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One-way versions of the Kirchhoff integral

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ABSTRACT

The conventional Kirchhoff integral, based on the *two-way* wave equation, states how the acoustic pressure at a point A inside a closed surface S can be calculated when the acoustic wave field is known on S. In its general form, the integrand consists of two terms: one term contains the gradient of a Green's function and the acoustic pressure; the other term contains a Green's function and the gradient of the acoustic pressure.

The integrand can be simplified by choosing reflecting boundary conditions for the two-way Green's functions in such a way that either the first term or the second term vanishes on S. This conventional approach to deriving Rayleigh-type integrals has practical value only for media with small contrasts, so that the two-way Green's functions do not contain significant multiple reflections. We present a modified approach for simplifying the integrand of the Kirchhoff integral by choosing absorbing boundary conditions for the *one-way* Green's functions. The resulting Rayleigh-type integrals are the theoretical basis for true amplitude one-way wave-field extrapolation techniques in inhomogeneous media with significant contrasts.

INTRODUCTION

An essential step in seismic modeling and inversion is the simulation or elimination of propagation effects by means of wave-field extrapolation (forward or backward propagation). Wave-field extrapolation techniques can be subdivided into *one-way* and *two-way* approaches. The one-way approach is robust, but handles primary waves only. The two-way approach handles both multiples and primaries but requires an accurate description of the source wave field and the geology of the subsurface. Although a significant amount of research is being carried out on exact inversion techniques based on two-way wave-field extrapolation, we expect that, at least in the coming decade, the seismic industry will depend mainly on partial but robust inversion techniques based on one-way wave-field extrapolation. Therefore, we can justify giving a high priority to the development of accurate extrapolation algorithms for primary waves only, particularly for *elastic* wave-field extrapolation. Our philosophy on this important issue can be summarized as follows: In practical situations significant multiples are always *surface-related*, so the first step should be elimination of all surface-related multiples (Berkhout, 1986). Note that for this preprocessing step no information on the subsurface is needed. Next, one-way wave-field extrapolation can start as part of the inversion process. Now information about the subsurface is required in terms of a macro model.

In this paper we analyze Rayleigh-type integrals. In the frequency (ω) domain, the conventional Rayleigh integral represents the pressure at a point A of the subsurface ($z > 0$) for a given pressure distribution at the surface ($z = 0$):

$$P_A = \iint_{-\infty}^{\infty} \frac{1}{\rho} \left[\frac{\partial G}{\partial z} P \right]_{z=0} dx\, dy,$$

where G is the *two-way* Green's wave field generated by two identical monopoles (with opposite signatures) at (x_A, y_A, z_A) and $(x_A, y_A, -z_A)$ in a reference medium that is symmetric with respect to $z = 0$.

$$\rho(x, y, z) = \rho(x, y, -z)$$

and

$$c(x, y, z) = c(x, y, -z).$$

(ρ and c represent the density and the propagation velocity, respectively.) Hence, for a given lower half-space (the subsurface), the upper half-space is defined as the mirror image of the lower half-space. Note that for an inhomogeneous subsurface, G may become very complicated, since all multiple reflections between the lower and upper half-spaces must be included (Figure 1). Note also that G is, in principle, two-way.

For one-way techniques, it is common practice to substitute in the conventional Rayleigh integral a one-way Green's function. In this paper, it is shown that this substitution is fundamentally wrong because the conventional Rayleigh integral

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could be derived from the Kirchhoff integral only by making use of the two-way properties of the Green's function. We give a correct derivation that starts with the Kirchhoff integral. The result, called the "one-way version of the Rayleigh integral," is the theoretical basis for all one-way techniques in inhomogeneous media.

THE KIRCHHOFF INTEGRAL FOR INHOMOGENEOUS FLUIDS

Consider an inhomogeneous fluid, described by the space-dependent propagation velocity $c(\mathbf{r})$ and the mass density $\rho(\mathbf{r})$, where $\mathbf{r}$ is a shorthand notation for the Cartesian coordinates (x, y, z). In this fluid we consider a volume V enclosed by a surface S with an outward pointing normal vector $\mathbf{n}$ (see Figure 2). Given a monochromatic acoustic wave field $P(\mathbf{r}, \omega)$ radiated by sources outside S, the wave field $P(\mathbf{r}_A, \omega)$ at any point A inside S can be computed from $P(\mathbf{r}, \omega)$ and its normal derivative $\partial P(\mathbf{r}, \omega)/\partial n$ on S with the aid of the Kirchhoff integral:

$$P(\mathbf{r}_A, \omega) = -\oint_S \frac{1}{\rho(\mathbf{r})}\left[\frac{\partial G(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial n} P(\mathbf{r}, \omega) - G(\mathbf{r}, \mathbf{r}_A, \omega)\frac{\partial P(\mathbf{r}, \omega)}{\partial n}\right] dS \quad (1)$$

(Morse and Feshbach, 1953; Burridge and Knopoff, 1964; Aki and Richards, 1980; Berkhout, 1985). The acoustic wave field $P(\mathbf{r}, \omega)$ satisfies in V the two-way wave equation

$$\rho\nabla\cdot\left(\frac{1}{\rho}\nabla P\right) + k^2 P = 0, \quad (2a)$$

where the wavenumber $k(\mathbf{r}, \omega)$ is defined as

$$k(\mathbf{r}, \omega) = \omega/c(\mathbf{r}). \quad (2b)$$

Similarly, the Green's function $G(\mathbf{r}, \mathbf{r}_A, \omega)$ satisfies in V the two-way wave equation with a point source at A

$$\rho\nabla\cdot\left(\frac{1}{\rho}\nabla G\right) + k^2 G = -\rho\delta(\mathbf{r} - \mathbf{r}_A). \quad (3)$$

Hence, the Green's function can be interpreted as the spatial impulse response of the medium in V. The actual solution of equation (3) depends on the choice of the boundary conditions for G on S. If we choose

$$\frac{\partial G(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial n} = 0 \quad \text{on } S \quad (4a)$$

(Neumann boundary condition), the Kirchhoff integral (1) simplifies to

$$P(\mathbf{r}_A, \omega) = \oint_S \left[\frac{1}{\rho(\mathbf{r})} G_I(\mathbf{r}, \mathbf{r}_A, \omega)\frac{\partial P(\mathbf{r}, \omega)}{\partial n}\right] dS. \quad (4b)$$

However, the price we must pay is that the Green's function G_I may become very complicated because equation (4a) means that S is acoustically hard; i.e., perfectly reflecting ($R = +1$). Similarly, if we choose

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = 0 \quad \text{on } S \quad (5a)$$

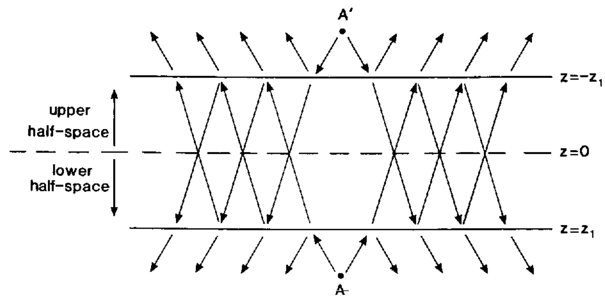

FIG. 1. A two-way Green's function for an inhomogeneous medium contains all multiple reflections between the symmetric lower and upper half-spaces. The case of one horizontal reflector at $z = z_1$ is shown.

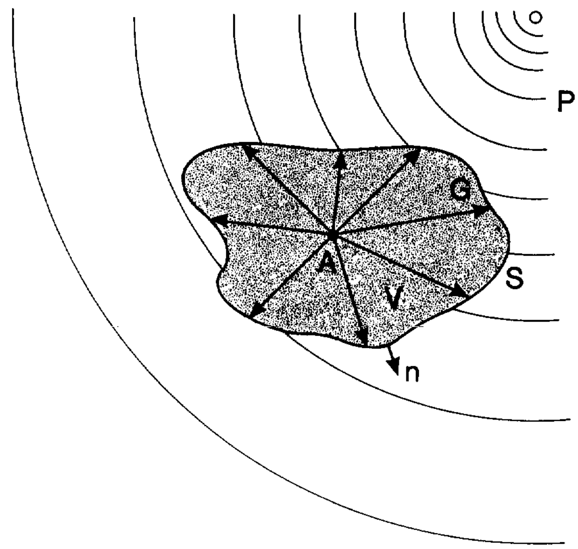

FIG. 2. Assuming sources outside S, the Kirchhoff integral (1) states that the acoustic pressure at any point A inside S can be calculated using the spatial impulse response G, when the acoustic wave field and its normal derivative are known on S.

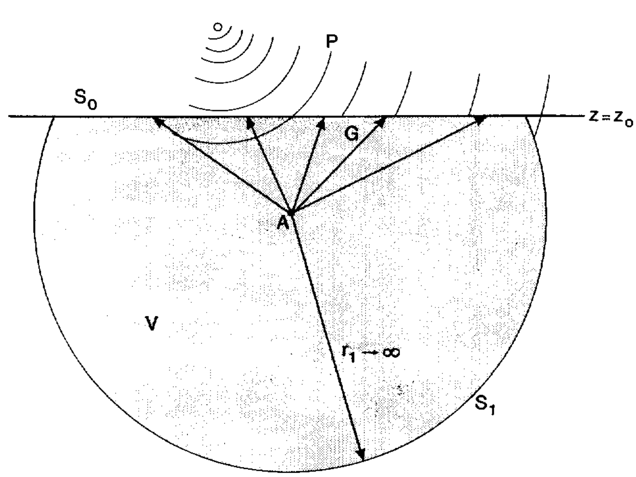

FIG. 3. Configuration for which the closed-surface integral (1) may be replaced by the open-surface integral (6). The lower half-space is assumed to be source-free. This configuration is the basis for the derivation of several types of Rayleigh integrals.

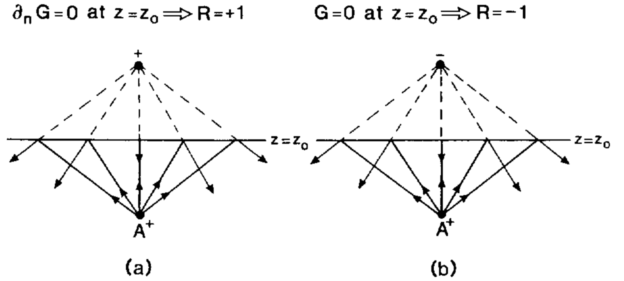

FIG. 4. (a) Neumann boundary condition: surface $z = z_0$ is a perfect reflector with $R = +1$. (b) Dirichlet boundary condition: surface $z = z_0$ is a perfect reflector with $R = -1$.

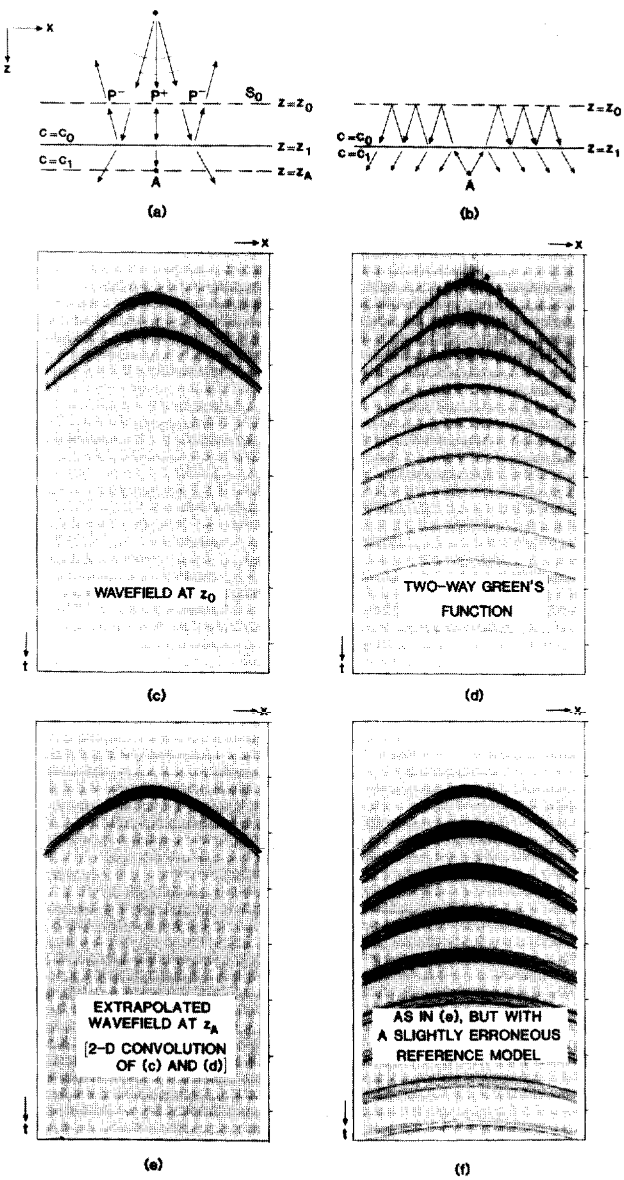

FIG. 5. Application of the two-way Rayleigh II integral: (a) Raypaths in the actual medium for the wave field $P(\mathbf{r}, \omega)$. (b) Raypaths in the reference medium for the two-way Green's function.

(Dirichlet boundary condition), the Kirchhoff integral (1) simplifies to

$$P(\mathbf{r}_A, \omega) = -\oint_S \left[\frac{1}{\rho(\mathbf{r})} \frac{\partial G_{II}(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial n} P(\mathbf{r}, \omega) \right] dS. \quad (5b)$$

In a manner similar to G_I, $\partial G_{II}/\partial n$ may become very complicated because equation (5a) means that S is acoustically soft; i.e., perfectly reflecting ($R = -1$).

The Kirchhoff integral (1), as well as its simplified versions equations (4b) and (5b), are, in principle, two-way expressions. In the next section we consider equations (4b) and (5b) for a special configuration and illustrate the two-way properties with an example. Next we divert from what is generally done in the literature and, instead of fully reflecting boundary conditions, we choose fully absorbing boundary conditions for G on S. Following this alternative route we show how to derive elegant one-way expressions.

CONVENTIONAL DERIVATION OF RAYLEIGH-TYPE INTEGRALS (TWO-WAY APPROACH)

Consider the half-space geometry of Figure 3. Closed surface S consists of a horizontal flat surface S_0 at $z = z_0$ and a hemisphere S_1 with midpoint A and radius r_1 in the lower half-space $z \geq z_0$. Assuming that the sources of the wave field P are situated in the upper half-space $z < z_0$, the contribution of the Kirchhoff integral over S_1 to the pressure in A vanishes if r_1 goes to infinity (Sommerfeld radiation condition, Bleistein, 1984). For this situation, equation (1) may be replaced by

$$P(\mathbf{r}_A, \omega) = \int_{-\infty}^{\infty}\int \frac{1}{\rho(\mathbf{r})} \left[\frac{\partial G(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial z} P(\mathbf{r}, \omega) - G(\mathbf{r}, \mathbf{r}_A, \omega) \frac{\partial P(\mathbf{r}, \omega)}{\partial z} \right]_{z_0} dx\, dy. \quad (6)$$

Consider again the Neumann and Dirichlet boundary conditions for G,

$$\frac{\partial G(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial z} = 0 \quad \text{at } z = z_0 \quad (7a)$$

or

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = 0 \quad \text{at } z = z_0, \quad (7b)$$

which are satisfied if we assume for G either a rigid surface or a free surface at z_0. In both cases, the surface acts as a perfect reflector, so we may alternatively interpret the Green's function as if it were caused by two monopoles situated symmetrically with respect to $z = z_0$ in a reference medium which is also symmetric with respect to $z = z_0$ (classical representation). If these monopoles have identical source functions (Figure 4a), Neumann's condition (7a) is satisfied, and the Kirchhoff integral (6) may be replaced by the Rayleigh I integral

$$P(\mathbf{r}_A, \omega) = -\int_{-\infty}^{\infty}\int \left[\frac{1}{\rho(\mathbf{r})} G_I(\mathbf{r}, \mathbf{r}_A, \omega) \frac{\partial P(\mathbf{r}, \omega)}{\partial z} \right]_{z_0} dx\, dy. \quad (8a)$$

On the other hand, if the two monopoles have opposite source functions (Figure 4b), Dirichlet's condition (7b) is satisfied and the Kirchhoff integral (6) may be replaced by the Rayleigh II integral

$$P(\mathbf{r}_A, \omega) = \iint_{-\infty}^{\infty} \left[\frac{1}{\rho(\mathbf{r})} \frac{\partial G_{\mathrm{II}}(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial z} P(\mathbf{r}, \omega) \right]_{z_0} dx\, dy. \quad (8b)$$

Note that expressions (8a) and (8b) are exact, but for practical use, are inconvenient because in the general inhomogeneous case the two-way Green's functions G_{I} or $\partial G_{\mathrm{II}}/\partial z$ may contain strong surface-related multiple reflections. This means that these Green's functions must be designed very accurately, which is best illustrated with a simple example. Referring to Figure 5a, we consider a 2-D medium which consists of two homogeneous half-spaces $z \geq z_1$ (with propagation velocity c_1) and $z < z_1$ (with propagation velocity c_0). The surface S_0 lies in the homogeneous upper half-space at $z = z_0$, with $z_0 < z_1$. The wave field $P(\mathbf{r}, \omega)$ propagating in this medium is radiated by a source above z_0. Figure 5a shows the raypaths for $P(\mathbf{r}, \omega)$. In Figure 5b the raypaths for the two-way Green's function $\partial G_{\mathrm{II}}(\mathbf{r}, \mathbf{r}_A, \omega)/\partial z$ are shown, including the surface-related multiples. Figure 5c is a space-time domain representation of the 2-D wave field $p(\mathbf{r}, t)$ at z_0. Note that the direct and the scattered wave fields can be clearly distinguished.

Figure 5d is a space-time domain representation of the 2-D two-way Green's function $\partial g_{\mathrm{II}}(\mathbf{r}, \mathbf{r}_A, t)/\partial z$, at z_0. Note that the multiples between the reflector at $z = z_1$ and the surface are clearly visible. Figure 5e is a space-time domain representation of the 2-D wave field $p(\mathbf{r}_A, t)$, obtained by applying the 2-D version of two-way Rayleigh II integral (8b) for all x_A and taking the inverse Fourier transform. Note that this exact result represents the transmitted downgoing wave at $z = z_A$ (see also Figure 5a).

In practice we do not often have available an exact description of the medium response $\partial g_{\mathrm{II}}(\mathbf{r}, \mathbf{r}_A, t)/\partial z$. Figure 5f is a space-time domain representation of the 2-D wave field $p(\mathbf{r}_A, t)$ which was obtained by applying the Rayleigh II integral with a Green's function that was only slightly in error (the reflector depth $z = z_1$ in Figure 5b was one-half of a wavelength in error for the central frequency). Note that this result contains many spurious reflections. We may conclude that two-way Rayleigh integrals (8a) and (8b) in general require very accurate generation of the multiple reflections. This accuracy can be realized only if the actual medium where the extrapolation occurs is accurately known. In practice, this knowledge is generally not available. Therefore the two-way Rayleigh integrals (8a) and (8b) have practical use only in the situation where the reference model below the surface does not contain reflectors but contains only smooth transition zones. In that case, the two-way Green's functions do not include significant multiple reflections.

MODIFIED DERIVATION OF RAYLEIGH-TYPE INTEGRALS (ONE-WAY APPROACH)

An important property of the Kirchhoff integral is that the choice of medium for the Green's function $G(\mathbf{r}, \mathbf{r}_A, \omega)$ is not unique.

Inside volume V the medium for the Green's function is the medium for the acoustic wave field $P(\mathbf{r}, \omega)$.

Outside volume V the medium for the Green's function may be chosen in any convenient way.

We made use of this property in the previous section, where we chose a reference medium for G that was different from the actual medium for P (compare Figure 5b with Figure 5a). By choosing a reference medium with a fully reflecting boundary, we got a Green's function containing many significant multiple reflections. Therefore, let us now choose a reference medium for G which is fully nonreflecting outside V. For the half-space geometry of Figure 3, this means that we choose a nonreflecting upper half-space $z < z_0$. Hence,

$$\rho(x, y, z < z_0) = \rho(x, y, z_0) \qquad \text{for all } z < z_0 \quad (9a)$$

and

$$c(x, y, z < z_0) = c(x, y, z_0) \qquad \text{for all } z < z_0. \quad (9b)$$

With this choice, no energy returns from the upper half-space, so G is purely upgoing at $z = z_0$:

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = G^-(\mathbf{r}, \mathbf{r}_A, \omega) \qquad \text{at } z = z_0. \quad (10a)$$

In terms of boundary conditions, we may say that surface $z = z_0$ is an absorbing boundary for G.

The sources for the acoustic wave field $P(\mathbf{r}, \omega)$ are situated in the upper half-space, so at z_0 the wave field consists of the downgoing incident wave field (including higher order terms) $P^+(\mathbf{r}, \omega)$ and the upgoing scattered wave field (including higher order terms) $P^-(\mathbf{r}, \omega)$ according to

$$P(\mathbf{r}, \omega) = P^+(\mathbf{r}, \omega) + P^-(\mathbf{r}, \omega) \qquad \text{at } z = z_0. \quad (10b)$$

Substitution of equations (10a) and (10b) into the Kirchhoff integral (6) yields (in shorthand notation):

$$P(\mathbf{r}_A, \omega) = \iint_{-\infty}^{\infty} \frac{1}{\rho} \left[\frac{\partial G^-}{\partial z} (P^+ + P^-) - G^- \left(\frac{\partial P^+}{\partial z} + \frac{\partial P^-}{\partial z} \right) \right]_{z_0} dx\, dy \quad (11a)$$

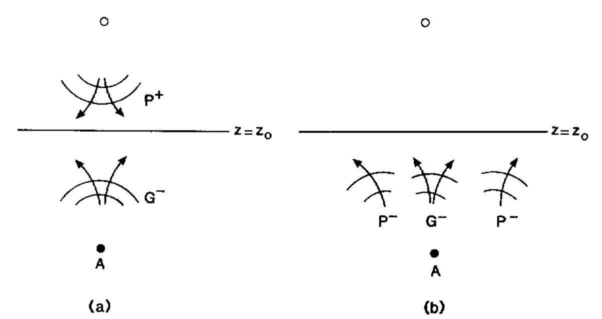

FIG. 6. Choosing a reflection-free upper half-space for G, the Kirchhoff integral (11) consists of (a) a term containing P^+ and G^- on $z = z_0$ and (b) a term containing P^- and G^- on $z = z_0$. Only the term with *opposite* propagating wave fields on $z = z_0$ (a) contributes to the result $P(\mathbf{r}_A, \omega)$.

or

$$P(\mathbf{r}_A, \omega) = \int\!\!\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\frac{\partial G^-}{\partial z} P^+ - G^- \frac{\partial P^+}{\partial z}\right]_{z0} dx\, dy$$

$$+ \int\!\!\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\frac{\partial G^-}{\partial z} P^- - G^- \frac{\partial P^-}{\partial z}\right]_{z0} dx\, dy. \quad (11b)$$

In the Appendix, we show that the only contribution to $P(\mathbf{r}_A, \omega)$ comes from the first integral in the right-hand side of equation (11b). In other words, only the wave fields P^+ and G^- propagating in opposite directions through z_0 contribute to the wave field $P(\mathbf{r}_A, \omega)$; see also Figure 6. Hence,

$$P(\mathbf{r}_A, \omega) = \int\!\!\int_{-\infty}^{\infty} \frac{1}{\rho}\left[\frac{\partial G^-}{\partial z} P^+ - G^- \frac{\partial P^+}{\partial z}\right]_{z0} dx\, dy \quad (12a)$$

or

$$P(\mathbf{r}_A, \omega) = \int\!\!\int_{-\infty}^{\infty} \left[\frac{1}{\rho}\frac{\partial G^-}{\partial z} P^+\right]_{z0} dx\, dy$$

$$+ \int\!\!\int_{-\infty}^{\infty} \left[\frac{-1}{\rho} G^- \frac{\partial P^+}{\partial z}\right]_{z0} dx\, dy. \quad (12b)$$

We also show in the Appendix that the two integrals in the right-hand side of equation (12b) are identical. Hence, equation (12b) may be rewritten as

$$P(\mathbf{r}_A, \omega) = -2\int\!\!\int_{-\infty}^{\infty} \left[\frac{1}{\rho(\mathbf{r})} G^-(\mathbf{r}, \mathbf{r}_A, \omega) \frac{\partial P^+(\mathbf{r}, \omega)}{\partial z}\right]_{z0} dx\, dy \quad (13a)$$

or, equivalently,

$$P(\mathbf{r}_A, \omega) = 2\int\!\!\int_{-\infty}^{\infty} \left[\frac{1}{\rho(\mathbf{r})} \frac{\partial G^-(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial z} P^+(\mathbf{r}, \omega)\right]_{z0} dx\, dy. \quad (13b)$$

By analogy with equations (8a) and (8b), we call equations (13a) and (13b) the one-way versions of the Rayleigh I and Rayleigh II integrals, respectively. For the special case of a homogeneous medium, G_I at z_0 in equation (8a) equals $2G^-$ at z_0 in equation (13a); similarly, $\partial G_{II}/\partial z$ at z_0 in equation (8b) equals $2\partial G^-/\partial z$ at z_0 in equation (13b). Hence, for the special case of a homogeneous medium, the two-way Rayleigh integrals (8a) and (8b) are identical to the one-way Rayleigh integrals (13a) and (13b), respectively.

For arbitrarily inhomogeneous media, however, the one-way Rayleigh integrals (13a) and (13b) are very different from the two-way Rayleigh integrals (8a) and (8b). Because the one-way Green's functions do not contain surface-related multiples, one-way Rayleigh integrals (13a) and (13b) are rather insensitive to small errors in the reference model. This is illustrated in Figure 7, where the experiment of Figure 5 was repeated for the downgoing wave field P^+, using the one-way Rayleigh II integral for downgoing wave fields, as given by equation (13b). Note that a small error in the reference medium has a minor effect on the extrapolation result (Figure 7f). This result is typical for one-way techniques.

PRACTICAL ASPECTS

In one-way wave-field extrapolation techniques as used in processes such as nonrecursive redatuming and nonrecursive depth migration, a one-way Green's function must be computed for an inhomogeneous medium. Considering forward extrapolation, one starts with the two-way Rayleigh integral (8b) and computes the upward traveling part of the Green's function at the surface with a ray-tracing or Gaussian beam method. Although this intuitive approach leads to the correct results for primary events, it is a procedure that cannot be justified theoretically for an inhomogeneous medium, since

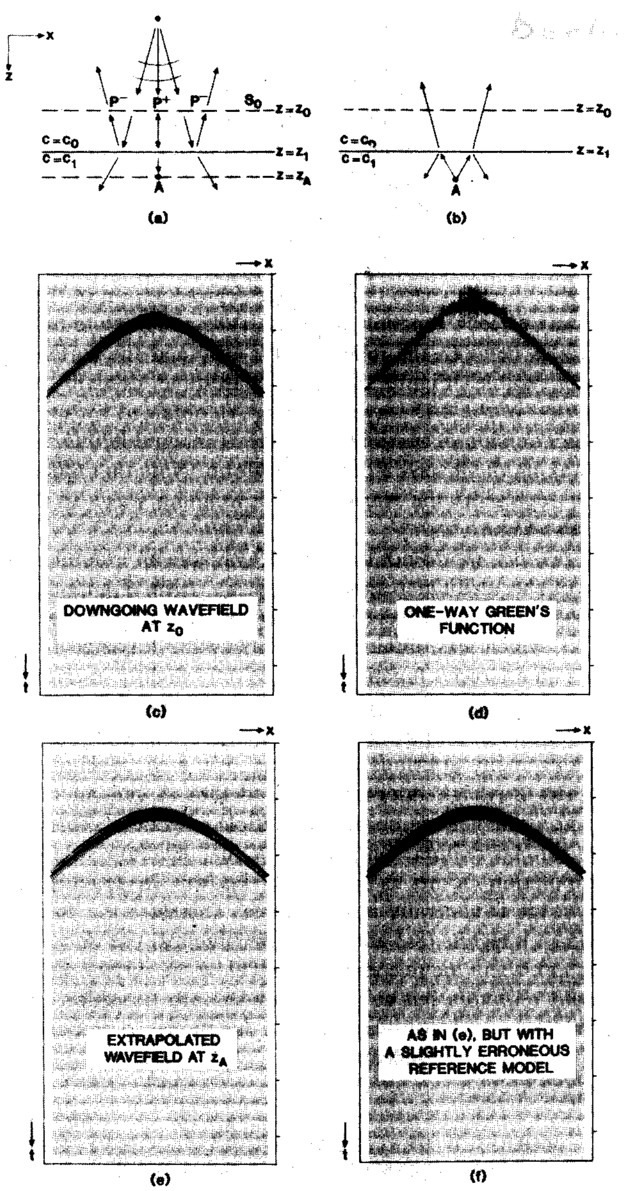

FIG. 7. Application of the one-way Rayleigh II integral. (a) Raypaths in the actual medium for the wave field $P(\mathbf{r}, \omega)$. (b) Raypaths in the reference medium (actual medium) for the one-way Green's function.

equation (8b) applies for a reference medium with a fully reflecting surface.

In summary, if only the upward traveling part of the Green's function is used (characteristic of one-way techniques), then the two-way Rayleigh integral (8b) should be replaced by the one-way Rayleigh integral (13b). The difference between integrals (8b) and (13b) largely clarifies the confusion that still exists as to the justification of using one-way Green's functions in inhomogeneous media. Note that G^- in integral (13b) includes reflections and internal multiples in the lower half-space ($z > z_0$). However, in one-way techniques, these higher-order contributions are generally neglected, meaning that the outcome of integral (13b) is a downgoing wave field ($P = P^+$). In a subsequent paper, we shall show that the elastic counterpart of integral (13b) plays a key role in elastic one-way wave-field extrapolation.

CONCLUSIONS

(1) A formulation has been given for the acoustic version of the Kirchhoff integral, where both the velocity and the density may be inhomogeneous:

$$P_A = \iint_{-\infty}^{\infty} \frac{1}{\rho}\left[\frac{\partial G}{\partial z} P - G\frac{\partial P}{\partial z}\right]_{z_0} dx\, dy. \qquad (14)$$

The medium properties of one half-space (the half-space of A) should be the same for P and G. The medium properties of the other half-space may be chosen for G in any convenient way.

(2) The conventional approach for transforming the Kirchhoff integral into Rayleigh-type integrals produces

$$P_A = -\iint_{-\infty}^{\infty} \frac{1}{\rho}\left[G_I \frac{\partial P}{\partial z}\right]_{z_0} dx\, dy \qquad (15a)$$

or

$$P_A = \iint \frac{1}{\rho}\left[\frac{\partial G_{II}}{\partial z} P\right]_{z_0} dx\, dy, \qquad (15b)$$

where z_0 defines a reflection-free surface of a reference medium that is symmetric with respect to z_0. The two-way Green's functions G_I and G_{II} represent the pressure responses of two monopoles which have equal and opposite polarities. G_I and G_{II} were interpreted as the pressure responses of a single monopole for z_0 being a rigid surface ($R_0 = +1$) and a free surface ($R_0 = -1$), respectively (Figure 4). In this way, it can be easily seen that G_I and G_{II} contain surface-related multiples for media with large contrasts. Therefore, expressions (15a) and (15b) have practical value only for media with small contrasts.

(3) An alternative approach to transform the Kirchhoff integral into Rayleigh-type integrals yields

$$P_A = -2\iint_{-\infty}^{\infty} \frac{1}{\rho}\left[G^- \frac{\partial P^+}{\partial z}\right]_{z_0} dx\, dy, \qquad (16a)$$

or

$$P_A = 2\iint_{-\infty}^{\infty} \frac{1}{\rho}\left[\frac{\partial G^-}{\partial z} P^+\right]_{z_0} dx\, dy, \qquad (16b)$$

where the upper half-space ($z < z_0$) of the reference medium is assumed to be reflection-free, P^+ is the downgoing source wave field, and the one-way Green's function G^- represents the pressure response at $z = z_0$ due to a single monopole situated at A (Figure 6a). Expressions (16a) and (16b) are potentially valuable in practical situations where an inhomogeneous half-space ($z \geq z_0$) has significant contrasts. They form the theoretical basis of one-way extrapolation techniques.

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APPENDIX

PROOF OF THE CONCLUSIONS DRAWN FROM EQUATIONS (11) AND (12)

First we present a simple proof, assuming $\nabla\rho = 0$ and $\nabla c = 0$ at $z = z_0$. Later we generalize the proof for arbitrary ρ and c, assuming only $\partial\rho/\partial z = 0$ and $\partial c/\partial z = 0$ at $z = z_0$.

We define the 2-D spatial Fourier transform $\tilde{A}(k_x, k_y)$ of a space-dependent function $A(x, y)$ by

$$\tilde{A}(k_x, k_y) = \iint_{-\infty}^{\infty} A(x, y)e^{i(k_x x + k_y y)}\, dx\, dy. \qquad (A\text{-}1a)$$

Similarly, we define

$$\tilde{B}(k_x, k_y) = \iint_{-\infty}^{\infty} B(x, y)e^{i(k_x x + k_y y)}\, dx\, dy. \qquad (A\text{-}1b)$$

Using equations (A-1a) and (A-1b), the 2-D version of Parseval's theorem is

$$\iint_{-\infty}^{\infty} A(x, y)B(x, y)\, dx\, dy$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \tilde{A}(k_x, k_y)\tilde{B}(-k_x, -k_y)\, dk_x\, dk_y \qquad \text{(A-1c)}$$

(Dudgeon and Mersereau, 1984).

Applying this theorem to Kirchhoff integral (11b) yields

$$P(\mathbf{r}_A, \omega) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \frac{1}{\rho}\left[\frac{\partial \tilde{G}_0^-}{\partial z}\tilde{P}^+ - \tilde{G}_0^- \frac{\partial \tilde{P}^+}{\partial z}\right]_{z_0} dk_x\, dk_y$$

$$+ \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \frac{1}{\rho}\left[\frac{\partial \tilde{G}_0^-}{\partial z}\tilde{P}^- - \tilde{G}_0^- \frac{\partial \tilde{P}^-}{\partial z}\right]_{z_0} dk_x\, dk_y, \qquad \text{(A-2a)}$$

where

$$\tilde{P}^{\pm} = \tilde{P}^{\pm}(k_x, k_y, z; \omega) \qquad \text{(A-2b)}$$

and

$$\tilde{G}_0^- = \tilde{G}^-(-k_x, -k_y, z; x_A, y_A, z_A; \omega). \qquad \text{(A-2c)}$$

$\tilde{P}^{\pm}$ and $\tilde{G}_0^-$ satisfy the one-way wave equations

$$\frac{\partial \tilde{P}^{\pm}}{\partial z} = \mp i k_z \tilde{P}^{\pm} \qquad \text{at } z = z_0 \qquad \text{(A-3a)}$$

and

$$\frac{\partial \tilde{G}_0^-}{\partial z} = +i k_z \tilde{G}_0^- \qquad \text{at } z = z_0, \qquad \text{(A-3b)}$$

where

$$k_z = \sqrt{\omega^2/c^2 - k_x^2 - k_y^2} \qquad \text{at } z = z_0. \qquad \text{(A-3c)}$$

Now it can be easily verified that

$$\left[\frac{\partial \tilde{G}_0^-}{\partial z}\tilde{P}^- - \tilde{G}_0^- \frac{\partial \tilde{P}^-}{\partial z}\right]_{z_0} = 0, \qquad \text{(A-4)}$$

meaning that the only contribution to $P(\mathbf{r}_A, \omega)$ comes from the first integral in the right-hand side of equation (A-2a). Hence,

$$P(\mathbf{r}_A, \omega) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \frac{1}{\rho}\left[\frac{\partial \tilde{G}_0^-}{\partial z}\tilde{P}^+ - \tilde{G}_0^- \frac{\partial \tilde{P}^+}{\partial z}\right]_{z_0} dk_x\, dk_y \qquad \text{(A-5a)}$$

or

$$P(\mathbf{r}_A, \omega) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \left[\frac{1}{\rho}\frac{\partial \tilde{G}_0^-}{\partial z}\tilde{P}^+\right]_{z_0} dk_x\, dk_y$$

$$+ \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \left[\frac{-1}{\rho}\tilde{G}_0^- \frac{\partial \tilde{P}^+}{\partial z}\right]_{z_0} dk_x\, dk_y. \qquad \text{(A-5b)}$$

From one-way wave equations (A-3), it follows that

$$\left[\frac{\partial \tilde{G}_0^-}{\partial z}\tilde{P}^+\right]_{z_0} = -\left[\tilde{G}_0^- \frac{\partial \tilde{P}^+}{\partial z}\right]_{z_0}, \qquad \text{(A-6)}$$

meaning that the two integrals in the right-hand side of equation (A-5b) are identical. Hence, according to Parseval's theorem (A-1c), the two integrals in the right-hand side of equation (12b) are also identical. This completes the proof for the simple case that $\nabla\rho = 0$ and $\nabla c = 0$ at $z = z_0$.

We now consider a more general case, assuming only $\partial\rho(x, y, z)/\partial z = 0$ and $\partial c(x, y, z)/\partial z = 0$ at $z = z_0$. We introduce scaled pressure functions

$$P_s = P/\sqrt{\rho} \qquad \text{at } z = z_0 \qquad \text{(A-7a)}$$

and

$$G_s = G/\sqrt{\rho} \qquad \text{at } z = z_0, \qquad \text{(A-7b)}$$

so the Kirchhoff integral (11b) can be rewritten as

$$P(\mathbf{r}_A, \omega) = \iint_{-\infty}^{\infty} \left[\frac{\partial G_s^-}{\partial z} P_s^+ - G_s^- \frac{\partial P_s^+}{\partial z}\right]_{z_0} dx\, dy$$

$$+ \iint_{-\infty}^{\infty} \left[\frac{\partial G_s^-}{\partial z} P_s^- - G_s^- \frac{\partial P_s^-}{\partial z}\right]_{z_0} dx\, dy. \qquad \text{(A-8)}$$

Substitution of P_s and G_s into equations (2) and (3) yields

$$\nabla^2 P_s + k_s^2 P_s = 0 \qquad \text{at } z = z_0 \qquad \text{(A-9a)}$$

and

$$\nabla^2 G_s + k_s^2 G_s = 0 \qquad \text{at } z = z_0, \qquad \text{(A-9b)}$$

with

$$k_s^2 = \frac{\omega^2}{c^2} - \frac{3}{4}\left|\frac{\nabla\rho}{\rho}\right|^2 + \frac{\nabla^2\rho}{2\rho} \qquad \text{at } z = z_0; \qquad \text{(A-9c)}$$

see also Brekhovskikh (1980). Equation (A-9a) can be rewritten as

$$\frac{\partial^2 P_s}{\partial z^2} = -\left[k_s^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]P_s \qquad \text{at } z = z_0. \qquad \text{(A-10)}$$

The spatial differentiations $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ can be written as spatial convolutions along the x-axis and y-axis, respectively, according to

$$\frac{\partial^2 A(x, y)}{\partial x^2} = \int_{-\infty}^{\infty} d_2(x - x')A(x', y)\, dx' \qquad \text{(A-11a)}$$

and

$$\frac{\partial^2 A(x, y)}{\partial y^2} = \int_{-\infty}^{\infty} d_2(y - y')A(x, y')\, dy', \qquad \text{(A-11b)}$$

with

$$d_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{D}_2(k_x)e^{-ik_x x}\, dk_x \qquad \text{(A-11c)}$$

and

$$d_2(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{D}_2(k_y)e^{-ik_y y}\, dk_y, \qquad \text{(A-11d)}$$

where $\tilde{D}_2(k_x)$ and $\tilde{D}_2(k_y)$ represent band-limited versions of $-k_x^2$ and $-k_y^2$, respectively (Berkhout, 1985). Equation (A-11) is exact when $A(x, y)$ is a spatially band-limited function. With definitions (A-11c) and (A-11d), equation (A-10) can be rewritten as

$$\left.\frac{\partial^2 P_s(\mathbf{r}, \omega)}{\partial z^2}\right|_{z0} = -\iint_{-\infty}^{\infty} \left[H_2(\mathbf{r}', \mathbf{r}, \omega) P_s(\mathbf{r}', \omega)\right]_{z0} dx'\, dy', \tag{A-12a}$$

where

$$H_2(\mathbf{r}', \mathbf{r}, \omega) = k_s^2(\mathbf{r}, \omega)\delta(x - x')\delta(y - y') + d_2(x - x')\delta(y - y') + \delta(x - x')d_2(y - y'), \tag{A-12b}$$

with

$$\mathbf{r} = (x, y, z) \tag{A-12c}$$

and

$$\mathbf{r}' = (x', y', z' = z). \tag{A-12d}$$

Note that operator H_2 is symmetric in $\mathbf{r}$ and $\mathbf{r}'$:

$$H_2(\mathbf{r}', \mathbf{r}, \omega) = H_2(\mathbf{r}, \mathbf{r}', \omega).$$

Let us now implicitly define an operator H_1 according to

$$H_2(\mathbf{r}'', \mathbf{r}, \omega) = \iint_{-\infty}^{\infty} H_1(\mathbf{r}'', \mathbf{r}', \omega) H_1(\mathbf{r}', \mathbf{r}, \omega)\, dx'\, dy', \tag{A-13a}$$

with

$$\mathbf{r}'' = (x'', y'', z'' = z). \tag{A-13b}$$

Note that operator H_1 is symmetric in $\mathbf{r}$ and $\mathbf{r}'$:

$$H_1(\mathbf{r}', \mathbf{r}, \omega) = H_1(\mathbf{r}, \mathbf{r}', \omega). \tag{A-13c}$$

From equations (A-12) and (A-13), we obtain the following *one-way wave equations* for the scaled pressure functions:

$$\left.\frac{\partial P_s^{\pm}(\mathbf{r}, \omega)}{\partial z}\right|_{z0} = \mp i \iint_{-\infty}^{\infty} \left[H_1(\mathbf{r}', \mathbf{r}, \omega) P_s^{\pm}(\mathbf{r}', \omega)\right]_{z0} dx'\, dy'. \tag{A-14a}$$

Similarly,

$$\left.\frac{\partial G_s^{-}(\mathbf{r}', \mathbf{r}_A, \omega)}{\partial z}\right|_{z0} = +i \iint_{-\infty}^{\infty} \left[H_1(\mathbf{r}, \mathbf{r}', \omega) G_s^{-}(\mathbf{r}, \mathbf{r}_A, \omega)\right]_{z0} dx\, dy. \tag{A-14b}$$

Substitution of equation (A-14) into equation (A-8) yields

$$\begin{aligned}
P(\mathbf{r}_A, \omega) &= \iint_{-\infty}^{\infty} \left\{ \iint_{-\infty}^{\infty} \left[iH_1(\mathbf{r}, \mathbf{r}', \omega) G_s^{-}(\mathbf{r}, \mathbf{r}_A, \omega)\right] dx\, dy\, P_s^{+}(\mathbf{r}', \omega) \right\}_{z0} dx'\, dy' \\
&+ \iint_{-\infty}^{\infty} \left\{ G_s^{-}(\mathbf{r}, \mathbf{r}_A, \omega) \iint_{-\infty}^{\infty} \left[iH_1(\mathbf{r}', \mathbf{r}, \omega) P_s^{+}(\mathbf{r}', \omega)\right] dx'\, dy' \right\}_{z0} dx\, dy \\
&+ \iint_{-\infty}^{\infty} \left\{ \iint_{-\infty}^{\infty} \left[iH_1(\mathbf{r}, \mathbf{r}', \omega) G_s^{-}(\mathbf{r}, \mathbf{r}_A, \omega)\right] dx\, dy\, P_s^{-}(\mathbf{r}', \omega) \right\}_{z0} dx'\, dy' \\
&- \iint_{-\infty}^{\infty} \left\{ G_s^{-}(\mathbf{r}, \mathbf{r}_A, \omega) \iint_{-\infty}^{\infty} \left[iH_1(\mathbf{r}', \mathbf{r}, \omega) P_s^{-}(\mathbf{r}', \omega)\right] dx'\, dy' \right\}_{z0} dx\, dy.
\end{aligned} \tag{A-15}$$

From symmetry property (A-13c), it follows that the last two terms in the right-hand side of equation (A-15) cancel, meaning that the only contribution to $P(\mathbf{r}_A, \omega)$ comes from the first two terms. Furthermore, from the same symmetry property (A-13c), it follows that the first two terms in equation (A-15) are identical. Hence, the two integrals in the right-hand side of equation (12b) are identical. This completes the proof for the case of $\partial\rho(x, y, z)/\partial z = 0$ and $\partial c(x, y, z)/\partial z = 0$ at $z = z_0$.