2) We find it convenient to choose the constants A and B in the equation $\tau = A\ln t + B$ so that τ is equal to 1, 2, ... n at the new sample points. There is no loss of generality in this choice; what is important is the value of t that corresponds to each new sample point. To determine A and B we assign sample n to a chosen time t_n and sample n-1 to the time t_n -dt, where dt is the maximum sampling interval for the Nyquist frequency. This determines A. We assign sample 1 to the time t_0; B and n are then determined.

3) The angle theta was used in Appendix A as a parameter in defining the DMO ellipse. It has no physical significance and must be eliminated to get the equation of the ellipse. Unfortunately we also used it in the paper for the angle of incidence. There is no connection and we apologize for the confusion.

4) Mohan et al. raise a question as to the phase of the DMO operator. The expression for G [equation (CG-2) in the discussion] can be approximated when

$$h^2K^2 / A^2F^2$$

is small. The result is

$$G \rightarrow \exp\left(\frac{-i\pi h^2 K^2}{AF}\right).$$

There is no singularity when h or K approaches zero. The only singularity is when F approaches zero, as stated in Appendix B.

We thank Dr. Mohan and his colleagues for their comments and hope they will publish their results in the area of chaos/fractals etc.

Gerald H. F. Gardner

Discussion

On: "Green's function implementation of common-offset wave equation migration," (A. Ehinger, P. Lailly and K. J. Marfurt, November-December 1996 GEOPHYSICS, 61, p. 1813-1821)

The paper by Ehinger and co-authors deals with migration in complex media in which severe multi-pathing may occur. I wholeheartedly agree with their remark that the use of complete Green's functions is preferred over approximate Green's functions containing first arrivals only.

In their Appendix the authors derive that the inverse extrapolation operator for downward-propagating waves is given by the complex conjugate of the forward extrapolation operator for upward-propagating waves, according to

$$W^{-1}(0 \rightarrow z) = \overline{W(z \rightarrow 0)}, \tag{1}$$

and vice versa

$$W^{-1}(z \rightarrow 0) = \overline{W(0 \rightarrow z)}. \tag{2}$$

My first remark is a minor one and concerns the normalization. The authors claim that their results are due to an adequate formulation of the one-way wave equation, which implies a flux conservation property (I will call this *flux-normalization*). However, we obtained the same expressions without this normalization [see for instance Wapenaar and Berkhout (1989, equations 7.74b and 7.74a), or our paper on inverse wave field extrapolation in GEOPHYSICS in 1989]. Apparently equations (1) and (2) above apply to operators with or without flux-normalization. The authors needed this normalization because they derived their inverse operators via adjoint operators, which is alright, but not essential. Nevertheless, flux-normalization has an added value, because it implies reciprocity of the one-way operators, as I will discuss below.

My main "concern" is that the authors based their derivation on a paraxial approximation of the one-way wave equations. Their remark that equations (1) and (2) are exact for W's obeying these paraxial equations is correct but of limited value, since the considered W's are not exact.

In the references mentioned above we showed that, for exact W's, equations (1) and (2) involve the following approximations:

1) evanescent waves are neglected,

2) transmission losses and other second and higher order scattering terms are neglected.

The neglect of evanescent waves is quite fundamental: any attempt to treat evanescent waves correctly in inverse extrapolation leads by definition to unstable results, even in homogeneous media (Berkhout and van Wulfften Palthe, 1979). The neglect of second and higher order scattering terms may become significant in complex media. In the paper by Ehinger et al. these approximations are "hidden" in their initial assumption (i.e., the paraxial approximation of the one-way wave equation).

In the next section I review a number of theorems for the one-way wave equation that avoid the paraxial approximations, and I indicate how these theorems lead to improved inverse operators for complex media.

Reciprocity theorems for one-way wave fields and their application in inverse wave field extrapolation

Consider a wave vector **P** and a source vector **S**, defined as

$$\mathbf{P} = \begin{pmatrix} \mathcal{P}^+ \\ \mathcal{P}^- \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} \mathcal{S}^+ \\ \mathcal{S}^- \end{pmatrix}, \tag{3}$$

where $\mathcal{P}^+$ and $\mathcal{P}^-$ represent downgoing and upgoing waves, respectively, and where $\mathcal{S}^+$ and $\mathcal{S}^-$ represent source functions for these downgoing and upgoing waves. In the following it is assumed that the one-way wave fields $\mathcal{P}^+$ and $\mathcal{P}^-$ are flux-normalized [note that for exact one-way wave fields this normalization involves pseudo-differential operators rather than scalar factors; see de Hoop (1992) or Wapenaar and Grimbergen (1996)]. The one-way wave equation for these vectors reads (in the frequency domain)

$$\frac{\partial \mathbf{P}}{\partial z} - \underset{\sim}{\mathbf{B}}\mathbf{P} = \mathbf{S}, \tag{4}$$

where $\hat{\underset{\sim}{\mathbf{B}}}$ is a full 2 x 2 matrix, containing functions of the modified square-root operator [for details, see Fishman et al. (1987) and the references given above]. Note that the anti-diagonal of $\hat{\underset{\sim}{\mathbf{B}}}$ accounts for the coupling between the downgoing and upgoing waves.

In the following I consider two acoustical states, denoted by the subscripts A and B (Ehinger et al. do something similar in equation (A-19)). In general, the wave fields, sources and material parameters in two different states are interrelated via reciprocity theorems [de Hoop (1988), Fokkema and van den Berg (1993)]. For the one-way wave fields considered here, the reciprocity theorems read (Wapenaar and Grimbergen, 1996)

$$\int_{\Sigma_1 \cup \Sigma_2} \mathbf{P}_A^T \underset{\sim}{\mathbf{N}} \mathbf{P}_B n_z d^2\mathbf{x} = \int_{\mathcal{V}} \mathbf{P}_A^T \underset{\sim}{\mathbf{N}} \left\{ \underset{\sim}{\mathbf{B}}_B - \underset{\sim}{\mathbf{B}}_A \right\} \mathbf{P}_B d^3\mathbf{r} + \int_{\mathcal{V}} \left\{ \mathbf{P}_A^T \underset{\sim}{\mathbf{N}} \mathbf{S}_B + \mathbf{S}_A^T \underset{\sim}{\mathbf{N}} \mathbf{P}_B \right\} d^3\mathbf{r} \tag{5}$$

[T denotes transposition, $\mathbf{r} = (x, y, z)$, $\mathbf{x}=(x, y)$] and

$$\int_{\Sigma_1 \cup \Sigma_2} \mathbf{P}_A^H \underset{\sim}{\mathbf{J}} \mathbf{P}_B n_z d^2\mathbf{x} \approx \int_{\mathcal{V}} \mathbf{P}_A^H \underset{\sim}{\mathbf{J}} \left\{ \underset{\sim}{\mathbf{B}}_B - \underset{\sim}{\mathbf{B}}_A \right\} \mathbf{P}_B d^3\mathbf{r} + \int_{\mathcal{V}} \left\{ \mathbf{P}_A^H \underset{\sim}{\mathbf{J}} \mathbf{S}_B + \mathbf{S}_A^H \underset{\sim}{\mathbf{J}} \mathbf{P}_B \right\} d^3\mathbf{r}, \tag{6}$$

(H denotes transposition and complex conjugation), with

$$\underset{\sim}{\mathbf{N}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{and } \underset{\sim}{\mathbf{J}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

In equations (5) and (6), the integration volume $\mathcal{V}$ is "enclosed" by two infinite parallel surfaces Σ_1 and Σ_2 normal to the z-axis, see Figure 1. These surfaces need not be physical boundaries. The outward pointing normal vector is denoted by $\mathbf{n} = (0,0,n_z)$, with $n_z = -1$ at the upper surface Σ_1 and $n_z = +1$ at the lower surface Σ_2.

The one-way reciprocity theorem of the convolution type (equation 5) is exact and provides a basis for general representations of one-way wave fields. One of the consequences of equation (5) is that the kernels of the one-way extrapolation operators obey reciprocity, according to

$$\mathcal{W}^+(\mathbf{r}_2,\mathbf{r}_1) = \mathcal{W}^-(\mathbf{r}_1,\mathbf{r}_2) \tag{8}$$

The one-way reciprocity theorem of the correlation type (equation 6) has been derived under the assumption that evanescent wave modes can be ignored. For the special situation that the media in states A and B are identical and source-free in $\mathcal{V}$, the integrals on the right-hand side of equation (6) vanish, leaving

$$\int_{\Sigma_1} \left(\{\mathcal{P}_A^+(\mathbf{r})\} * \mathcal{P}_B^+(\mathbf{r}) - \{\mathcal{P}_A^-(\mathbf{r})\} * \mathcal{P}_B^-(\mathbf{r}) \right) d^2\mathbf{x} \approx \int_{\Sigma_2} \left(\{\mathcal{P}_A^+(\mathbf{r})\} * \mathcal{P}_B^+(\mathbf{r}) - \{\mathcal{P}_A^-(\mathbf{r})\} * \mathcal{P}_B^-(\mathbf{r}) \right) d^2\mathbf{x}, \tag{9}$$

(in my notation, * denotes complex conjugation). Note that when $\mathcal{P}_A^-$ and $\mathcal{P}_B^-$ are negligible in comparison with $\mathcal{P}_A^+$ and $\mathcal{P}_B^+$, equation (A-19) of Ehinger et al. is obtained. With this approximation, equations (1) and (2) above can be derived straightforwardly. In complex media with significant scattering it is not justified to ignore $\mathcal{P}_A^-$ and $\mathcal{P}_B^-$. In its full form, equation (9) provides the basis for deriving improved inverse wave field extrapolation operators. For the kernel $\mathcal{F}^+(\mathbf{r}_1,\mathbf{r}_2)$ of the inverse extrapolation operator for downward propagating waves the following Neumann series expansion is obtained

$$\{\mathcal{F}^+(\mathbf{r}_1,\mathbf{r}_2)\}^{(k)} \approx \{\mathcal{F}^+(\mathbf{r}_1,\mathbf{r}_2)\}^{(0)} + \int_{\Sigma_1} \mathcal{C}(\mathbf{r}_1,\mathbf{r}) \{\mathcal{F}^+(\mathbf{r},\mathbf{r}_2)\}^{(k-1)} d^2\mathbf{x}, \tag{10}$$

with $\mathbf{r}_1 \in \Sigma_1, \mathbf{r}_2 \in \Sigma_2$ and

$$\{\mathcal{F}^+(\mathbf{r}_1,\mathbf{r}_2)\}^{(0)} \approx \{\mathcal{W}^-(\mathbf{r}_1,\mathbf{r}_2)\}^* = \{\mathcal{W}^+(\mathbf{r}_2,\mathbf{r}_1)\}^*, \tag{11}$$

[we obtained equation (10) in a slightly different form in Wapenaar and Berkhout (1989, Chapter 9); Herman (1992) obtained a similar result]. Note that, according to equation (11), the leading term of equation (10) is equivalent to Ehinger's result [equation (1) above]. The higher order terms are obtained by iteratively applying equation (10), where $\mathcal{C}(\mathbf{r}_1,\mathbf{r})$ represents the multi-dimensional cross-correlation of the deconvolved seismic reflection measurements at Σ_1. Apparently the seismic data itself can be used to modify the inverse operators. Intuitively this can be well understood: the "missing energy" at Σ_2 [i.e., the transmission losses etc. in $\mathcal{W}^+(\mathbf{r}_2,\mathbf{r}_1)$] is contained in the reflection measurements at Σ_1 (assuming no anelastic losses). Using reciprocity [equation (8)], the kernel $\mathcal{F}^-(\mathbf{r}_2,\mathbf{r}_1)$ of the inverse extrapolation operator for upward propagating waves is given by

$$\mathcal{F}^-(\mathbf{r}_2,\mathbf{r}_1)=\mathcal{F}^+(\mathbf{r}_1,\mathbf{r}_2). \tag{12}$$

SUMMARY

For weakly inhomogeneous media the kernels of the inverse extrapolation operators read

$$\{\mathcal{F}^+(\mathbf{r}_1,\mathbf{r}_2)\} \approx \{\mathcal{W}^-(\mathbf{r}_1,\mathbf{r}_2)\}^*, \tag{13}$$

$$\{\mathcal{F}^-(\mathbf{r}_2,\mathbf{r}_1)\} \approx \{\mathcal{W}^+(\mathbf{r}_2,\mathbf{r}_1)\}^*, \tag{14}$$

where * denotes complex conjugation [these expressions are equivalent to equations (1) and (2) above]. These results are valid with or without flux-normalization, no paraxial approximation is involved and to some extent multi-pathing is included. However, evanescent waves are neglected and transmission losses and other second and higher order scattering terms are neglected as well. For flux-normalized operators equations (13) and (14) may be generalized to

$$\{\mathcal{F}^+(\mathbf{r}_1,\mathbf{r}_2)\} \approx \{\mathcal{W}^-(\mathbf{r}_1,\mathbf{r}_2)\}^* = \{\mathcal{W}^+(\mathbf{r}_2,\mathbf{r}_1)\}^*, \tag{15}$$

$$\{\mathcal{F}^-(\mathbf{r}_2,\mathbf{r}_1)\} \approx \{\mathcal{W}^+(\mathbf{r}_2,\mathbf{r}_1)\}^* = \{\mathcal{W}^-(\mathbf{r}_1,\mathbf{r}_2)\}^*. \tag{16}$$

Equations (13) through (16) break down in complex media with significant scattering. Improved inverse operator kernels can be obtained via a Neumann expansion; the correction terms can be derived directly from the seismic reflection measurements [equations (10) through (12)]. These improved inverse operator kernels account for multi-pathing, transmission losses and other second and higher order scattering terms. The only approximation is that evanescent wave modes are neglected, hence, they are stable and no paraxial approximation is involved.

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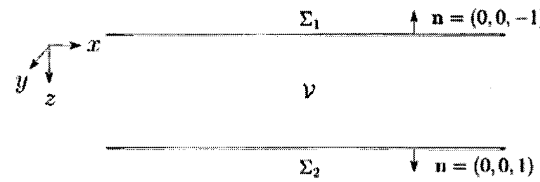

Figure 1. The configuration for the reciprocity theorems.

Kees Wapenaar

Reply by the authors to the discussion by K. Wapenaar

In our paper, the equalities

$$W^{-1}(z \to 0) = \overline{W(0 \to z)} \qquad (1)$$
$$W^{-1}(0 \to z) = \overline{W(z \to 0)}$$

are of central importance for the mathematical correctness of our proposed common offset wave-equation migration algorithm. In his comment, Kees Wapenaar outlines two "concerns" regarding these equalities:

1) a minor one, concerning the importance of the flux normalization for obtaining (1);

2) a major one, concerning the physical relevance of our W operators.

Regarding the first "concern," we do not claim that flux normalization is essential to obtain equalities of type (1). We just mean that our very specific choice of W (based on a specific paraxial wave equation) implies flux normalization and reciprocity, which results in (1) to be true.

Regarding Kees Wapenaar's major "concern," we agree that our inverse extrapolation operators are the inverse operators of non-exact forward extrapolation (propagation) operators. However, there seems to be some misunderstanding between Kees Wapenaar and us about the kind of "approximation" we use. In fact, our paraxial operators W certainly hide some approximations; however, contrarily to what Kees Wapenaar says, they give rise—in heterogeneous media—to transmission losses and, to some extent, to multiple scattering.

At this stage it seems thus important to clarify what we mean by "approximation." We therefore recall the origin of our paraxial extrapolators: they are based on specific paraxial wave equations proposed by Bamberger et al., 1984 (see also Bamberger et al., 1988). These authors derived a "partial differential equation that extends to heterogeneous media the parabolic approximation of the wave equation in such a way that

1) the Cauchy problem is mathematically well posed (i.e. it admits a unique solution that depends continuously on the initial conditions and on the velocity model)

2) its solution matches at best in the paraxial direction and for slowly varying media the propagation and transmission properties of the solution of the full wave equation" (Bamberger et al., 1984, page 34.)

Clearly, transmission losses are, to some extent, taken into account. The physical properties of the solution of this equation regarding propagation have been the subject of extensive studies. Besides the many geophysical papers published on the subject we would like to mention in particular

1) the many papers published in the field of underwater acoustics (Tappert, 1977, gives an overview) about propagation in weakly heterogeneous media,

2) Bamberger et al., 1984, who gives, for general heterogeneous media, the general properties regarding the propagation of energy and who studies in detail reflection and transmission properties at an interface[1],

3) Duquet, 1996, who establishes conditions on the seismic source to be used for the paraxial wavefield to approximate the true wavefield.

Nevertheless, and we fully agree with Kees Wapenaar at that point, the extrapolation operators are not exact. They are approximations that work under severe physical assumptions (on the velocity model, on the seismic source and on the propagation range) and which, when applied in models with severe velocity contrasts, can give rise to propagation artifacts (because multiple scattering is modeled in a non-exact way). Despite of these shortcomings, these approximations have turned out to give very good results even when applied to the very complex Marmousi model (see our paper).

Another key issue is, of course, numerical stability and this is an interesting feature of the specific paraxial extrapolators we use (as shown e.g. by Etgen, 1994, such numerical stability does not hold for some other extrapolators). This numerical stability results from the great care taken, regarding the mathematics, to derive the paraxial equations[2].

The paraxial wave equations thus turns out to be a quite attractive tool for cheap and stable wavefield modeling in strongly heterogeneous media. This is the reason why they are so widely used in the geophysical industry, especially in the context of shot record and poststack depth migration. The purpose of our paper was to enlarge the realm of these equations by illustrating their use for the implementation of a cost-effective and stable common offset wave-equation migration algorithm.

Kees Wapenaar outlines that the same could be done with other extrapolation operators that better approximate the full

[1]note that reflection implies multiple scattering

[2]The proof of the numerical stability (Collino, 1987) uses a discrete energy inequality which relies on the above mentioned considerations about the continuous dependance of the solution both on the initial conditions and on the velocity distribution.

(Continued on p. 1346)

(Continued from p. 1335)
wave equation. This is good news and, if numerical stability applies, it should make common offset wave-equation migration even more attractive.

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Andreas Ehinger, Patrick Lailly, and Kurt J. Marfurt

Errata

To: "2-D magnetic interpretation using the vertical integral," J. B. C. Silva (March-April 1996 GEOPHYSICS, 61, p. 387-393)

Equation (2), instead of

$$a_{ij} = 2\frac{\left(u_{ij}^2 + v_{ij}^2\right)\left(\alpha_j\alpha_o + \gamma_j\gamma_o\right) + 2\left(\alpha_j u_{ij} - \gamma_j v_{ij}\right)\left(\alpha_o u_{ij} - \gamma_o v_{ij}\right)}{\left(u_{ij}^2 + v_{ij}^2\right)^2} , \tag{1}$$

should read

$$a_{ij} = 2\frac{-\left(u_{ij}^2 + v_{ij}^2\right)\left(\alpha_j\alpha_o + \gamma_j\gamma_o\right) + 2\left(\alpha_j u_{ij} + \gamma_j v_{ij}\right)\left(\alpha_o u_{ij} + \gamma_o v_{ij}\right)}{\left(u_{ij}^2 + v_{ij}^2\right)^2} , \tag{2}$$

and equation (5), instead of

$$b_{ij} = 2\frac{v_{ij}\left(\gamma_j\gamma_o - \alpha_j\alpha_o\right) - u_{ij}\left(\alpha_j\gamma_o + \gamma_j\alpha_o\right)}{u_{ij}^2 + v_{ij}^2} + C , \tag{3}$$

should read

$$b_{ij} = 2\frac{-v_{ij}\left(\gamma_j\gamma_o - \alpha_j\alpha_o\right) - u_{ij}\left(\alpha_j\gamma_o + \gamma_j\alpha_o\right)}{u_{ij}^2 + v_{ij}^2} + C . \tag{4}$$

To: "Automatic 3-D interpretation of potential-field data using analytic signal derivatives," by N. Debeglia and J. Corpel (January-February 1997 GEOPHYSICS, 62, p. 87-96)

When we sent the last revision of our paper to GEOPHYSICS, we had not yet received the March-April 1996 issue of GEOPHYSICS and read the paper by Hsu et al. Thereby it could not be included in the references used to assess the method and write the paper. We note some convergences between the two approaches despite the fact that the depth computation algorithms are quite different.

S. -K. Hsu, J. -C., Sibuet and C. -T. Shyu have first introduced the generalized form of the analytic signal in a 3-D case, and we are pleased to cite their paper, "High-resolution detection of geologic boundaries from potential-field anomalies: An enhanced analytic signal technique," in March-April 1996 GEOPHYSICS, **61,** p. 373-386. As well as ours, their paper points out the improvements in resolution resulting from the use of derivatives in analytic signal interpretation.

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