Modal expansion of one-way operators in laterally varying media

Joris L. T. Grimbergen*, Frank J. Dessing*, and Kees Wapenaar[‡]

ABSTRACT

One of the main benefits of prestack depth migration in seismic processing is its ability to handle complicated medium configurations. When considerable lateral variations in the acoustic parameters are present in the subsurface, prestack depth migration is necessary for optimal lateral resolution. However, most migration algorithms still deal with lateral variations in an approximate manner because these variations are in many cases moderate compared to the profound variations in the depth direction.

From other areas of science (e.g., optics, oceanography, and seismology), it is known that lateral variations can be dealt with by a decomposition of the wavefield into wave modes. In this paper, we explore the possibility of applying this concept to the construction of oneway wavefield operators for depth migration. We expand the Helmholtz operator on an orthogonal basis of wave modes and obtain one-way wavefield operators that are unconditionally stable and significantly increase the lateral resolution of the result.

INTRODUCTION

An important requirement for current seismic migration schemes is the ability to deal accurately with lateral as well as depth variations in the subsurface. In many cases, the subsurface shows profound variations in the depth direction, while lateral changes are less rapid. In the past, this characteristic has been exploited by a number of wavefield extrapolation algorithms that use a one-way decomposition of the wavefield (Claerbout, 1971; Berkhout, 1982; Holberg, 1988; Blacquière et al., 1989; Hale, 1991).

A one-way (or directional) decomposition comprises the splitting of the wavefield with respect to a certain direction

of preference. In surface seismic applications, the direction of preference is usually the depth direction. The axes perpendicular to the direction of preference are referred to as *lateral* coordinates. Hence, in surface seismic applications the lateral coordinates are the horizontal coordinates, i.e., parallel to the surface (in well-to-well seismics the coordinate along the borehole is chosen as the lateral coordinate). Mathematically, lateral variations and depth variations are dealt with separately. Depth variations result in coupling between up- and downgoing waves, whereas horizontal scattering, due to lateral changes, is in principle included in the downward or upward extrapolation (continuation) of the one-way wavefield. In Claerbout (1971) and Berkhout (1982), the extrapolation operators are constructed via series expansions. The other references use optimized operators that are derived from the phase-shift operator in the Fourier domain. The operators of the latter class are further referred to as local explicit operators.

In all of the above-mentioned references, the assumption is made that the medium is homogeneous within the spatial length of the extrapolation operators. If the medium varies laterally, the operator is applied locally for each gridpoint, according to the values of the acoustic parameters at that gridpoint. This approximation is only acceptable for smooth, lateral variations of the medium. However, if the lateral variations in the medium are no longer small on a wavelength scale, the results of these methods become unreliable. In the case of the local explicit operator, the extrapolation results can even become unstable (Etgen, 1994).

In this paper, the possibility of an improved handling of lateral medium variations is investigated. From optics, seismology, and specific seismic applications, it is known that an expansion of the wavefield into wave modes proves to be an appropriate method of dealing with predominantly laterally varying media (e.g., Weinberg and Burridge, 1974; Collin, 1991; Ernst and Herman, 1995). These applications are usually limited to waveguides or other structures with comparatively small variations in the direction of preference. Moreover, the medium

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^{*}Formerly Centre for Technical Geoscience, Laboratory of Seismics and Acoustics, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands; presently Shell International Exploration & Production B.V., Research and Technical Services, Volmerlaan 8, 2288 GD Rijswijk, ZH, The Netherlands. E-mail: j.l.t.grimbergen@siep.shell.com. E-mail: f.j.dressing@siep.shell.com.

[†]Centre for Technical Geoscience, Laboratory of Seismics and Acoustics, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands. E-mail: c.p.a.wapenaar@ctg.tudelft.nl.

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parameters vary in such a way that the guided wave modes dominate the wavefield. Clearly, this is not the case in reflection seismics, where wave-guiding situations seldom occur as a result of lateral variations. Here, the radiating part of the wavefield is more important than the guided wave modes.

An extrapolation scheme, based on a modal expansion of the wavefield into both guided and radiating wavefield constituents, significantly increases the lateral resolution of the result. The method is tested in a synthetic migration example, where the subsurface model contains a high-velocity domal structure (typical of salt) and a number of faults. It is shown that by using the modal expansion for the construction of one-way wavefield operators, significant improvements can be achieved compared to local explicit methods.

Close links exist between the method presented in this paper and the work of Pai (1985) and Kosloff and Kessler (1987). In Pai (1985), a modal decomposition in the wavenumber-frequency domain is carried out for laterally varying media and applied to the two-way wave equation. For laterally invariant media the resulting extrapolation operator reduces to the phase-shift operator. Kosloff and Kessler (1987) mention the possibility of a modal decomposition applied to the two-way wave equation. Both references take the discretized wave equation as a point of departure, whereas in the present paper a derivation from the continuous formulation is carried out. This derivation leans upon functional calculus (Reed and Simon 1978, 1979).

ONE-WAY OPERATORS AND KERNELS

In this section, the relevant concept of one-way wavefield operators for lossless source-free inhomogeneous fluids is reviewed briefly. We consider the one-way wave equation in the space-frequency (\mathbf{x}, ω) domain. In the following, the temporal frequency ω is suppressed in the notation, for reasons of convenience. Since in this paper we are interested primarily in the impact of lateral variations on propagation, we will neglect the vertical variations within each extrapolation step, similar to the local explicit method. Internal multiples and other secondorder effects related to the vertical variations will be neglected as well. [For a recent discussion on the one-way wave equation and its properties in arbitrarily inhomogeneous media, we refer to Wapenaar and Grimbergen (1996).] At depth level x_3 we may thus write

$$\frac{\partial P^{\pm}(\mathbf{x}_H, x_3)}{\partial x_3} = \mp j \hat{\mathcal{H}}_1 P^{\pm}(\mathbf{x}_H, x_3), \qquad (1)$$

where $\mathbf{x}_H = (x_1, x_2)$ represents the horizontal coordinates. $P^{\pm}(\mathbf{x}_H, x_3)$ is the monochromatic one-way upgoing (⁻) or downgoing (⁺) flux-normalized acoustic wavefield (de Hoop, 1992), *j* the imaginary unit, and $\hat{\mathcal{H}}_1$ the so-called square root operator (Claerbout, 1971). In this paper, operators are distinguished from other variables by a circumflex. The square-root operator relates to the Helmholtz operator $\hat{\mathcal{H}}_2$, according to

$$\hat{\mathcal{H}}_2 = \hat{\mathcal{H}}_1 \hat{\mathcal{H}}_1. \tag{2}$$

The Helmholtz operator may be written as

$$\hat{\mathcal{H}}_2 = \left(\frac{\omega}{c'}\right)^2 + \boldsymbol{\nabla}_H^2, \tag{3}$$

where $\nabla_H = (\partial/\partial x_1, \partial/\partial x_2)$. In equation (3), lateral variations in the density ρ are incorporated in the modified velocity c', satisfying

$$\left(\frac{\omega}{c'}\right)^2 = \left(\frac{\omega}{c}\right)^2 - \frac{3(\nabla_H \varrho) \cdot (\nabla_H \varrho)}{4\varrho^2} + \frac{(\nabla_H^2 \varrho)}{2\varrho}, \quad (4)$$

with $c = c(\mathbf{x})$ and $\varrho = \varrho(\mathbf{x})$ (Wapenaar and Grimbergen, 1996). From equations (2) and (3), it can be seen that one cannot come up with an ordinary partial differential operator $\hat{\mathcal{H}}_1$ that satisfies these equations. The square root operator belongs to the more general class of pseudodifferential operators (Calderón and Zygmund, 1957; Kumano-go, 1974; Treves, 1980; Taylor, 1981).

The wavefield $P^{\pm}(\mathbf{x}_H, x_3)$ at depth level x_3 can be expressed in terms of the wavefield at depth level x'_3 according to

$$P^{\pm}(\mathbf{x}_{H}, x_{3}) = \int_{\mathbb{R}^{2}} W^{\pm}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}', x_{3}') P^{\pm}(\mathbf{x}_{H}', x_{3}') d^{2}\mathbf{x}_{H}',$$
(5)

where the propagator $W^{\pm}(\mathbf{x}_H, x_3; \mathbf{x}'_H, x'_3)$ is the solution of the one-way wave equation (1),

$$\frac{\partial W^{\pm}(\mathbf{x}_H, x_3; \mathbf{x}'_H, x'_3)}{\partial x_3} = \mp j \hat{\mathcal{H}}_1 W^{\pm}(\mathbf{x}_H, x_3; \mathbf{x}'_H, x'_3), \quad (6)$$

with initial condition

$$W^{\pm}(\mathbf{x}_{H}, x_{3} = x'_{3}; \mathbf{x}'_{H}, x'_{3}) = \delta(\mathbf{x}_{H} - \mathbf{x}'_{H}).$$
(7)

We choose $x_3 > x'_3$ for downgoing waves because $W^+(\mathbf{x}_H, x_3; \mathbf{x}'_H, x'_3)$ represents the forward propagator. Similarly, we choose $x_3 < x'_3$ for the upward propagator $W^-(\mathbf{x}_H, x_3; \mathbf{x}'_H, x'_3)$. From equation (7), the propagator $W^{\pm}(\mathbf{x}_H, x_3; \mathbf{x}'_H, x'_3)$ can be solved by a Taylor series expansion with respect to $(x_3 - x'_3)$:

$$W^{\pm}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}', x_{3}') = \sum_{k=0}^{\infty} \frac{(x_{3} - x_{3}')^{k}}{k!} \times \left[\frac{\partial^{k} W^{\pm}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}', x_{3}')}{\partial x_{3}^{k}} \right]_{x_{3} = x_{3}'}.$$
(8)

Using equation (6) with equation (7), we have

$$W^{\pm}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}', x_{3}') = \sum_{k=0}^{\infty} \frac{(x_{3} - x_{3}')^{k}}{k!} (\mp j)^{k} \hat{\mathcal{H}}_{1}^{k} \,\delta(\mathbf{x}_{H} - \mathbf{x}_{H}').$$
(9)

The series above is recognized as the series expansion of an exponential. Hence, we may write symbolically

$$W^{\pm}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}', x_{3}') = \exp\{\mp j(x_{3} - x_{3}')\hat{\mathcal{H}}_{1}\}\delta(\mathbf{x}_{H} - \mathbf{x}_{H}').$$
(10)

At this point, it is convenient to introduce the kernel $\mathcal{A}(\mathbf{x}_H, \mathbf{x}'_H)$ of some operator $\hat{\mathcal{A}}$, according to

$$\hat{\mathcal{A}}F(\mathbf{x}_H) = \int_{\mathbb{R}^2} \mathcal{A}(\mathbf{x}_H, \mathbf{x}'_H) F(\mathbf{x}'_H) \, d^2 \mathbf{x}'_H, \qquad (11)$$

where $F(\mathbf{x}_H)$ is a function of \mathbf{x}_H on which the operator is active. In our case this function represents a monochromatic wavefield at a fixed depth level. From equation (11) it is clear that the following symbolic relation holds between operator \hat{A} and its kernel $\mathcal{A}(\mathbf{x}_H, \mathbf{x}'_H)$:

$$\mathcal{A}(\mathbf{x}_H, \mathbf{x}'_H) = \hat{\mathcal{A}}\delta(\mathbf{x}_H - \mathbf{x}'_H).$$
(12)

Considering equation (10), we conclude that $W^{\pm}(\mathbf{x}_H, x_3; \mathbf{x}'_H, x'_3)$ can be identified as the kernel of an operator $\hat{W}^{\pm}(x_3, x'_3)$:

$$\hat{W}^{\pm}(x_3, x_3') = \exp\{\mp j(x_3 - x_3')\hat{\mathcal{H}}_1\}.$$
 (13)

Analogous to equation (11), we introduce the kernel $\mathcal{H}_2(\mathbf{x}_H, x_3; \mathbf{x}'_H)$ of the Helmholtz operator $\hat{\mathcal{H}}_2$, according to

$$\hat{\mathcal{H}}_2 F(\mathbf{x}_H) = \int_{\mathbb{R}^2} \mathcal{H}_2(\mathbf{x}_H, x_3; \mathbf{x}'_H) F(\mathbf{x}'_H) d^2 \mathbf{x}'_H.$$
(14)

The kernel of the square root operator $\hat{\mathcal{H}}_1$ can be related to the kernel of the Helmholtz operator $\hat{\mathcal{H}}_2$ by applying it twice to a function $F(\mathbf{x}_H)$:

$$\begin{aligned} \hat{\mathcal{H}}_2 F(\mathbf{x}_H) &= \hat{\mathcal{H}}_1 \hat{\mathcal{H}}_1 F(\mathbf{x}_H) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{H}_1(\mathbf{x}_H, x_3; \mathbf{x}'_H) \mathcal{H}_1(\mathbf{x}''_H, x_3; \mathbf{x}'_H) \\ &\times F(\mathbf{x}'_H) d^2 \mathbf{x}''_H d^2 \mathbf{x}'_H. \end{aligned}$$
(15)

Comparing equation (15) with equation (14), we note that the kernels of the operators $\hat{\mathcal{H}}_2$ and $\hat{\mathcal{H}}_1$ are interrelated according to

$$\mathcal{H}_{2}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}')$$

$$= \int_{\mathbb{R}^{2}} \mathcal{H}_{1}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}'') \mathcal{H}_{1}(\mathbf{x}_{H}'', x_{3}; \mathbf{x}_{H}') d^{2}\mathbf{x}_{H}''. \quad (16)$$

This relation is the equivalent of equation (2) in terms of operator kernels.

EXPANDING THE HELMHOLTZ OPERATOR

The problem of finding expressions for one-way operators is essentially a problem of finding the square root operator $\hat{\mathcal{H}}_1$ from equation (2) or, similarly, from equation (16). As will become clear later in this section, this problem can be solved if the Helmholtz operator $\hat{\mathcal{H}}_2$ is expanded in terms of its eigenfunctions.

Consider again the Helmholtz operator $\hat{\mathcal{H}}_2$ at a fixed depth level x_3 :

$$\hat{\mathcal{H}}_2 = \left(\frac{\omega}{c'(\mathbf{x}_H, x_3)}\right)^2 + \boldsymbol{\nabla}_H^2 = k^2(\mathbf{x}_H, x_3) + \boldsymbol{\nabla}_H^2, \quad (17)$$

where $k(\mathbf{x}_H, x_3)$ is the wavenumber at that depth level. In the remainder of this section, x_3 is suppressed for notational convenience. The eigenfunctions $\phi(\mathbf{x}_H)$ of $\hat{\mathcal{H}}_2$ satisfy

$$\hat{\mathcal{H}}_2 \phi(\mathbf{x}_H) = \lambda \phi(\mathbf{x}_H). \tag{18}$$

Because $\hat{\mathcal{H}}_2$ acts on an unbounded lateral space, the parameter λ in equation (18) represents an operator spectrum rather than a set of eigenvalues. Regarding its mathematical properties, the Helmholtz operator is similar to the Hamiltonian operator

from nonrelativistic quantum mechanics. For this reason, we will lean on this well-developed theory in deriving the spectral properties of $\hat{\mathcal{H}}_2$ (Reed and Simon, 1978). From this theory it can be shown that $\hat{\mathcal{H}}_2$ is a self-adjoint operator if it is defined on an appropriate domain of functions. As a result, the eigenfunctions are orthogonal and complete and the spectrum is real valued.

At this point, we assume the lateral variations in the medium vanish outside some arbitrary but finite range of \mathbf{x}_H . This means we assume some lateral background medium with phase velocity c_0 , corresponding to a wavenumber $k_0 = \omega/c_0$. Under this condition, the spectrum σ of $\hat{\mathcal{H}}_2$ generally consists of a discrete part and a continuous part:

$$\sigma(\hat{\mathcal{H}}_2) = \sigma_{\text{discr}}(\hat{\mathcal{H}}_2) \cup \sigma_{\text{cont}}(\hat{\mathcal{H}}_2).$$
(19)

The outline of the spectrum of $\hat{\mathcal{H}}_2$ is given in Appendix A. Figure 1 shows the structure of the spectrum.

Due to the completeness of the basis of eigenfunctions, any function $F(\mathbf{x}_H)$ in the domain of $\hat{\mathcal{H}}_2$ can be expanded in terms of the eigenfunctions of $\hat{\mathcal{H}}_2$:

$$F(\mathbf{x}_{H}) = \int_{\mathbb{R}^{2}} \phi(\mathbf{x}_{H}, \kappa) \bar{F}(\kappa) d^{2}\kappa + \sum_{\lambda_{i} \in \sigma_{\text{discr}}} \phi^{(i)}(\mathbf{x}_{H}) \bar{F}^{(i)}.$$
(20)

The expansion coefficients $\bar{F}^{(i)}$ in the second term on the right side of equation (20) correspond to the discrete eigenvalues λ_i . The first term on the right side contains the expansion coefficients $\bar{F}(\kappa)$, where the integration variable κ is related to the continuous spectrum variable λ , according to

$$\lambda(\boldsymbol{\kappa}) = k_0^2 - \boldsymbol{\kappa} \cdot \boldsymbol{\kappa}, \qquad \lambda \in \sigma_{\text{cont}}.$$
(21)

From this relation, we can see that the continuous part of the spectrum is degenerate because for fixed $\lambda(\kappa) \in \sigma_{\text{cont}}$, an infinite number of solutions for κ exists. Equation (20) can be interpreted as an inverse transformation from the modal domain to the space domain. Using the orthogonality of the eigenfunctions and a proper normalization, the related forward transform can thus be written as

$$\bar{F}(\boldsymbol{\kappa}) = \int_{\mathbb{R}^2} F(\mathbf{x}_H) \phi^*(\mathbf{x}_H, \boldsymbol{\kappa}) d^2 \mathbf{x}_H$$
(22)

and

$$\bar{F}^{(i)} = \int_{\mathbb{R}^2} F(\mathbf{x}_H) \phi^{(i)}(\mathbf{x}_H) \, d^2 \mathbf{x}_H, \qquad (23)$$



FIG. 1. Spectrum of the Helmholtz operator $\hat{\mathcal{H}}_2$ in the complex plane.

where the asterisk (*) denotes complex conjugation. As an example, consider the laterally invariant case. In this case, the discrete spectrum disappears. It is easily seen, then, that the following complex exponential functions satisfy equation (18):

$$\phi(\mathbf{x}_H, \kappa) = \frac{1}{2\pi} \exp\{-j\kappa \cdot \mathbf{x}_H\}, \qquad (24)$$

in which a plane wave can be recognized. However, since $\hat{\mathcal{H}}_2$ is a real-valued self-adjoint operator having a real spectrum, we can alternatively choose the eigenfunctions to be real valued:

$$\phi(\mathbf{x}_H, \boldsymbol{\kappa}) = \frac{1}{\pi\sqrt{2}} \cos\{\boldsymbol{\kappa} \cdot \mathbf{x}_H - \pi/4\}.$$
 (25)

In this equation, the $\pi/4$ phase shift is essential for the construction of both odd-as-even functions $F(\mathbf{x}_H)$. Substitution of equations (24) or (25) in equations (20) and (22) yields the inverse and forward spatial Fourier and Hartley transformations, respectively (Bracewell, 1986).

We return to the laterally variant situation. By definition, $\hat{\mathcal{H}}_2$ becomes a multiplication operator in the domain constituted by its eigenfunctions. Therefore, according to equations (18) and (20), we may write

$$\hat{\mathcal{H}}_{2}F(\mathbf{x}_{H}) = \int_{\mathbb{R}^{2}} \lambda(\boldsymbol{\kappa})\phi(\mathbf{x}_{H},\boldsymbol{\kappa})\bar{F}(\boldsymbol{\kappa}) d^{2}\boldsymbol{\kappa} + \sum_{\lambda_{i}\in\sigma_{\text{discr}}} \lambda_{i}\phi^{(i)}(\mathbf{x}_{H})\bar{F}^{(i)}.$$
 (26)

The expansion coefficients can be eliminated from expression (26), by using the modal transform [equations (22) and (23)], yielding

$$\hat{\mathcal{H}}_2 F(\mathbf{x}_H) = \int_{\mathbb{R}^2} \mathcal{H}_2(\mathbf{x}_H, \mathbf{x}'_H) F(\mathbf{x}'_H) d^2 \mathbf{x}'_H, \qquad (27)$$

where the kernel $\mathcal{H}_2(\mathbf{x}_H, \mathbf{x}'_H)$ can be expressed according to

$$\mathcal{H}_{2}(\mathbf{x}_{H}, \mathbf{x}_{H}') = \int_{\mathbb{R}^{2}} \phi(\mathbf{x}_{H}, \kappa) \lambda(\kappa) \phi^{*}(\mathbf{x}_{H}', \kappa) d^{2}\kappa + \sum_{\lambda_{i} \in \sigma_{\text{discr}}} \phi^{(i)}(\mathbf{x}_{H}) \lambda_{i} \phi^{(i)}(\mathbf{x}_{H}').$$
(28)

Equation (28) and other expansions of following kernels should be understood in the sense of distributions (Zemanian, 1965).

EXPANDING THE ONE-WAY PROPAGATOR

Using equations (16) and (28) as well as the orthonormality of the eigenfunctions, the kernel of the square root operator can be written as

$$\mathcal{H}_{1}(\mathbf{x}_{H}, \mathbf{x}_{H}') = \int_{\mathbb{R}^{2}} \phi(\mathbf{x}_{H}, \kappa) \lambda^{\frac{1}{2}}(\kappa) \phi^{*}(\mathbf{x}_{H}', \kappa) d^{2}\kappa + \sum_{\lambda_{i} \in \sigma_{\text{discr}}} \phi^{(i)}(\mathbf{x}_{H}) \lambda^{\frac{1}{2}}_{i} \phi^{(i)}(\mathbf{x}_{H}'), \quad (29)$$

where for later convenience the signs of the square root are chosen according to

$$\operatorname{Re}(\lambda^{\frac{1}{2}}) \ge 0 \quad \text{for} \quad \lambda \ge 0$$
 (30)

and

$$\operatorname{Im}(\lambda^{\frac{1}{2}}) < 0 \quad \text{for} \quad \lambda < 0.$$
 (31)

Figure 2 shows the spectrum of the square root operator. Similarly, for the primary propagator as defined in equation (10), we may write

$$W^{\pm}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}', x_{3}')$$

$$= \int_{\mathbb{R}^{2}} \phi(\mathbf{x}_{H}, \kappa) \exp\left\{ \mp j(x_{3} - x_{3}')\lambda^{\frac{1}{2}}(\kappa) \right\}$$

$$\times \phi^{*}(\mathbf{x}_{H}', \kappa) d^{2}\kappa$$

$$+ \sum_{\lambda_{i} \in \sigma_{\text{discr}}} \phi^{(i)}(\mathbf{x}_{H}) \exp\left\{ \mp j(x_{3} - x_{3}')\lambda_{i}^{\frac{1}{2}} \right\} \phi^{(i)}(\mathbf{x}_{H}').$$
(32)

NUMERICAL IMPLEMENTATION IN 2-D

The discretization of the wavefield operators and variables leads to matrix operators and (column) vectors. The one-way matrix operators differ fundamentally from their continuous counterparts as the "spectrum" becomes fully discrete due to the finite dimensions of the matrix. The important properties of the continuous operators (e.g., self-adjointness) and eigenfunctions (orthogonality and completeness), however, translate elegantly into similar properties for matrix operators and eigenvectors (Golub and Van Loan, 1989).

For the 2-D situation, the transition from the Helmholtz operator $\hat{\mathcal{H}}_2$ to the corresponding matrix operator can be clarified using the operator kernel $\hat{\mathcal{H}}_2(x_1, x_3; x'_1)$. From equations (12) and (17) we have

$$\mathcal{H}_{2}(x_{1}, x_{3}; x_{1}') = \left(\frac{\omega}{c'(x_{1}, x_{3})}\right)^{2} \delta(x_{1} - x_{1}') + \frac{\partial^{2}}{\partial x_{1}^{2}} \delta(x_{1} - x_{1}').$$
(33)



FIG. 2. Spectrum of the square root operator $\hat{\mathcal{H}}_1$ in the complex plane.

The continuous variables (x_1, x'_1) above relate to discrete indices of an $M \times M$ matrix operator according to

$$\mathbf{H}_2 = \mathbf{C} + \mathbf{D}_2. \tag{34}$$

Here, $\mathbf{\underline{C}}$ is a diagonal matrix corresponding to the first term in equation (33):

$$\mathbf{\underline{C}} = \begin{pmatrix} \left(\frac{\omega}{c_1'}\right)^2 & 0 & \cdots & 0\\ 0 & \left(\frac{\omega}{c_2'}\right)^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \left(\frac{\omega}{c_M'}\right)^2 \end{pmatrix}, \quad (35)$$

where $c'_n = c'(n\Delta x_1, x_3)$ and Δx_1 is the lateral discretization interval. Furthermore, the matrix operator \mathbf{D}_2 represents the second-order differentiation filter, which may be implemented as

$$\mathbf{D}_{2} = \frac{1}{\Delta x_{1}^{2}} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0\\ 1 & -2 & 1 & \cdots & 0 & 0\\ 0 & 1 & -2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -2 & 1\\ 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}.$$
 (36)

However, in practice a matrix operator is applied which contains more nonzero diagonals. The matrix elements of this operator are calculated using a least-squares optimization algorithm (Thorbecke and Rietveld, 1994). As can be seen by inspection of equations (35) and (36), both matrix operators in equation (34) are real valued and symmetric; hence, \mathbf{H}_2 is self-adjoint ($\mathbf{H}_2 = \mathbf{H}_2^H$, where *H* denotes transposition and complex conjugation). A well-known result from matrix algebra (Golub and van Loan, 1989) states that, for a self-adjoint matrix, the following decomposition can be applied:

$$\mathbf{H}_2 = \mathbf{L} \mathbf{\Lambda} \mathbf{L}^{-1} = \mathbf{L} \mathbf{\Lambda} \mathbf{L}^H, \qquad (37)$$

where \mathbf{L} contains the discrete equivalents of the normalized eigenfunctions ϕ in its columns and $\underline{\Lambda}$ is a diagonal matrix containing the eigenvalues of \mathbf{H}_2 . This equation is the discrete counterpart of equation (28). Hence, the eigenvalues in $\underline{\Lambda}$ replace both the discrete and the continuous part of the spectrum (see also Figure 1). The square root matrix operator \mathbf{H}_1 consequently can be written as

$$\mathbf{H}_{1} = \mathbf{L} \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{L}^{-1} = \mathbf{L} \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{L}^{H}, \qquad (38)$$

with the signs of the square root of the eigenvalues in $\underline{\Lambda}^{\frac{1}{2}}$ chosen according to equations (30) and (31). This result is evident because, as an equivalent of equations (2) and (16), we have $\underline{H}_2 = \underline{H}_1 \underline{H}_1$. The procedure described above is illustrated for a laterally invariant and a laterally variant medium. Figure 3 shows the laterally variant medium profile. At a fixed value of x_3 , this medium contains a lateral perturbation (i.e., a velocity dip) of the laterally invariant medium. In Figure 4 the spectrum of the square root operator $\hat{\mathcal{H}}_1$ and eigenfunctions of both profiles are compared. The common part of the spectra [Figures 4(a) and 4(d)] corresponds to the continuous part in Figure 2. These eigenvalues would condense into a continuous branch if the lateral aperture were increased to infinity. The isolated eigenvalues on the right in Figure 4(d) are caused by the velocity dip in the medium, which acts as a waveguide. In Figures 4(b) and 4(c), two eigenfunctions are shown for the laterally homogeneous profile. In Figures 4(e) and 4(f), one guided wave mode and one radiating wave mode are shown.

Finally, the primary propagator matrix $\mathbf{W}^{\pm}(x_3, x'_3)$ can be expressed according to

$$\mathbf{W}^{\pm}(x_3, x_3') = \mathbf{L}(x_3') \exp\left\{ \pm j(x_3 - x_3') \mathbf{\Lambda}^{\frac{1}{2}} \right\} \mathbf{L}^{H}(x_3').$$
(39)

In equation (39), the amplitude of the exponential function equals unity for propagating wave modes and is lower than unity for evanescent wave modes. This is demonstrated in Figure 5, where the amplitude of the exponential for each eigenvalue in Figure 4(a) is depicted. Hence, the wavefield is extrapolated accurately and in a stable manner up to 90° and beyond (the evanescent wavefield). The stability of the extrapolator is an important distinctive feature of the described method compared to local explicit methods, which have been reported to exhibit unstable behavior in various situations (Etgen, 1994).

NUMERICAL IMPLEMENTATION IN 3-D

As in the 2-D case, we now take the kernel of the Helmholtz operator for the 3-D situation as a point of departure:

$$\mathcal{H}_{2}(\mathbf{x}_{H}, x_{3}; \mathbf{x}_{H}') = \left(\frac{\omega}{c'(\mathbf{x}_{H}, x_{3})}\right)^{2} \delta(\mathbf{x}_{H} - \mathbf{x}_{H}') + \nabla_{H}^{2} \delta(\mathbf{x}_{H} - \mathbf{x}_{H}').$$
(40)

Evidently, the eigenvectors of this operator are in this case two dimensional in space, which raises the question if equation (40) can still be expressed in terms of a matrix operator. Kinneging et al. (1989) show that monochromatic 3-D data can indeed be



FIG. 3. Laterally variant medium profile at a fixed depth level x_3 .

organized in vectors such that the corresponding 3-D wavefield operators again become matrices. Following this procedure, a one-way monochromatic wavefield at depth level x_3 can be written as a vector, according to

$$\mathbf{P}^{\pm}(x_{3}) = \begin{pmatrix} P^{\pm}(\Delta x_{1}, \Delta x_{2}, x_{3}) \\ P^{\pm}(2\Delta x_{1}, \Delta x_{2}, x_{3}) \\ \vdots \\ P^{\pm}(M\Delta x_{1}, \Delta x_{2}, x_{3}) \\ P^{\pm}(\Delta x_{1}, 2\Delta x_{2}, x_{3}) \\ \vdots \\ P^{\pm}(M\Delta x_{1}, N\Delta x_{2}, x_{3}) \end{pmatrix}.$$
(41)

This way of organizing the data can be used to derive a matrix operator from equation (40), which represents the Helmholtz operator at a fixed depth level x_3 for the 3-D situation. As in the 2-D case, this matrix operator is extremely sparse. To illustrate, we have computed a number of modes. One medium is laterally invariant with a velocity of 2500 m/s; the other medium profile is the circular symmetric extension of the profile shown in Figure 3. Figure 6 shows the 2-D eigenfunctions at a fixed x_3 .

EXAMPLES

Well-to-well extrapolation

As a first illustration of the one-way operators that have been constructed, a crosswell configuration is considered. A point source in one well generates a wavefield that is recorded in another well. The medium is assumed to be depth dependent only. In this example, the direction of preference is horizontal, while the lateral dimension represents depth. Figure 7 shows the 1-D subsurface model and the result of finite-difference modeling, which is used as a benchmark. The results of model expansion and the local explicit method are compared in Figure 8. Not surprisingly, the results of the local explicit method are very poor in this example because of the considerable velocity



FIG. 5. Amplitude of the eigenvalues of the propagator matrix for an extrapolation distance of 30 m.



FIG. 4. (a) Spectrum of the square root operator $\hat{\mathcal{H}}_1$ for a laterally invariant medium with a velocity of $c_0 = 2500$ m/s. The frequency equals 25 Hz; hence, $k_0 = \omega/c_0 = 0.063$ m⁻¹. (b) and (c) Two radiating wave modes at fixed x_3 . Note that the radiating wave modes in the homogeneous profile are harmonic functions. (d) Spectrum of the square root operator $\hat{\mathcal{H}}_1$ for the laterally variant medium of Figure 3. (e) Guided wave mode. (f) Radiating wave mode. The squares in the spectrum of the square root operator denote the eigenvalues corresponding to the plotted eigenfunctions.

changes perpendicular to the propagation direction. On the other hand, note the high resemblance between the results of modal expansion and finite-difference modeling.

Migration example

As a second example, we have migrated a set of synthetic shot records that have been generated by finite-difference modeling using the subsurface model in Figure 9. In this case both lateral (i.e., horizontal) variations and depth variations are present in the model. The model contains a high-velocity layer (salt) piercing through a number of layers. To the right of this structure, the block-shaped structure implies yet another lateral discontinuity. The acquisition parameters are summarized in Table 1. Because we are now dealing with depth variations, the wavefield is extrapolated in small steps, using the complex conjugate transposed of \mathbf{W}^{\pm} , as defined in equation (39) (hence, the evanescent waves are suppressed).

The stacked result of the separate shot record migrations using modal expansion is shown in Figure 10. The flanks of the salt as well as the faults are clearly imaged. Note the overall crisp character of the result. An unambiguous comparison with



FIG. 6. (a)–(c) Two-dimensional eigenfunctions at fixed x_3 in a homogenous 3-D medium with $c_0 = 2500$ m/s. (d)–(f) Two guided modes and one radiating mode in a radially symmetric extension of the profile of Figure 3.



FIG. 7. Velocity-depth profile of the medium with discontinous transitions (left) and the corresponding result of well-to-well finite-difference modeling (right). The source is located 135 m below the surface and is denoted by a bullet.

the available local explicit method is not straightforward because both the length and the optimization angle can be varied. Choosing these parameters leads to conflicting requirements. In case of strong lateral variations, short operators are needed to avoid instabilities. Imaging of steep dips, however, asks for longer operators that allow for higher optimization angles. Figure 11 shows the results for several choices of these parameters. Note that a higher optimization angle improves the imaging of steep dips but causes stronger artifacts at the same time. These artifacts are caused by the increased spatial length of the operator. All results in Figure 11 are inferior to the model expansion result in Figure 10.

DISCUSSION

We have shown that the proposed method to calculate oneway operators has desirable properties such as the absence of dip limitation, the accurate handling of lateral variations, and the unconditional stability of the operators. The obvious drawback of the method is the computational cost of a full eigenvalue decomposition, which is considerable compared to the construction of the local explicit operators. However, the following considerations may help to overcome this problem.

1) The Helmholtz matrix operator is a sparse symmetric band matrix. For a full symmetric $M \times M$ matrix, the number of floating-point operations necessary to calculate all eigenvalues and all eigenvectors will increase with the third power of M. However, in case of a matrix

 Table 1. The acquisition parameters for the migration example.

Parameter	Value
Geometry	Fixed spread
Number of shots	11
Shot spacing	500 m
Number of detectors per shot	251
Receiver spacing	20 m
Recording time	3 s
Time sampling	4 ms
Frequency content wavelet	Up to 35 Hz

operator with a fixed number of nonzero diagonals independent of M, the number of floating-point operations will increase only with the square of M (Golub and Van Loan, 1989).

- 2) Not all eigenvalues need to be calculated (Druskin and Knizhnerman, 1994). Calculating only the positive eigenvalues (propagating modes) still leads to accurate results because the evanescent part of the wavefield decays exponentially with the extrapolation distance. (In inverse extrapolation, the evanescent field is suppressed anyway to obtain stable operators.) This argument holds in particular for low temporal frequencies where a large number of the eigenvalues are associated to evanescent wave modes.
- 3) Hybrid methods may be implemented. Local explicit operators and modal expansion operators (in regions of significant lateral variations) can be applied in combination. This will be a subject of future research.

The modal expansion method also provides an interesting scope for turning-wave migration. In some references (Claerbout, 1985; Hale et al., 1992), the phase-shift method is applied because it has no dip limitation, which is an essential requirement for turning-wave migration. However, the phase-shift method is applicable only in laterally invariant media (which was acknowledged by the authors). The modal expansion method combines both the ability to deal with lateral variations and the ability to handle dips up to 90°.



FIG. 9. Velocity model for the migration. Velocities are indicated in the corresponding layers.



FIG. 8. Results of well-to-well extrapolation using modal expansion (left) and the local explicit method (right).

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FIG. 10. Migrated section using modal expansion extrapolation operators.



FIG. 11. Migrated sections using local explicit operators. (a) Operator length 27, optimization up to 60° (top).
(b) Operator length 27, optimization up to 80°. (c) Operator length 37, optimization up to 80°.

CONCLUSION

From the modal expansion of the Helmholtz operator, oneway wavefield operators can be constructed that have the capacity to handle strong lateral variations in the medium parameters. The spectrum of the Helmholtz operator can be derived along the same lines as the Hamiltonian operator from the field of quantum mechanics. It provides clear insight into the components of the wavefield, guided and radiating wave modes. In case of a laterally invariant medium, the method is equivalent to a plane-wave decomposition. The analysis based upon functional calculus justifies the subsequent 2-D and 3-D discrete implementations.

The modal expansion divides the wavefield into constituents (wave modes) distinguished by their vertical-phase velocity. The construction of the square root operator and the primary propagator is straightforward in the modal domain because these operators turn into multiplication operators. The expansion of the wavefield into modes implies the exact solution of the horizontal scattering process, which is related to the lateral variations of the medium parameters. The one-way extrapolation operators obtained by this method are intrinsically stable. The migration result that was presented clearly shows an improved lateral resolution and a quality superior to local explicit operators that have been optimized for steeply dipping events.

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APPENDIX A THE SPECTRUM OF $\hat{\mathcal{H}}_2$

Consider the second-order differential operator $\hat{\mathcal{H}}_2$ on the (Hilbert) space of square integrable functions, defined on the lateral coordinates \mathbf{x}_{H} . We demand the outcome of the action of $\hat{\mathcal{H}}_2$ on a test function to be square integrable as well. This limits the domain of $\hat{\mathcal{H}}_2$ to a so-called Sobolev space. For a (bounded) wavefield in an inhomogeneous fluid, this does not lead to any restrictions in the analysis.

We first examine the analogy of $\hat{\mathcal{H}}_2$ with the Hamiltonian from nonrelativistic quantum mechanics. The similarities between both operators allow for a quantitative description of the spectral properties of the Helmholtz operator using the results for the Hamiltonian operator \hat{H} (Reed and Simon, 1978, 1979).

The 2-D Hamiltonian operator can be written according to

$$\hat{H} = -\Delta + V(\mathbf{x}_H), \tag{A-1}$$

where Δ is the Laplacian $(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)$ and $V(\mathbf{x}_H)$ is the potential function. For our purpose, we assume this function to have compact support, i.e., it vanishes outside some bounded domain. According to Von Neumann's definition of the spectrum $\sigma(\hat{H})$, it can be described as a perturbation of the spectrum of the Laplacian Δ . On an appropriately chosen Sobolev space, the Laplacian is a self-adjoint operator, implying that its spectrum is real valued. Moreover, the related space of eigenfunctions is orthogonal and complete.

Using the Fourier transform, it can be shown that the spectrum of $-\Delta$ is purely continuous. It covers the interval $[0, \infty)$. The perturbation by $V(\mathbf{x}_H)$ does not affect the continuous spectrum or the property of self-adjointness, provided that $V(\mathbf{x}_{H})$ is a real function. However, resulting from the perturbation, a finite number of real eigenvalues in the interval $[-\min\{V(\mathbf{x}_H)\}, 0)$ may occur (Reed and Simon, 1978).

We now return to the Helmholtz operator. Rewriting equation (17), we obtain

$$\hat{\mathcal{H}}_{2} = k_{0}^{2} - \underbrace{\left[\left\{k_{0}^{2} - k^{2}(\mathbf{x}_{H})\right\} - \nabla_{H}^{2}\right]}_{2\text{-D Hamiltonian}}.$$
(A-2)

As indicated in equation (A-2), the Hamiltonian can be recognized on the righthand side of the equation, provided $k(\mathbf{x}_H)$ is real (no losses). A "background" wavenumber k_0 has been introduced to obtain a potential function $k_0^2 - k^2(\mathbf{x}_H)$ with compact support.

Equation (A-2) shows that the Helmholtz operator $\hat{\mathcal{H}}_2$ is the sum of a multiplication operator k_0^2 and the Hamiltonian (with

reversed polarity). Therefore, the spectrum of $\hat{\mathcal{H}}_2$ is obtained if the spectrum of \hat{H} is first mirrored around the imaginary axis and then shifted with k_0^2 to the right. The result of this procedure is depicted in Figure 1. Physically, the eigenvalues λ in equation (18) correspond to k_3^2 , the square of the vertical wavenumber. The continuous part stretches over the interval $(-\infty, k_0^2)$, where k_0 is the background wavenumber and the discrete eigenvalues are located within the interval $(k_0^2, \max\{k^2(\mathbf{x}_H)\})$. Because the eigenvalues λ relate directly to the square of k_3 , we conclude that, for positive λ , the corresponding eigenfunctions $\phi(\mathbf{x}_H)$ must represent wavefield constituents that propagate with a vertical phase velocity $\omega/\sqrt{\lambda}$; for negative λ , the corresponding eigenfunctions $\phi(\mathbf{x}_H)$ represent evanescent wave modes.