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Inverse extrapolation of primary seismic waves

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ABSTRACT

Forward wave-field extrapolation operators *simulate* propagation effects from one depth level to another. Inverse wave-field extrapolation operators *eliminate* those propagation effects. Since forward wave-field extrapolation can be described in terms of spatial *convolution*, inverse wave-field extrapolation can be described in terms of spatial *deconvolution*. A simple approximation to a stable, spatially band-limited deconvolution operator is obtained by taking the complex conjugate of the convolution operator. A one-way version of the Kirchhoff integral that contains the conjugate complex Green's function is derived. Unlike the situation with respect to the forward problem, the modification of the *closed* surface integral into an *open* boundary integral involves an approximation that is identical to the approximation in the conjugate complex deconvolution approach. This approximation neglects the evanescent field and causes a second-order amplitude error.

For a plane acquisition surface, the one-way Kirchhoff integral is transformed into a one-way Rayleigh integral. For media with small to moderate contrasts, the one-way Rayleigh integral with the conjugate complex Green's function describes true amplitude inverse extrapolation of primary waves. This is illustrated with an example, in which the Green's function has been modeled with the Gaussian beam method.

INTRODUCTION

In one-way seismic modeling and inversion, it is common practice to consider the incident (downgoing) wave field and the scattered (upgoing) wave field separately. One-way seismic modeling is essentially based on *forward* extrapolation of downgoing and upgoing waves (*simulation* of propagation effects, see Figures 1a and 1b). One-way seismic inversion techniques, such as migration (Berkhout and Van Wulfften Palthe, 1979; Berkhout, 1985), inverse scattering (Bleistein, 1984), or redatuming (Berryhill, 1984), are essentially based on *inverse* extrapolation of downgoing and upgoing waves (*elimination* of propagation effects, see Figures 1c and 1d).

Berkhout and Wapenaar (1989, hereafter referred to as paper I) derived one-way versions of the Kirchhoff integral for *forward* extrapolation of primary waves through inhomogeneous acoustic media. In this paper we derive one-way versions of the Kirchhoff integral for *inverse* extrapolation of primary waves through inhomogeneous acoustic media. The expressions that we obtain (commonly known as generalized Kirchhoff summation) have been used previously by various authors (Schneider, 1978; Clayton and Stolt, 1981; Castle, 1982; Carter and Frazer, 1984; Wiggins, 1984; Berryhill, 1984). The main point of our paper is that we critically analyze the approximations that are involved. One important conclusion is that the inverse extrapolation results are necessarily spatially band-limited. A second conclusion is that second-order amplitude errors are made. Hence, for large contrasts, where the errors would be large, a refinement is required.

THE KIRCHHOFF INTEGRALS WITH FORWARD AND BACKWARD PROPAGATING GREEN'S FUNCTIONS

We consider an inhomogeneous lossless fluid, which is described by the space-dependent compression modulus $K(\mathbf{r})$ and the mass density $\rho(\mathbf{r})$, where $\mathbf{r}$ is a shorthand notation for the Cartesian coordinates (x, y, z). In this fluid we consider a (sub-)volume V enclosed by a surface S with an outward pointing normal vector $\mathbf{n}$. The space and frequency-dependent acoustic pressure $P(\mathbf{r}, \omega)$ satisfies in V the following equation:

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla P \right) + k^2 P = 0, \tag{1a}$$

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where the wavenumber $k(\mathbf{r}, \omega)$ is defined as

$$k(\mathbf{r}, \omega) = \frac{\omega}{\sqrt{K(\mathbf{r})/\rho(\mathbf{r})}} = \frac{\omega}{c(\mathbf{r})}, \tag{1b}$$

and it is assumed that the acoustic wave field is caused by sources outside V. $c(\mathbf{r})$ represents the space-dependent propagation velocity, and ω represents the radial frequency. We define a Green's wave field $G(\mathbf{r}, \mathbf{r}_A, \omega)$, which satisfies in V the following equation:

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla G \right) + k^2 G = -\rho \delta(\mathbf{r} - \mathbf{r}_A), \tag{2}$$

where $\mathbf{r}_A = (x_A, y_A, z_A)$ denotes the Cartesian coordinates of an internal point A in V.

Note that when $G(\mathbf{r}, \mathbf{r}_A, \omega)$ is a solution of equation (2), then the complex conjugated function $G^*(\mathbf{r}, \mathbf{r}_A, \omega)$ is also a solution of equation (2). Throughout this paper, $G(\mathbf{r}, \mathbf{r}_A, \omega)$ is the frequency-domain representation of the *causal* or *forward propagating* Green's wave field $g(\mathbf{r}, \mathbf{r}_A, t)$ according to

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = \int_{-\infty}^{\infty} g(\mathbf{r}, \mathbf{r}_A, t) e^{-i\omega t} \, dt, \tag{3a}$$

or

$$g(\mathbf{r}, \mathbf{r}_A, t) = \frac{1}{\pi} \operatorname{Re} \left[\int_0^{\infty} G(\mathbf{r}, \mathbf{r}_A, \omega) e^{i\omega t} \, d\omega \right], \tag{3b}$$

with

$$g(\mathbf{r}, \mathbf{r}_A, t) = 0 \qquad \text{for } t < 0. \tag{3c}$$

Consequently, $G^*(\mathbf{r}, \mathbf{r}_A, \omega)$ is the frequency-domain representation of the *anticausal* or *backward propagating* Green's wave field $g(\mathbf{r}, \mathbf{r}_A, -t)$.

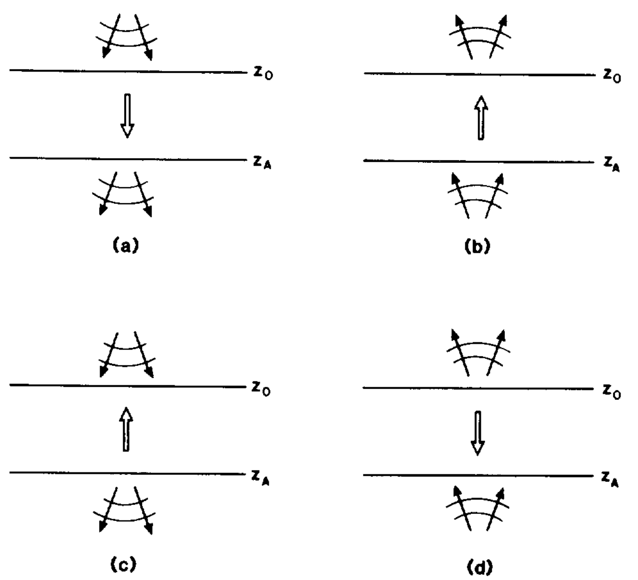

FIG. 1. Four cases of one-way wave-field extrapolation. (a) Forward extrapolation of downgoing waves. (b) Forward extrapolation of upgoing waves. (c) Inverse extrapolation of downgoing waves. (d) Inverse extrapolation of upgoing waves.

With these two Green's wave fields, two versions of the Kirchhoff integral can be derived. When we apply the theorem of Gauss,

$$\int_V \nabla \cdot \mathbf{Q} \, dV = \oint_S \mathbf{Q} \cdot \mathbf{n} \, dS \tag{4}$$

to the vector function

$$\mathbf{Q}_1 = \frac{\nabla G}{\rho} P - G \frac{\nabla P}{\rho}, \tag{5a}$$

where P and G satisfy equations (1a) and (2) in V, we obtain the first version of the Kirchhoff integral:

$$P(\mathbf{r}_A, \omega) = -\oint_S \frac{1}{\rho(\mathbf{r})} \left[\nabla G(\mathbf{r}, \mathbf{r}_A, \omega) P(\mathbf{r}, \omega) - G(\mathbf{r}, \mathbf{r}_A, \omega) \nabla P(\mathbf{r}, \omega) \right] \cdot \mathbf{n} \, dS \tag{5b}$$

(see Figure 2a).

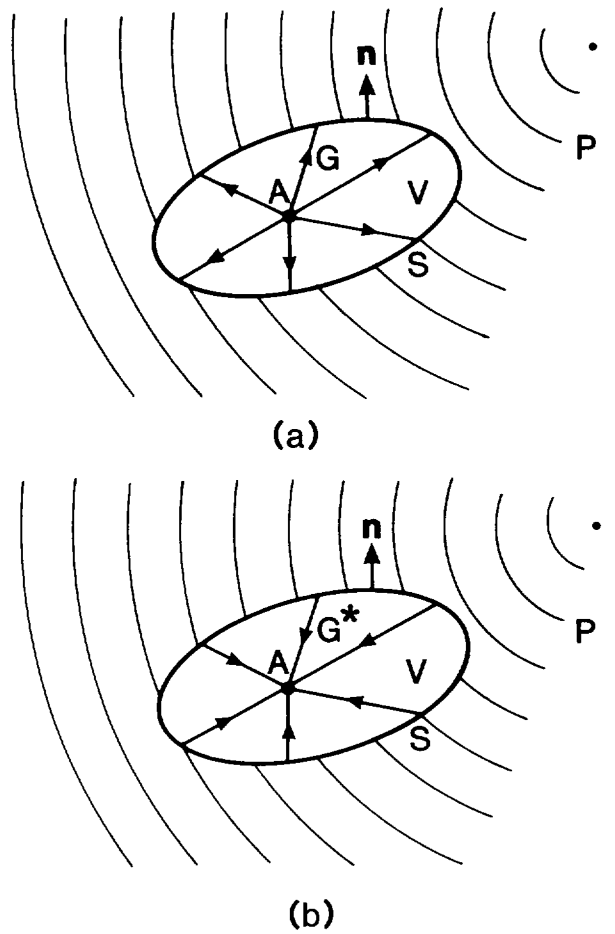

FIG. 2. Assuming sources outside S, the acoustic wave field at any point A inside S can be calculated when the wave field and its normal derivative are known on S. For this purpose, we may use either Kirchhoff integral (5b) with the forward propagating Green's wave field (a) or Kirchhoff integral (6b) with the backward propagating Green's wave field (b).

On the other hand, when we apply the theorem of Gauss to the vector function

$$\mathbf{Q}_2 = \frac{\nabla G^*}{\rho} P - G^* \frac{\nabla P}{\rho}, \tag{6a}$$

where P and G^* satisfy equations (1a) and (2) in V, we obtain the second version of the Kirchhoff integral:

$$P(\mathbf{r}_A, \omega) = -\oint_S \frac{1}{\rho(\mathbf{r})} \left[\nabla G^*(\mathbf{r}, \mathbf{r}_A, \omega) P(\mathbf{r}, \omega) - G^*(\mathbf{r}, \mathbf{r}_A, \omega) \nabla P(\mathbf{r}, \omega) \right] \cdot \mathbf{n}\, dS \tag{6b}$$

(see Figure 2b).

Both versions of the Kirchhoff integral are exact. In paper I, we used the version with the forward-propagating Green's functions (5b) to derive forward wave-field extrapolation operators. In this paper, we use the version with the backward-propagating Green's functions (6b) to derive inverse wave-field extrapolation operators.

MODIFYING THE KIRCHHOFF INTEGRAL FOR SEISMIC DATA

Consider Figure 3a, which symbolically represents seismic data acquisition. Our aim is to derive the scattered wave field at subsurface point A from the measured wave field at acquisition surface S_0. For this purpose, we want to apply the Kirchhoff integral, either with forward-propagating Green's functions (5b) or with backward-propagating Green's functions (6b). We construct a closed surface S surrounding a *source-free* volume V. Therefore, first we consider Figure 3b, where closed surface S consists of acquisition surface S_0 and a hemisphere S_1 with radius R in the lower half-space. Of course, no measurements are available on the hemisphere S_1.

Suppose we apply Kirchhoff integral (5b) with the forward propagating Green's functions. Then, the contribution of this integral over S_1 vanishes when R goes to infinity [Sommerfeld radiation condition, Bleistein (1984)], so the Kirchhoff integral (5b) need only be evaluated over the acquisition surface S_0. Note, however, that in order to compute the Green's function $G(\mathbf{r}, \mathbf{r}_A, \omega)$, knowledge of the medium is required in the entire half-space below S_0, including the "scattering objects" [$G(\mathbf{r}, \mathbf{r}_A, \omega)$ satisfies equation (2), which should hold throughout V]. This means that with the aid of Kirchhoff integral (5b), we can compute the scattered wave field at A (the objective) only when the scattering objects are known. Therefore this solution actually describes forward modeling; it has no practical value for inverse wave-field extrapolation.

Suppose, on the other hand, that we apply Kirchhoff integral (6b), with the backward-propagating Green's functions. Then the contribution of this integral over S_1 does not vanish when R goes to infinity [the Sommerfeld radiation condition requires that the Green's function propagates outward through S_1 in the same direction as the total wave field, Bleistein (1984)]. This important aspect is seldom recognized in the seismic literature. In fact, many authors use the Kirchhoff integral with the backward-propagating Green's functions for inverse extrapolation of the scattered wave field from S_0 to A. This approach is certainly not exact. However, it is a very practical solution and therefore we also use it in this paper, but first we critically analyze the approximations that necessarily occur because the radiation conditions are not satisfied. In our analysis, we consider the modified representation of seismic data as shown in Figure 3c. Here it is assumed that the scattered wave field is radiated by (unknown) secondary sources in the subsurface.

Closed surface S consists of acquisition surface S_0, a plane horizontal reference surface S_1 at $z = z_1$ (between A and the secondary sources of interest), and a cylindrical surface S_2 with a vertical axis through A and radius R. Note that volume V, enclosed by S, is assumed to be source-free.

Also note that because V does not cover the entire lower half-space (as in Figure 3b), knowledge of the medium (for the computation of the Green's functions) is required only in the region between acquisition surface S_0 and the secondary source. Consider Kirchhoff integral (6b) with the backward-propagating Green's functions. The contribution of this integral over S_2 vanishes when R goes to infinity (the cylindrical surface is proportional to R, the integrand is proportional to $1/R^2$). Consequently, for the geometry of Figure 3c, the Kirchhoff integral (6b) may be replaced by

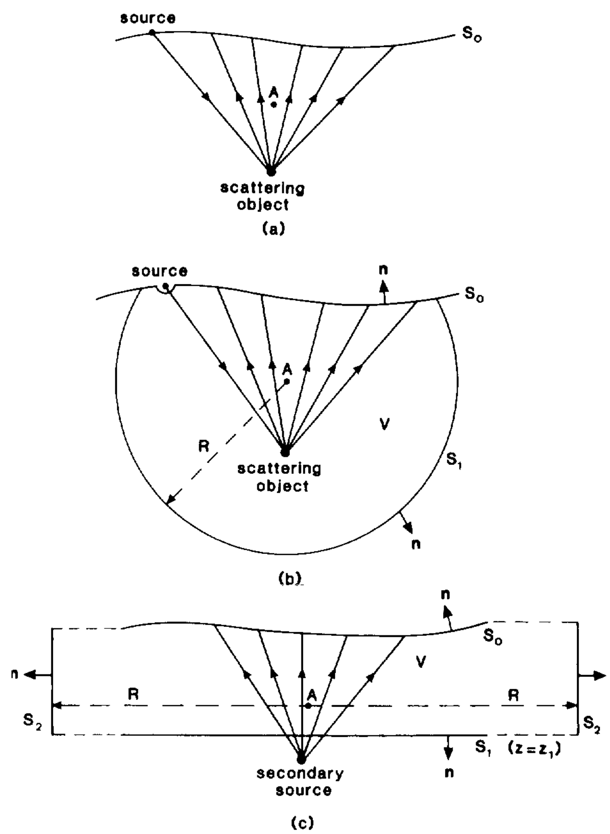

FIG. 3. (a) Simplified representation of the seismic situation. (b) Geometry for the Kirchhoff integral (5b) with the forward-propagating Green's functions. The scattered wave field in A can be computed only when the scattering objects are known. (c) Geometry for the Kirchhoff integral (6b) with the backward-propagating Green's functions. Under certain conditions (discussed in the text) the contribution of this integral over S_1 can be neglected.

$$P(\mathbf{r}_A, \omega) = P_0(\mathbf{r}_A, \omega) + \Delta P(\mathbf{r}_A, \omega), \tag{7a}$$

where

$$P_0(\mathbf{r}_A, \omega) = -\int_{S_0} \frac{1}{\rho}\left[\nabla G^* P - G^* \nabla P\right] \cdot \mathbf{n}\, dS_0 \tag{7b}$$

and

$$\Delta P(\mathbf{r}_A, \omega) = -\int_{S_1} \frac{1}{\rho}\left[\nabla G^* P - G^* \nabla P\right] \cdot \mathbf{n}\, dS_1. \tag{7c}$$

When $\Delta P(\mathbf{r}_A, \omega)$ as defined in equation (7c) may be neglected, then equation (7b) describes *inverse* wave-field extrapolation (toward the secondary sources) from acquisition surface S_0 to subsurface point A. In the following section we investigate the conditions under which $\Delta P(\mathbf{r}_A, \omega)$ may be neglected.

INVERSE EXTRAPOLATION

Homogeneous medium; curved acquisition surface

First we consider a homogeneous medium. At $z = z_1$ the acoustic wave field related to the secondary source below z_1 is purely upgoing:

$$P(\mathbf{r}, \omega) = P^-(\mathbf{r}, \omega) \quad \text{at} \quad z = z_1 \tag{8a}$$

(the positive z-axis is pointing downward). Similarly, the Green's wave field related to the source at A above z_1 is purely downgoing:

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = G^+(\mathbf{r}, \mathbf{r}_A, \omega) \quad \text{at} \quad z = z_1. \tag{8b}$$

We may now rewrite equation (7c) as

$$\Delta P(\mathbf{r}_A, \omega) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\left(\frac{\partial G^+}{\partial z}\right)^* P^- - (G^+)^* \frac{\partial P^-}{\partial z}\right]_{z_1} dx\, dy. \tag{9}$$

Following the derivation in the Appendix, setting $P^+ = G^- = 0$ and $\partial P^+/\partial z = \partial G^-/\partial z = 0$ at z_1 yields

$$\Delta P(\mathbf{r}_A, \omega) \approx 0. \tag{10}$$

The underlying assumption is that $\tilde{P}^-(k_x, k_y, z_1; \omega)$ as defined by equation (A-3a), $\tilde{G}^+(k_x, k_y, z_1; x_A, y_A, z_A; \omega)$ as defined by equation (A-3b), or both are negligible in the evanescent wavenumber area $k_x^2 + k_y^2 \geq k^2$. This assumption is satisfied when the (secondary) source of the acoustic wave field and the source (at A) of the Green's wave field are not both in the direct vicinity of S_1. With this result, we may rewrite equation (7) as

$$P(\mathbf{r}_A, \omega) \approx P_0(\mathbf{r}_A, \omega) = -\int_{S_0} \frac{1}{\rho(\mathbf{r})}\left[\nabla G^*(\mathbf{r}, \mathbf{r}_A, \omega) P(\mathbf{r}, \omega) - G^*(\mathbf{r}, \mathbf{r}_A, \omega) \nabla P(\mathbf{r}, \omega)\right] \cdot \mathbf{n}\, dS_0. \tag{11}$$

This Kirchhoff integral describes *inverse* wave-field extrapolation from acquisition surface S_0 to subsurface point A. It is interesting to note that, in order to arrive at this result, it was essential to make use of the backward-propagating Green's function G^* (if we had used G instead of G^*, then $P_0(\mathbf{r}_A, \omega)$ would have vanished instead of $\Delta P(\mathbf{r}_A, \omega)$, so we would have obtained a Kirchhoff integral for *forward* extrapolation from S_1 to A, see also paper I). The only approximation in equation (11) is a spatial band limitation (neglecting evanescent waves at z_1). This approximation imposes a restriction on the maximum obtainable spatial resolution (Berkhout, 1984). This fundamental aspect of inverse wave-field extrapolation is not always fully appreciated.

The validity of Kirchhoff integral (11) is demonstrated with a simple example. We consider 2-D wave propagation in the model shown in Figure 4a. The propagation velocity equals 2000 m/s. The acoustic pressure field of a buried dipole source, measured at the curved surface S_0, is shown in Figure 4b as a function of space and time. The normal derivative of the wave field at S_0 is shown in Figure 4c. Inverse wave-field extrapolation to depth level z_A is carried out by transforming the data from the time domain to the frequency domain and by applying the 2-D version of equation (11) for all points A at depth level z_A and for all frequencies within the seismic bandwidth. The result, transformed back to the time domain, is shown in Figure 5a. It represents a hyperbolically shaped dipole response. In Figure 5b the maximum of each trace is shown as a function of lateral position (dotted line). Note the perfect match with the analytically computed response (solid line). The very small deviations at the edges are due to the limited aperture (ideally S_0 should be of infinite extent).

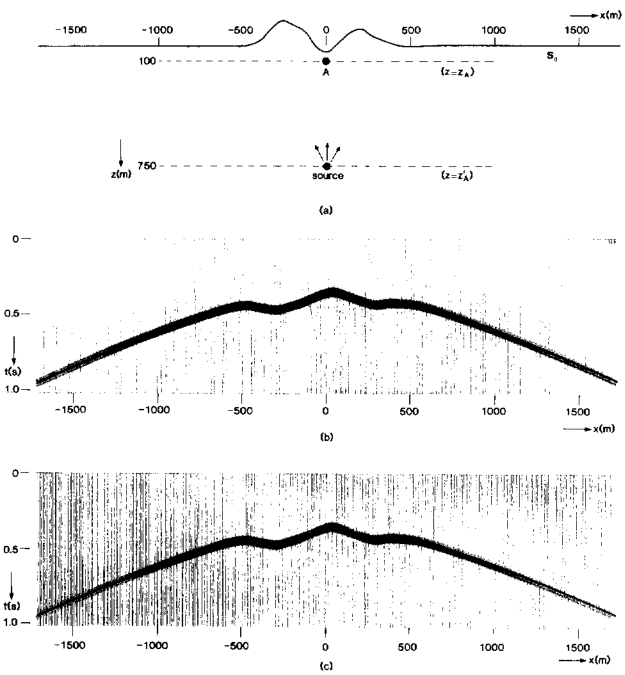

FIG. 4. (a) Homogeneous medium containing a buried dipole source. (b) Pressure field measured at surface S_0 as a function of space and time. (c) Normal derivative of the data in (b).

Next we used equation (11) to inverse extrapolate the data from S_0 to the source depth level z'_A, thus violating the condition that A should not be close to the source. Figure 5c shows the result in the space-time domain. The real part of the data at 35 Hz is shown as a function of lateral position in Figure 5d (the imaginary part is approximately zero). Note that the dipole source (ideally represented by a spatial delta function) is smeared out in space due to the inevitable neglecting of the evanescent wave field. Theoretically the width of the main lobe should be equal to the wavelength λ (Berkhout, 1984). In this example λ equals 2000/35 = 57 m, whereas the width of the main lobe in Figure 5d equals 74 m. The small difference is explained in the greater part by the limited aperture. Finally, we carried out inverse wave-field extrapolation based upon the Rayleigh approximation. That is, we assumed that only the pressure field $P(\mathbf{r}, \omega)$ at S_0 is available and approximated equation (11) by replacing $P(\mathbf{r}, \omega)$ by $2P(\mathbf{r}, \omega)$ and by omitting $\nabla P(\mathbf{r}, \omega)$. The result at z_A, transformed back to the time domain, is shown in Figure 5e. Note that, along with the expected hyperbolically shaped dipole response, some significant artifacts are present in these data. The maximum of each trace is shown as a function of lateral position in Figure 5f (dotted line). Note the significant amplitude deviations from the exact, analytically computed response (solid line). Obviously the Rayleigh approximation is not accurate for a curved recording surface S_0. We show below that the Rayleigh approximation is accurate for a flat surface S_0.

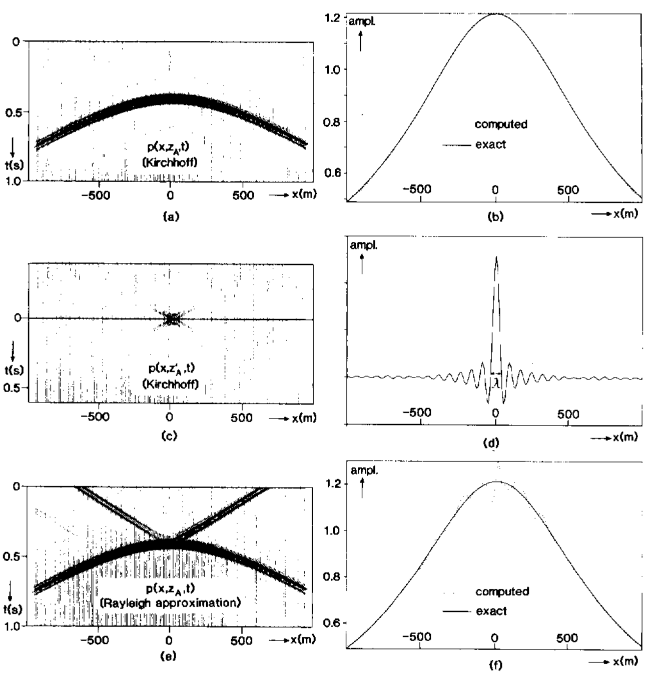

FIG. 5. (a) Inverse extrapolated data at z_A [Kirchhoff integral (11)]. (b) Maximum amplitude per trace of (a). (c) Inverse extrapolated data at z_A^1 [Kirchoff integral (11)]. (d) Real part of central frequency component from data in (c). (e) Inverse extrapolated data at z_A (Rayleigh approximation). (f) Maximum amplitude per trace of (e).

Inhomogeneous medium; curved acquisition surface

We analyze the error term $\Delta P(\mathbf{r}_A, \omega)$ given by equation (7c) for an inhomogeneous medium. At $z = z_1$ the acoustic wave field consists of upgoing waves (including higher order terms) related to the secondary source below z_1 and downgoing waves (including higher order terms) caused by scattering above z_1. Hence,

$$P(\mathbf{r}, \omega) = P^+(\mathbf{r}, \omega) + P^-(\mathbf{r}, \omega) \quad \text{at} \quad z = z_1, \qquad (12a)$$

(see Figure 6a).

For the Green's wave field, we may choose below $z = z_1$ (outside V) any convenient reference medium. We choose $c(x, y, z > z_1) = c(x, y, z_1)$ and $\rho(x, y, z > z_1) = \rho(x, y, z_1)$. With this choice the Green's wave field at z_1 is purely downgoing (no scattering occurs in the lower half-space below z_1); hence,

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = G^+(\mathbf{r}, \mathbf{r}_A, \omega) \quad \text{at} \quad z = z_1 \qquad (12b)$$

(see also Figures 6b and 6d). We may now rewrite equation (7c) as

$$\Delta P(\mathbf{r}_A, \omega) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\left(\frac{\partial G^+}{\partial z}\right)^* (P^+ + P^-) - (G^+)^*\left(\frac{\partial P^+}{\partial z} + \frac{\partial P^-}{\partial z}\right)\right]_{z=z_1} dx\, dy. \qquad (13)$$

To analyze this error term, we assume for simplicity $\nabla c = \nabla \rho = \mathbf{0}$ at $z = z_1$. Following the derivation in the Appendix, setting $G^- = 0$ and $\partial G^-/\partial z = 0$ at z_1 yields

$$\Delta P(\mathbf{r}_A, \omega) \approx -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\left(\frac{\partial G^+}{\partial z}\right)^* P^+\right]_{z=z_1} dx\, dy. \qquad (14)$$

Again the underlying assumption is that the (secondary) source of the acoustic wave field and the source (at A) of the Green's wave field are not both in the direct vicinity of S_1. Let us now write the Green's wave field G^+ at z_1 as the sum of a wave field G_d^+ which propagates *directly* from A to $z = z_1$ (Figure 6b) and a wave field G_s^+ which is *scattered* by the inhomogeneities above A before it arrives at $z = z_1$ (Figure 6d):

$$G^+(\mathbf{r}, \mathbf{r}_A, \omega) = G_d^+(\mathbf{r}, \mathbf{r}_A, \omega) + G_s^+(\mathbf{r}, \mathbf{r}_A, \omega). \qquad (15a)$$

Consequently,

$$[G^+(\mathbf{r}, \mathbf{r}_A, \omega)]^* = [G_d^+(\mathbf{r}, \mathbf{r}_A, \omega)]^* + [G_s^+(\mathbf{r}, \mathbf{r}_A, \omega)]^*. \qquad (15b)$$

Here $(G_d^+)^*$ propagates directly from $z = z_1$ and converges to A from below (Figure 6c). On the other hand, $(G_s^+)^*$ propagates from $z = z_1$, is scattered by the inhomogeneities above A, and converges to A from above (Figure 6e). With the subdivision made in equation (15b), we may rewrite equation (14) as

$$\Delta P(\mathbf{r}_A, \omega) = \Delta P_1(\mathbf{r}_A, \omega) + \Delta P_2(\mathbf{r}_A, \omega), \qquad (16a)$$

where

$$\Delta P_1(\mathbf{r}_A, \omega) \approx -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\left(\frac{\partial G_d^+}{\partial z}\right)^* P^+\right]_{z=z_1} dx\, dy \tag{16b}$$

and

$$\Delta P_2(\mathbf{r}_A, \omega) \approx -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\left(\frac{\partial G_s^+}{\partial z}\right)^* P^+\right]_{z=z_1} dx\, dy. \tag{16c}$$

Equation (16b) describes direct backpropagation of the *total* downgoing wave field P^+ at z_1 to A. Hence,

$$\Delta P_1(\mathbf{r}_A, \omega) \approx P^+(\mathbf{r}_A, \omega). \tag{17a}$$

Equation (16c) describes backpropagation of the downgoing wave field P^+ at z_1 via the scattering medium where the propagation direction changes, so ΔP_2 represents an upgoing wave field at A:

$$\Delta P_2(\mathbf{r}_A, \omega) = \Delta P^-(\mathbf{r}_A, \omega). \tag{17b}$$

Note that, according to equation (16c), $\Delta P^-(\mathbf{r}_A, \omega)$ is proportional to the product of the scattered wave P^+ (Figure 6a) and the scattered backpropagating Green's wave field $(G_s^+)^*$ (Figure 6e). Hence, the magnitude of $\Delta P^-(\mathbf{r}_A, \omega)$ is proportional to multiply reflected waves. Substituting equations (16a), (17a), and (17b) into equation (7a) yields

$$P(\mathbf{r}_A, \omega) \approx P_0(\mathbf{r}_A, \omega) + P^+(\mathbf{r}_A, \omega) + \Delta P^-(\mathbf{r}_A, \omega) \tag{18a}$$

or

$$P(\mathbf{r}_A, \omega) = P^+(\mathbf{r}_A, \omega) + P^-(r_A, \omega), \tag{18b}$$

where

$$P^-(\mathbf{r}_A, \omega) \approx P_0(\mathbf{r}_A, \omega) + \Delta P^-(\mathbf{r}_A, \omega), \tag{18c}$$

with $P_0(\mathbf{r}_A, \omega)$ defined by equation (7b). Hence, assuming $\Delta P^-(\mathbf{r}_A, \omega)$ may be neglected, we obtain

$$P^-(\mathbf{r}_A, \omega) \approx -\int_{S_0} \frac{1}{\rho}\left[\nabla G^* P - G^* \nabla P\right] \cdot \mathbf{n}\, dS_0. \tag{19}$$

This expression is used by many authors. It describes inverse wave-field extrapolation (toward the sources) from acquisition surface S_0 to subsurface point A (Figure 3c). Our analysis has shown that in equation (19) evanescent waves are neglected and that $\Delta P^-(\mathbf{r}_A, \omega)$, as defined by equations (16c) and (17b), is neglected. The magnitude of the latter term is proportional to multiply reflected waves. This not only means that multiply reflected waves are handled erroneously by equation (19) but also that the *primary* wave contribution to $P^-(\mathbf{r}_A, \omega)$ is not fully correct. Neglecting $\Delta P^-(\mathbf{r}_A, \omega)$ is justified only when the contrasts in the medium are moderate. In that case, equation (19) describes true amplitude inverse extrapolation of primary waves. This is illustrated below by an example.

When the contrasts in the medium are significant, $\Delta P^-(\mathbf{r}_A, \omega)$ may not be neglected and should be estimated in an iterative way. Further discussion is beyond the scope of this paper.

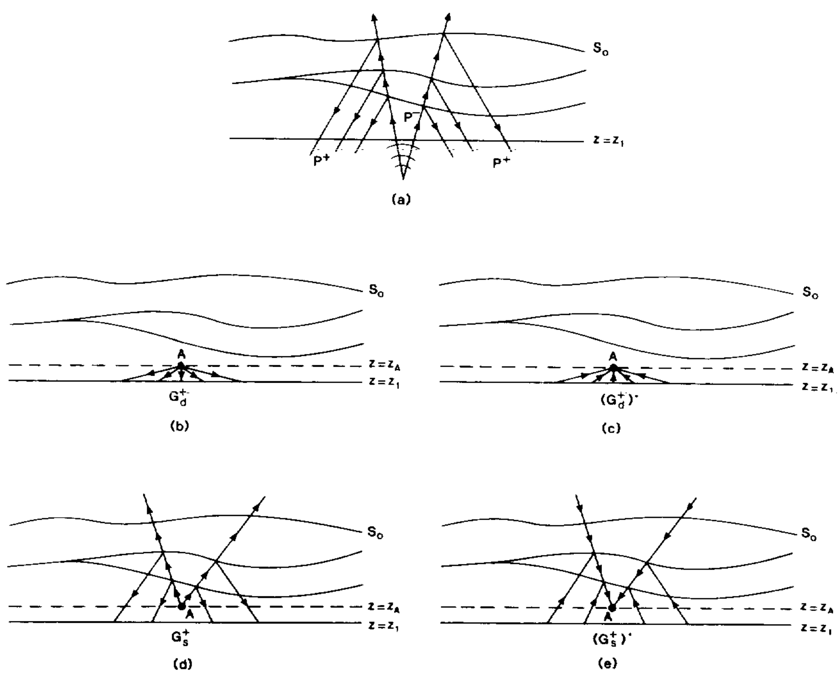

FIG. 6. (a) The wave field at z_1 consists of a direct upgoing wave field P^- and a scattered downgoing wave field P^+. (b) G_d^+ at z_1 represents a Green's wave field which propagates directly from A to z_1. (c) $(G_d^+)^*$ at z_1 represents a Green's wave field which propagates directly back from z_1 to A. (d) G_s^+ at z_1 represents a Green's wave field which is scattered during propagation from A to z_1. (e) $(G_s^+)^*$ at z_1 represents a Green's wave field which is scattered during backpropagation from z_1 to A.

Inhomogeneous medium; plane acquisition surface

When the acquisition surface S_0 is a plane surface at $z = z_0$, we may rewrite equation (19) as

$$P^-(\mathbf{r}_A, \omega) \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\left(\frac{\partial G}{\partial z}\right)^* P - G^* \frac{\partial P}{\partial z}\right]_{z=z_0} dx\, dy. \tag{20}$$

For the total wave field at $z = z_0$, we write

$$P(\mathbf{r}, \omega) = P^+(\mathbf{r}, \omega) + P^-(\mathbf{r}, \omega) \quad \text{at} \quad z = z_0. \tag{21a}$$

For the Green's wave field, we may choose above $z = z_0$ (outside V) any convenient reference medium. We choose $c(x, y, z < z_0) = c(x, y, z_0)$ and $\rho(x, y, z < z_0) = \rho(x, y, z_0)$. With this choice, the Green's wave field at z_0 is purely upgoing (no scattering occurs in the half-space above z_0); hence,

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = G^-(\mathbf{r}, \mathbf{r}_A, \omega) \quad \text{at} \quad z=z_0. \tag{21b}$$

We may now rewrite equation (20) as

$$P^-(\mathbf{r}_A, \omega) \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\left(\frac{\partial G^-}{\partial z}\right)^* (P^+ + P^-) - (G^-)^*\left(\frac{\partial P^+}{\partial z} + \frac{\partial P^-}{\partial z}\right)\right]_{z=z_0} dx\, dy. \tag{22}$$

Although it is not necessary (see also the Appendix of paper I), we assume for simplicity that $\nabla c = \nabla \rho = \mathbf{0}$ at $z = z_0$.

Following the derivation in the Appendix, setting $G^+ = 0$ and $\partial G^+/\partial z = 0$ and replacing z_1 by z_0 yields

$$P^-(\mathbf{r}_A, \omega) \approx 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{1}{\rho}\left(\frac{\partial G^-(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial z}\right)^* P^-(\mathbf{r}, \omega)\right]_{z_0} dx\, dy. \tag{23a}$$

Compare this result with equation (13b) in paper I:

$$P(\mathbf{r}_A, \omega) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{1}{\rho}\frac{\partial G^-(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial z} P^+(\mathbf{r}, \omega)\right]_{z_0} dx\, dy. \tag{23b}$$

Equation (23b) is the one-way version of the Rayleigh II integral for forward extrapolation (the sources are above z_0). Analogously, we refer to equation (23a) as the one-way version of the Rayleigh II integral for inverse extrapolation (the sources are below z_A). Equation (23b) yields the exact *total* wave field at A. Equation (23a), on the other hand, yields an approximate version of the *upgoing* wave field at A. The approximations involve neglecting evanescent waves and neglecting $\Delta P^-(\mathbf{r}_A, \omega)$, which is proportional to (but not restricted to) multiply reflected waves.

We demonstrate the validity of equation (23a) with the aid of a numerical 2-D example. Consider the inhomogeneous medium shown in Figure 7a. A line source is buried in the subsurface at a depth of $z = 600$ m. The response at $z = z_0$ of this source is shown in Figure 7b. This response was computed with a finite-difference modeling scheme. It represents the acoustic pressure p as a function of the lateral coordinate x and time t. Because the upper half-space $z < z_0$ is homogeneous and the acquisition surface z_0 is reflection-free, the recorded pressure represents an upgoing wave field; hence, $P = P^-(x, z_0, t)$. By applying a Fourier transform from time t to

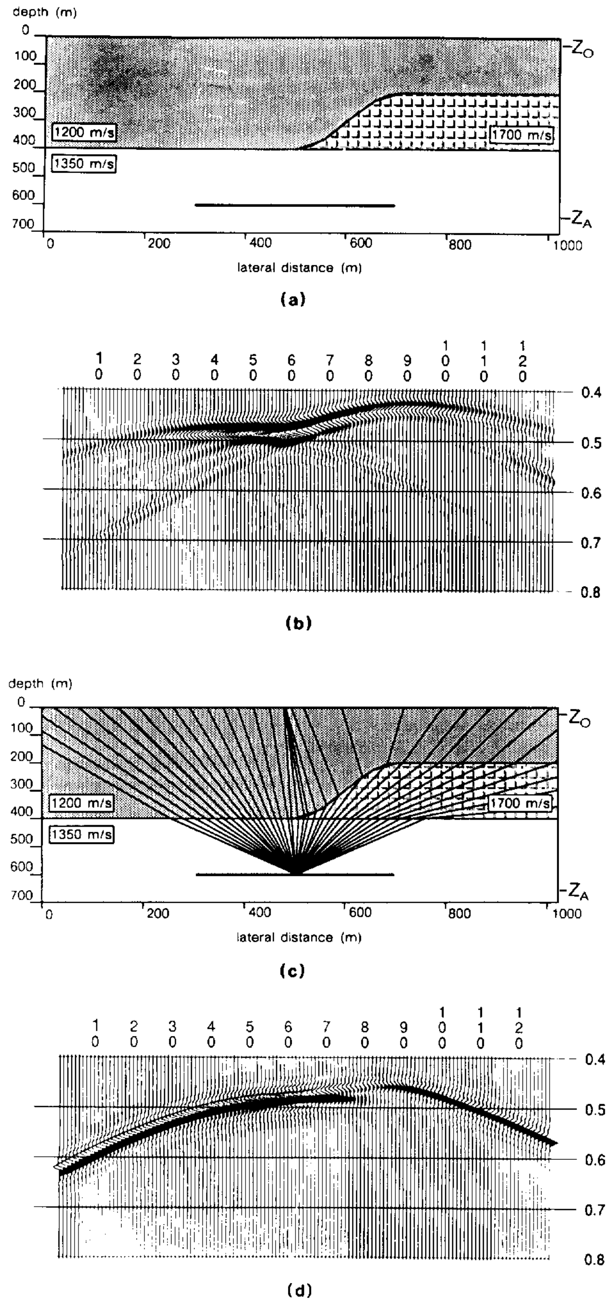

FIG. 7. (a) Inhomogeneous medium with a buried line source at $z = 600$ m. (b) Upgoing wave field registered at z_0. (c) Fan of rays used for Gaussian beam modeling of the Green's wave field. (d) Time-domain representation of a (band-limited) Green's wave field.

frequency ω, the data are decomposed into monochromatic wave fields $P^-(x, z_0, \omega)$. Inverse extrapolation of these data to depth level z_A is described by equation (23a); hence, we first need to compute the Green's wave fields at z_0 for many Green's source points A at $z=z_A$. This is actually a *forward modeling* problem. By solving the 2-D version of equation (2) numerically, one obtains the monochromatic monopole response $G(x, z; x_A, z_A; \omega)$. Of course, any accurate modeling algorithm can be used for this purpose. Here we use the Gaussian beam method. This method involves shooting a fan of rays (Figure 7c) and solving a high-frequency approximation to wave equation (2) around each ray. The resulting beams exhibit a Gaussian amplitude distribution around the rays. The response at any point (x, z) is now found by superimposing the contributions of the individual beams at that point. Figure 7d shows the time-domain representation of the (band-limited) Green's wave field $g^-(x, z_0; x_A, z_A; t)$. For more information on the Gaussian beam method, the reader is referred to Červeny et al. (1982).

Applying equation (23a) for all A at $z = z_A$ and for all frequencies within the seismic band yields, after applying an inverse Fourier transform, the space-time data $P^-(x, z_A, t)$ (see also Figure 8a).

Note that the distorting propagation effects of the overburden have been removed (compare with Figure 7b). Figure 8b shows the maximum amplitude of each trace as a function of lateral position x. Note the almost constant amplitude along the line source. Figure 8c shows the central trace of the inversely extrapolated data in Figure 8a. For comparison, in Figure 8d the wavelet is shown that was used for modeling the input data (Figure 7b). Apparently the inverse extrapolation restored the wavelet almost perfectly. For applications of true amplitude inverse wave-field extrapolation in 2-D and 3-D prestack redatuming, the reader is referred to Peels (1988) and Kinneging et al. (1989).

DISCUSSION

Inverse one-way wave-field extrapolation, as described by Rayleigh II integral (23a), is not directly applicable to seismic data where the particle velocity rather than the upgoing pressure wave field is measured. Berkhout and Wapenaar (1988) have proposed the following processing sequence:

(1) Decomposition of the seismic response into downgoing and upgoing waves.
(2) Elimination of surface related multiple reflections.
(3) Estimation of the macro subsurface model.
(4) Prestack migration, yielding angle-dependent reflectivity.
(5) Elastic inversion for velocity and density.
(6) Lithologic inversion for rock and pore parameters.

After the first two steps, the data may be interpreted as if upgoing waves (related to downgoing source waves) were measured at a reflection-free acquisition surface. Hence, the one-way Rayleigh II integrals (23a) and (23b) may be applied to these data in step (4) (prestack migration). The above processing sequence is also applicable to multicomponent seismic data. In the latter case, the acoustic one-way Rayleigh II integrals should be replaced by elastic one-way Rayleigh II integrals for P and S waves (Wapenaar and Haime; 1989).

CONCLUSIONS

(1) Two equivalent versions of the Kirchhoff integral have been reviewed for inhomogeneous fluids:

$$P(\mathbf{r}_A, \omega) = -\oint_S \frac{1}{\rho}\left[\nabla G P - G \nabla P\right] \cdot \mathbf{n}\, dS \qquad (24a)$$

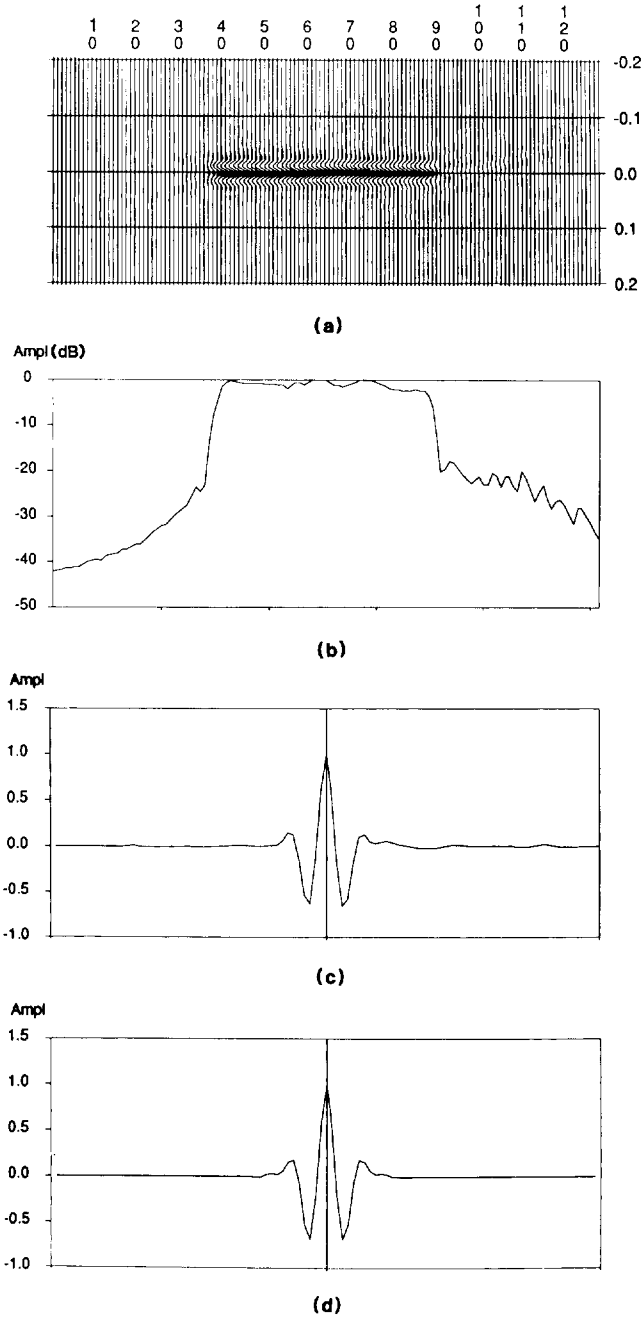

FIG. 8. (a) Inverse extrapolated data at z_A = 600 m. (b) Maximum amplitude per trace of (a) (logarithmic scale). (c) Central trace of (a). (d) Wavelet used for modeling the input data of Figure 7b.

and

$$P(\mathbf{r}_A, \omega) = -\oint_S \frac{1}{\rho}\left[\nabla G^* P - G^* \nabla P\right] \cdot \mathbf{n}\, dS. \quad (24b)$$

In equation (24a) $G = G(\mathbf{r}, \mathbf{r}_A, \omega)$ represents a *forward*-propagating Green's wave field. In equation (24b) $G^* = G^*(\mathbf{r}, \mathbf{r}_A, \omega)$ represents a *backward*-propagating Green's wave field.

(2) The Kirchhoff integral (24b) with the backward-propagating Green's wave field has been adapted to seismic data, yielding

$$P^-(\mathbf{r}_A, \omega) \approx -\int_{S_0} \frac{1}{\rho}\left[\nabla G^* P - G^* \nabla P\right] \cdot \mathbf{n}\, dS_0. \quad (25)$$

Here S_0 represents the acquisition surface and P and ∇P represent the wave field at S_0 related to (secondary) sources below S_0. Equation (25) describes inverse wave-field extrapolation (toward the sources) from S_0 to subsurface point A. Only the upgoing wave field is reconstructed. The solution is spatially band-limited (evanescent waves are neglected); hence, the maximum obtainable spatial resolution is limited (see Figure 5d). Amplitude errors in $P^-(\mathbf{r}_A, \omega)$ are of the same order as result from neglecting multiply reflected waves (second order). Assuming moderate contrasts, Kirchhoff integral (25) describes true amplitude inverse extrapolation of primary waves. When the contrasts are significant, an iterative procedure must be applied.

(3) For a planar acquisition surface, the Kirchhoff integral (25) can be transformed into a one-way Rayleigh-type integral for inverse wave-field extrapolation, yielding

$$P^-(\mathbf{r}_A, \omega) \approx 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\left(\frac{\partial G^-}{\partial z}\right)^* P^-\right]_{z_0} dx\, dy. \quad (26)$$

Assuming moderate contrasts, Rayleigh integral (26) describes true amplitude inverse extrapolation of primary waves (Figure 8).

(4) The one-way Rayleigh integral (26) fits very well into the seismic processing scheme as proposed by Berkhout and Wapenaar (1988).

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APPENDIX

ONE-WAY VERSIONS OF THE KIRCHHOFF INTEGRAL WITH BACKWARD PROPAGATING GREEN'S WAVE FIELDS

Consider the following expression:

$$\Omega = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\frac{\partial G^+}{\partial z} + \frac{\partial G^-}{\partial z}\right)^* (P^+ + P^-) - (G^+ + G^-)^* \left(\frac{\partial P^+}{\partial z} + \frac{\partial P^-}{\partial z}\right)\right]_{z=z_1} dx\, dy, \quad (A\text{-}1)$$

where

$$P^{\pm} = P^{\pm}(x, y, z; \omega) \quad (A\text{-}2a)$$

and

$$G^{\pm} = G^{\pm}(x, y, z; x_A, y_A, z_A; \omega). \quad (A\text{-}2b)$$

Define the 2-D spatial Fourier transforms of $P^{\pm}$ and $G^{\pm}$ by

$$\tilde{P}^{\pm}(k_x, k_y, z; \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^{\pm}(x, y, z; \omega) e^{i(k_x x + k_y y)}\, dx\, dy \quad (A\text{-}3a)$$

and

$$\tilde{G}^{\pm}(k_x, k_y, z; x_A, y_A, z_A; \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G^{\pm}(x, y, z; x_A, y_A, z_A; \omega) e^{i(k_x x + k_y y)}\, dx\, dy. \tag{A-3b}$$

According to the 2-D version of Parseval's theorem (Dudgeon and Mersereau, 1984), we may replace equation (A-1) by

$$\Omega = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left(\frac{\partial \tilde{G}^+}{\partial z} + \frac{\partial \tilde{G}^-}{\partial z}\right)^* (\tilde{P}^+ + \tilde{P}^-) - (\tilde{G}^+ + \tilde{G}^-)^* \left(\frac{\partial \tilde{P}^+}{\partial z} + \frac{\partial \tilde{P}^-}{\partial z}\right)\right]_{z=z_1} dk_x\, dk_y. \tag{A-4}$$

Let us now assume that P^+ and P^- represent downgoing and upgoing acoustic wave fields at $z = z_1$ and that G^+ and G^- represent downgoing and upgoing Green's wave fields at $z = z_1$. If we assume in addition that the acoustic medium parameters c and ρ satisfy

$$\nabla c = \nabla \rho = \mathbf{0} \quad \text{at} \quad z = z_1, \tag{A-5}$$

then $\tilde{P}^{\pm}$ and $\tilde{G}^{\pm}$ satisfy the following one-way equations:

$$\frac{\partial \tilde{P}^{\pm}}{\partial z} = \mp i k_z \tilde{P}^{\pm} \quad \text{at} \quad z = z_1, \tag{A-6a}$$

$$\frac{\partial \tilde{G}^{\pm}}{\partial z} = \mp i k_z \tilde{G}^{\pm} \quad \text{at} \quad z = z_1; \tag{A-6b}$$

and, consequently,

$$\left(\frac{\partial \tilde{G}^{\pm}}{\partial z}\right)^* = \pm i k_z^* (\tilde{G}^{\pm})^* \quad \text{at} \quad z = z_1. \tag{A-6c}$$

Here

$$k_z = \sqrt{k_1^2 - k_x^2 - k_y^2} \quad \text{for} \quad k_x^2 + k_y^2 \le k_1^2 \tag{A-7a}$$

and

$$k_z = -i\sqrt{k_x^2 + k_y^2 - k_1^2} \quad \text{for} \quad k_x^2 + k_y^2 > k_1^2, \tag{A-7b}$$

where

$$k_1 = \omega / c(z_1). \tag{A-7c}$$

Note that

$$k_z = k_z^* \quad \text{for} \quad k_x^2 + k_y^2 \le k_1^2 \tag{A-8a}$$

and

$$k_z = -k_z^* \quad \text{for} \quad k_x^2 + k_y^2 > k_1^2. \tag{A-8b}$$

Now for propagating waves, ($k_x^2 + k_y^2 \le k_1^2$), equations (A-6), (A-7), and (A-8) imply

$$\left[\left(\frac{\partial \tilde{G}^+}{\partial z} + \frac{\partial \tilde{G}^-}{\partial z}\right)^* (\tilde{P}^+ + \tilde{P}^-) - (\tilde{G}^+ + \tilde{G}^-)^* \left(\frac{\partial \tilde{P}^+}{\partial z} + \frac{\partial \tilde{P}^-}{\partial z}\right)\right]_{z=z_1} = 2\left[\left(\frac{\partial \tilde{G}^+}{\partial z}\right)^* \tilde{P}^+ + \left(\frac{\partial \tilde{G}^-}{\partial z}\right)^* \tilde{P}^-\right]_{z=z_1}. \tag{A-9a}$$

Similarly, for evanescent waves ($k_x^2 + k_y^2 > k_1^2$), equations (A-6), (A-7), and (A-8) imply

$$\left[\left(\frac{\partial \tilde{G}^+}{\partial z} + \frac{\partial \tilde{G}^-}{\partial z}\right)^* (\tilde{P}^+ + \tilde{P}^-) - (\tilde{G}^+ + \tilde{G}^-)^* \left(\frac{\partial \tilde{P}^+}{\partial z} + \frac{\partial \tilde{P}^-}{\partial z}\right)\right]_{z=z_1} = 2\left[\left(\frac{\partial \tilde{G}^-}{\partial z}\right)^* \tilde{P}^- + \left(\frac{\partial \tilde{G}^-}{\partial z}\right)^* \tilde{P}^+\right]_{z=z_1}. \tag{A-9b}$$

Substituting these results into equation (A-4) yields

$$\Omega = 2\left(\frac{1}{2\pi}\right)^2 \iint_{\text{propagating waves}} \left[\left(\frac{\partial \tilde{G}^+}{\partial z}\right)^* \tilde{P}^+ + \left(\frac{\partial \tilde{G}^-}{\partial z}\right)^* \tilde{P}^-\right]_{z=z_1} dk_x\, dk_y + 2\left(\frac{1}{2\pi}\right)^2 \iint_{\text{evanescent waves}} \left[\left(\frac{\partial \tilde{G}^+}{\partial z}\right)^* \tilde{P}^- + \left(\frac{\partial \tilde{G}^-}{\partial z}\right)^* \tilde{P}^+\right]_{z=z_1} dk_x\, dk_y. \tag{A-10}$$

The second integral over the evanescent wavenumber area ($k_x^2 + k_y^2 > k_1^2$) is negligible when the source of $\tilde{P}^{\pm}$ and the source of the Green's wave field $\tilde{G}^{\pm}$ (point A) are not both in the direct vicinity of z_1. Hence,

$$\Omega \approx 2\left(\frac{1}{2\pi}\right)^2 \iint_{k_x^2 + k_y^2 \le k_1^2} \left[\left(\frac{\partial \tilde{G}^+}{\partial z}\right)^* \tilde{P}^+ + \left(\frac{\partial \tilde{G}^-}{\partial z}\right)^* \tilde{P}^-\right]_{z=z_1} dk_x\, dk_y, \tag{A-11a}$$

or by adding a negligible integral over the evanescent wavenumber area,

$$\Omega \approx 2\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\frac{\partial \tilde{G}^-}{\partial z}\right)^* \tilde{P}^+ + \left(\frac{\partial \tilde{G}^-}{\partial z}\right)^* \tilde{P}^-\right]_{z=z_1} dk_x\, dk_y, \qquad \text{(A-11b)}$$

or, according to the 2-D version of Parseval's theorem,

$$\Omega \approx 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\frac{\partial G^+}{\partial z}\right)^* P^+ + \left(\frac{\partial G^-}{\partial z}\right)^* P^-\right]_{z=z} dx\, dy. \qquad \text{(A-12)}$$

This expression shows that interaction occurs only for acoustic wave fields and (back-propagating) Green's wave fields which propagate in *opposite* directions through $z = z_1$.