar, J. Fokkema, and R. Snieder, "Retrieving the Green's function in an open system by cross-correlation: A comparison of approaches (L)," J. Acoust. Soc. Am. **118**, 2783–2786 (2005).
[34]K. Wapenaar, "Green's functions, propagation invariants, reciprocity theorems, wave-field representations and propagator matrices in two-dimensional time-dependent materials," Proc. R. Soc. A **481**, 20240479 (2025).
[35]J. Aichele (2026). "Diffuse wave-field correlation in space: An instantaneous time-reversal mirror for Green's function retrieval of dense arrays," Zenodo. https://doi.org/10.5281/zenodo.20799704
[36]A. T. de Hoop, *Handbook of Radiation and Scattering of Waves* (Academic, London, UK, 1995).
[37]K. A. Lurie, *An Introduction to the Mathematical Theory of Dynamic Materials* (Springer, New York, 2007).
[38]G. Trainiti and M. Ruzzene, "Non-reciprocal elastic wave propagation in spatiotemporal periodic structures," New J. Phys. **18**, 083047 (2016).
[39]H. Nassar, X. C. Xu, A. N. Norris, and G. L. Huang, "Modulated phononic crystals: Non-reciprocal wave propagation and Willis materials," J. Mech. Phys. Solids **101**, 10–29 (2017).
[40]H. Nassar, H. Chen, A. N. Norris, M. R. Haberman, and G. L. Huang, "Non-reciprocal wave propagation in modulated elastic metamaterials," Proc. R. Soc. A **473**, 20170188 (2017).
[41]B. M. Goldsberry, S. P. Wallen, and M. R. Haberman, "Non-reciprocal wave propagation in mechanically-modulated continuous elastic metamaterials," J. Acoust. Soc. Am. **146**, 782–788 (2019).
[42]A. Sotoodehfar, S. Boshgazi, M. Memarian, B. Rejaei, and K. Mehrany, "Investigating non-reciprocity in time-periodic media using a perturbative approach," Opt. Express **31**, 12534–12548 (2023).
[43]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. I.
[44]R. P. Porter, "Diffraction-limited, scalar image formation with holograms of arbitrary shape," J. Opt. Soc. Am. **60**, 1051–1059 (1970).
[45]M. L. Oristaglio, "An inverse scattering formula that uses all the data," Inverse Probl. **5**, 1097–1105 (1989).
[46]G. A. McMechan, "Determination of source parameters by wavefield extrapolation," Geophys. J. **71**, 613–628 (1982).
[47]D. Gajewski and E. Tessmer, "Reverse modelling for seismic event characterization," Geophys. J. Int. **163**, 276–284 (2005).
[48]C. Larmat, R. A. Guyer, and P. A. Johnson, "Time-reversal methods in geophysics," Phys. Today **63** (8),31–35 (2010).
[49]C. Hemon, "Equations d'onde et modeles" ("Wave equations and models"), Geophys. Prospect. **26**, 790–821 (1978).
[50]N. D. Whitmore, "Iterative depth migration by backward time propagation," in *SEG Expanded Abstr.* (1983), pp. 382–385.
[51]G. A. McMechan, "Migration by extrapolation of time-dependent boundary values," Geophys. Prospect. **31**, 413–420 (1983).
[52]H.-W. Zhou, H. Hu, Z. Zou, Y. Wo, and O. Youn, "Reverse time migration: A prospect of seismic imaging methodology," Earth Sci. Rev. **179**, 207–227 (2018).
[53]M. Fink and C. Prada, "Acoustic time-reversal mirrors," Inverse Probl. **17**, R1–R38 (2001).
[54]C. Prada, J.-L. Thomas, and M. Fink, "The iterative time reversal process: Analysis of the convergence," J. Acoust. Soc. Am. **97**, 62–71 (1995).
[55]G. Montaldo, J. F. Aubry, M. Tanter, and M. Fink, "Spatio-temporal coding in complex media for optimum beamforming: The iterative time-reversal approach," IEEE Trans. Ultras. Ferroelectr. Freq. Contr. **52**, 220–230 (2005).
[56]L. Borcea, G. Papanicolaou, and C. Tsogka, "Optimal waveform design for array imaging," Inverse Probl. **23**, 1973–2020 (2007).
[57]J. H. Rose, "'Single-sided' autofocusing of sound in layered materials," Inverse Probl. **18**, 1923–1934 (2002).
[58]F. Broggini and R. Snieder, "Connection of scattering principles: A visual and mathematical tour," Eur. J. Phys. **33**, 593–613 (2012).
[59]K. Wapenaar, F. Broggini, E. Slob, and R. Snieder, "Three-dimensional single-sided Marchenko inverse scattering, data-driven focusing, Green's function retrieval and their mutual relations," Phys. Rev. Lett. **110**, 084301 (2013).
[60]A. E. Malcolm, J. A. Scales, and B. A. van Tiggelen, "Extracting the Green function from diffuse, equipartitioned waves," Phys. Rev. E **70**, 015601 (2004).
[61]P. Roux and W. A. Kuperman, and the NPAL Group, "Extracting coherent wave fronts from acoustic ambient noise in the ocean," J. Acoust. Soc. Am. **116**, 1995–2003 (2004).
[62]M. M. Haney, "Infrasonic ambient noise interferometry from correlations of microbaroms," Geophys. Res. Lett. **36**, L19808, https://doi.org/10.1029/2009GL040179 (2009).
[63]K. G. Sabra, S. Conti, P. Roux, and W. A. Kuperman, "Passive *in vivo* elastography from skeletal muscle noise," Appl. Phys. Lett. **90**, 194101 (2007).
[64]K. Wapenaar and J. Fokkema, "Green's function representations for seismic interferometry," Geophysics **71**, SI33–SI46 (2006).
[65]A. Bakulin and R. Calvert, "The virtual source method: Theory and case study," Geophysics **71**, SI139–SI150 (2006).
[66]G. T. Schuster, *Seismic Interferometry* (Cambridge University Press, London, UK, 2009).
[67]J. Van der Neut, J. Thorbecke, K. Mehta, E. Slob, and K. Wapenaar, "Controlled-source interferometric redatuming by cross-correlation and multidimensional deconvolution in elastic media," Geophysics **76**, SA63–SA76 (2011).
[68]L. Stehly, M. Campillo, B. Froment, and R. L. Weaver, "Reconstructing Green's function by correlation of the coda of the correlation (C^3) of ambient seismic noise," J. Geophys. Res. **113**, B11306, https://doi.org/10.1029/2008JB005693 (2008).
[69]K. Wapenaar and J. van der Neut, "A representation for Green's function retrieval by multidimensional deconvolution," J. Acoust. Soc. Am. **128**, EL366–EL371 (2010).
[70]A. Curtis and D. Halliday, "Directional balancing for seismic and general wavefield interferometry," Geophysics **75**, SA1–SA14 (2010).
[71]A. Fichtner, L. Stehly, L. Ermert, and C. Boehm, "Generalized interferometry: I. Theory for interstation correlations," Geophys. J. Int. **208**, 603–638 (2017).
[72]J. Corones, "Bremmer series that correct parabolic approximations," J. Math. Anal. Appl. **50**, 361–372 (1975).
[73]D. D. Kosloff and E. Baysal, "Migration with the full acoustic wave equation," Geophysics **48**, 677–687 (1983).
[74]L. Fishman and J. J. McCoy, "Derivation and application of extended parabolic wave theories: I. The factorized Helmholtz equation," J. Math. Phys. **25**, 285–296 (1984).
[75]B. Ursin, "Review of elastic and electromagnetic wave propagation in horizontally layered media," Geophysics **48**, 1063–1081 (1983).
[76]C. P. A. Wapenaar and A. J. Berkhout, "Wave-field extrapolation techniques for inhomogeneous media which include critical angle events: II. Methods using the two-way wave equation," Geophys. Prospect. **34**, 147–179 (1986).
[77]L. O. Løseth and B. Ursin, "Electromagnetic fields in planarly layered anisotropic media," Geophys. J. Int. **170**, 44–80 (2007).
[78]W. T. Thomson, "Transmission of elastic waves through a stratified solid medium," J. Appl. Phys. **21**, 89–93 (1950).
[79]N. A. Haskell, "The dispersion of surface waves on multilayered media," Bull. Seism. Soc. Am. **43**, 17–34 (1953).
[80]F. Gilbert and G. E. Backus, "Propagator matrices in elastic wave and vibration problems," Geophysics **31**, 326–332 (1966).
[81]J. H. Woodhouse, "Surface waves in a laterally varying layered structure," Geophys. J. **37**, 461–490 (1974).
[82]B. L. N. Kennett and N. J. Kerry, "Seismic waves in a stratified half-space," Geophys. J. **57**, 557–584 (1979).
[83]V. Pacheco-Peña and N. Engheta, "Effective medium concept in temporal metamaterials," Nanophotonics **9**, 379–391 (2020).

[84]D. Peng, Y. Fan, R. Liu, X. Guo, and S. Wang, "Time-reversed water waves generated from an instantaneous time mirror," J. Phys. Commun. **4**, 105013 (2020).
[85]K. Aki, "Space and time spectra of stationary stochastic waves, with special reference to micro-tremors," Bull. Earthquake Res. Inst. **35**, 415–457 (1957).
[86]H. Cox, "Spatial correlation in arbitrary noise fields with application to ambient sea noise," J. Acoust. Soc. Am. **54**, 1289–1301 (1973).
[87]M. W. Asten, "On bias and noise in passive seismic data from finite circular array data processed using SPAC methods," Geophysics **71**, V153–V162 (2006).