

**WAVE FIELD EXTRAPOLATION TECHNIQUES
FOR INHOMOGENEOUS MEDIA WHICH INCLUDE
CRITICAL ANGLE EVENTS.
PART I: METHODS USING THE ONE-WAY
WAVE EQUATIONS***

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ABSTRACT

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Part I of this series starts with a brief review of the fundamental principles underlying wave field extrapolation. Next, the total wave field is split into downgoing and upgoing waves, described by a set of coupled one-way wave equations. In cases of limited propagation angles and weak inhomogeneities these one-way wave equations can be decoupled, describing primary waves only. For large propagation angles (up to and including 90°) an alternative choice of sub-division into downgoing and upgoing waves is presented. It is shown that this approach is well suited for modeling as well as migration and inversion schemes for seismic data which include critical angle events.

1. INTRODUCTION

In modeling as well as in migration schemes it is generally assumed that the seismic wave field may be split into downgoing and upgoing waves which propagate without interaction. This assumption is only justified for waves traveling with limited propagation angles, in seismic literature referred to as sub-critical events. The wave equations which govern these sub-critical events are commonly known as the one-way wave equations for downgoing and upgoing waves. This paper starts with a review of the one-way wave equations, the underlying assumptions and the solutions. Furthermore, an extensive discussion of the physical interpretation is presented.

The treatment of waves with large propagation angles is significantly different from the treatment of sub-critical events. In the literature much attention has been

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paid to the behavior of the wave field in the vicinity of so-called turning points. An extensive historical survey is presented by McHugh (1971). At a turning point the propagation direction of the energy flow of an incident wave field is fully changed from downwards to upwards or vice versa, due to a vertical gradient in the propagation velocity. In the following, such events will be referred to as "critical". Unlike sub-critical data, the treatment of seismic data including critical events is rather complicated, since propagating downgoing and upgoing waves are *coupled* near the turning point. This implies that modeling and migration schemes for sub-critical data cannot simply be extended for the incorporation of critical events if the coupling of propagating downgoing and upgoing waves is neglected. Several alternative modeling approaches have been proposed, based on the WKBJ-technique, which give an adequate treatment of critical events. Although satisfactory for modeling, these approaches are not suitable for migration and inversion applications.

We introduce an alternative subdivision of the total wave field near the turning point into downgoing and upgoing waves. Following this approach, we are able to define one-way forward and inverse wave field extrapolation operators which include critical events for stratified media. In part II of this series the one-way results will be compared with two-way wave field extrapolation operators which include critical events for arbitrary inhomogeneous media. In part III the application of the above wave field extrapolation operators in modeling, migration and inversion schemes will be discussed. In particular the inversion scheme for one dimensional media, which includes critical events, is of great practical importance: this inversion scheme computes velocity trends which can be used for a background medium in multi-dimensional linearized inverse scattering techniques as discussed by Berkhout (1984). The material in this part of the paper is presented in the following sections:

2. The wave equation modified for wave field extrapolation.
3. Decomposition in a homogeneous medium.
4. Decomposition in an arbitrarily inhomogeneous medium.
5. Decoupled one-way wave equations for sub-critical events.
6. The WKBJ-technique for continuously layered 1-D inhomogeneous media.
7. Decoupled one-way wave equations for critical events.
8. Conclusions.

For notational convenience we often denote a wave field $P(x, y, z, \omega)$ in the space-frequency domain as $P(z)$ or P , while the same wave field $\tilde{P}(k_x, k_y, z, \omega)$ in the wavenumber-frequency domain is often denoted as $\tilde{P}(z)$ or \tilde{P} . Similar conventions are used for operators.

2. THE WAVE EQUATION MODIFIED FOR WAVE FIELD EXTRAPOLATION

Consider the acoustic wave equation for inhomogeneous media, in the frequency domain given by

$$\nabla^2 P + k^2 P = \nabla \ln \rho \cdot \nabla P, \quad (2.1)$$

where

$P = P(x, y, z, \omega)$ is the acoustic pressure,

$k = \omega/c$ is the wave number,

$c = c(x, y, z)$ is the propagation velocity,

$\rho = \rho(x, y, z)$ is the mass density, and

ω is the circular frequency.

In order to describe wave field extrapolation in depth, it is convenient to rewrite this wave equation in such a way that the derivative with respect to depth (z is pointing downward) be expressed as a lateral operator working on P at level z . Straightforward manipulations lead to

$$\left[-\rho \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial P}{\partial z} \right) \right]_z = [H_2 * P]_z, \quad (2.2a)$$

with

$$H_2(x, y, z, \omega) = [k^2 d_0(x, y) + d_2(x) + d_2(y) - \frac{\partial \ln \rho}{\partial x} d_1(x) - \frac{\partial \ln \rho}{\partial y} d_1(y)]_z. \quad (2.2b)$$

In the following we delete the subscript z .

Notice that we use the notation introduced by Berkhout (1982) where the symbol $*$ refers to spatial convolutions. The operators $d_m(x)$ and $d_m(y)$ represent bandlimited one-dimensional spatial differential operators to x and y , respectively, where m represents the order of differentiation. In its present formulation, the wave equation is not yet suitable for wave field extrapolation, since it is a second-order differential equation which requires two boundary conditions. In principle there are two ways to transform equation (2.2a) into first-order differential equations:

(i) decomposition of the total wave field into downgoing and upgoing waves P^+ and P^- :

$$P = P^+ + P^-$$

such that

$$\frac{\partial P^+}{\partial z} = -jH_1^+ * P^+, \quad (2.3a)$$

$$\frac{\partial P^-}{\partial z} = +jH_1^- * P^-. \quad (2.3b)$$

(ii) reformulation of the scalar wave equation into a matrix equation

$$\frac{\partial}{\partial z} \begin{bmatrix} P \\ \frac{1}{\rho} \frac{\partial P}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & \rho \\ -\frac{1}{\rho} H_2 * & 0 \end{bmatrix} \begin{bmatrix} P \\ \frac{1}{\rho} \frac{\partial P}{\partial z} \end{bmatrix}. \quad (2.4)$$

In (2.3) we formulated *one-way wave equations*. The definition of operators H_1^+ and H_1^- depends on the approach to decomposition. We first discuss the conventional approach where H_1^+ and H_1^- cannot be decoupled for critical angle events. Later we introduce an alternative approach where H_1^+ and H_1^- decouple for sub-critical as well as critical angle events.

In (2.4) we formulated the matrix representation of the *two-way wave equation*. This equation is valid for sub-critical as well as critical events. Solutions will be discussed in part II of this series.

3. DECOMPOSITION IN A HOMOGENEOUS MEDIUM

In a homogeneous medium, (2.2a) can be simplified to

$$\frac{\partial P}{\partial z} = \pm j H_1 * P, \quad (3.1a)$$

where the *square root operator* H_1 is implicitly defined according to

$$H_1 * H_1 = H_2. \quad (3.1b)$$

We now study the propagation direction of P . We choose $j = +\sqrt{-1}$. In the wavenumber-frequency domain (Berkhout 1982), (3.1a) reads

$$\frac{\partial \tilde{P}}{\partial z} = \pm j \tilde{H}_1 \tilde{P}, \quad (3.2a)$$

where the square root operator \tilde{H}_1 is explicitly defined according to

$$\tilde{H}_1^2 = \tilde{H}_2, \quad (3.2b)$$

or

$$\tilde{H}_1 = \sqrt{k^2 - k_x^2 - k_y^2}. \quad (3.2c)$$

The symbol $\tilde{}$ refers to a double spatial Fourier transformation from x to k_x and y to k_y , where k_x and k_y represent the horizontal components of the wave vector \mathbf{k} . Consequently, in spherical coordinates

$$\tilde{H}_1 = k \cos \alpha, \quad (3.2d)$$

where α is the angle between the propagation direction and the z -axis.

The solution of equation (3.2a) is given by

$$\tilde{P}(z) = \tilde{W}(z, z_0) \tilde{P}(z_0), \quad (3.3a)$$

with

$$\tilde{W}(z, z_0) = \exp(\pm j\tilde{H}_1\Delta z), \quad (3.3b)$$

$$\Delta z = z - z_0. \quad (3.3c)$$

For propagating waves ($\tilde{H}_1^2 \geq 0$) it follows that $-j\tilde{H}_1$ refers to downgoing waves, since the argument of \tilde{W} decreases with depth. Similarly, $+j\tilde{H}_1$ refers to upgoing waves. The operator \tilde{W} is commonly known as the phase-shift operator (Gazdag 1978). The inverse double spatial Fourier transform of relation (3.3a) is given by

$$P(z) = W(z, z_0) * P(z_0), \quad (3.4)$$

where W represents the well-known Rayleigh II operator (Schneider 1978). Note that (3.4) describes wave field extrapolation in the z -direction; convolution is carried out in the x - and y -directions. The operators W and \tilde{W} are extensively discussed for both propagating and evanescent waves by Berkhouit (1982).

Summarizing, in a *homogeneous medium* the one-way wave equations for downgoing and upgoing waves P^+ and P^- , respectively, read

$$\frac{\partial P^+}{\partial z} = -jH_1^+ * P^+, \quad (3.5a)$$

$$\frac{\partial P^-}{\partial z} = +jH_1^- * P^-, \quad (3.5b)$$

with

$$H_1^+ = H_1^- = H_1. \quad (3.5c)$$

Notice that these relations are exact.

4. DECOMPOSITION IN AN ARBITRARILY INHOMOGENEOUS MEDIUM

In an arbitrarily inhomogeneous medium, one-way wave equations (3.5a) and (3.5b) do not hold. It has been shown by Brekhovskikh (1980), amongst others, that decomposition into downgoing and upgoing waves is not uniquely defined. Here we extend the approach which is usually followed for 1-D inhomogeneous media ($c = c(z)$, $\rho = \rho(z)$) to arbitrarily inhomogeneous media. Later we discuss an alternative approach which properly incorporates critical events.

In the following we choose the decomposition such that the particle velocity of our choice of the downgoing wave is related to the pressure of this downgoing wave only. A similar choice is taken for the upgoing wave. For the *total* wave field we have the following relations between particle velocity and pressure:

$$j\omega\rho V_z = -\frac{\partial P}{\partial z}, \quad (4.1a)$$

$$j\omega\rho \frac{\partial V_z}{\partial z} = H_2 * P, \quad (4.1b)$$

where relation (4.1a) describes the z -component of the equation of motion ($V_z = z$ -component of particle velocity), and where relation (4.1b) is obtained after substituting the x - and y -components of the equation of motion into the stress/strain relation. Notice that (4.1a) and (4.1b) are consistent with wave equation (2.2a).

Now we *define* downgoing and upgoing waves such that (according to the homogeneous situation)

$$j\omega\rho V_z^+ \stackrel{\text{def}}{=} +jH_1 * P^+, \tag{4.2a}$$

$$j\omega\rho V_z^- \stackrel{\text{def}}{=} -jH_1 * P^-, \tag{4.2b}$$

where

$$H_1 * H_1 = H_2, \tag{4.2c}$$

with H_2 defined by relation (2.2b).

Furthermore, we demand that

$$P = P^+ + P^- \tag{4.3a}$$

and

$$V_z = V_z^+ + V_z^-, \tag{4.3b}$$

where P and V_z satisfy (4.1a) and (4.1b). Straightforward manipulations yield

$$\frac{\partial P^+}{\partial z} = -jH_1 * P^+ - \frac{1}{2}H_1^{-1} * \left[\rho \frac{\partial}{\partial z} \left(\frac{1}{\rho} H_1 \right) \right] * (P^+ - P^-), \tag{4.4a}$$

$$\frac{\partial P^-}{\partial z} = +jH_1 * P^- + \frac{1}{2}H_1^{-1} * \left[\rho \frac{\partial}{\partial z} \left(\frac{1}{\rho} H_1 \right) \right] * (P^+ - P^-), \tag{4.4b}$$

where

$$H_1^{-1} * H_1 = \delta(x)\delta(y). \tag{4.4c}$$

Notice that these relations are exact, but our choice of downgoing and upgoing waves are coupled by the reflectivity properties of the medium. The relations (4.4a, b) can be written in the desired formulation (2.3a, b), when we define

$$P = P^+ + P^- = [\delta(x)\delta(y) + \Gamma^+] * P^+ \tag{4.5a}$$

for downward propagation, and

$$P = P^+ + P^- = [\delta(x)\delta(y) + \Gamma^-] * P^- \tag{4.5b}$$

for upward propagation. Now the one-way wave equations read

$$\frac{\partial P^+}{\partial z} = -jH_1^+ * P^+ \tag{4.6a}$$

and

$$\frac{\partial P^-}{\partial z} = +jH_1^- * P^-, \tag{4.6b}$$

with

$$jH_1^+ = jH_1 + \frac{1}{2}H_1^{-1} * \left[\rho \frac{\partial}{\partial z} \left(\frac{1}{\rho} H_1 \right) \right] * [\delta(x)\delta(y) - \Gamma^+], \quad (4.6c)$$

$$jH_1^- = jH_1 - \frac{1}{2}H_1^{-1} * \left[\rho \frac{\partial}{\partial z} \left(\frac{1}{\rho} H_1 \right) \right] * [\delta(x)\delta(y) - \Gamma^-]. \quad (4.6d)$$

Notice that the coupling of these one-way wave equations is described by the operators Γ^+ and Γ^- . In the following we consider three cases in which the one-way wave equations decouple.

- A. The medium properties be functions of the lateral coordinates only: $c = c(x, y)$, $\rho = \rho(x, y)$. For this situation (4.6a) and (4.6b) decouple since H_1^+ and H_1^- both simplify to H_1 . Notice that this decoupling is exact. A special case was already found in section 3 for a homogeneous medium.
- B. Only primary waves are considered, that is, Γ^+ and Γ^- are neglected in (4.5a) and (4.5b). The one-way wave equations (4.6a) and (4.6b) decouple as the coupling operators Γ^+ and Γ^- in (4.6c) and (4.6d) vanish. This approach is valid for sub-critical angle events only, as will be seen in section 5.
- C. Multiple reflections are incorporated in the primary waves, as described by (4.5a) and (4.5b). Now a general solution cannot be given. In section 7 we discuss a special case which is based on the WKB approach with a particular choice of upgoing and downgoing wave functions, which, well away from the critical region, reduce to the P^+ - and P^- -terms as defined above for $\Gamma^+ = \Gamma^- = 0$. This approach allows decoupled one-way wave equations to be constructed for sub-critical as well as critical angle events.

5. DECOUPLED ONE-WAY WAVE EQUATIONS FOR SUB-CRITICAL EVENTS

The least complicated case of decoupled one-way wave equations occurs when the medium properties are functions of the lateral coordinates only, as was shown in the previous section. Berkhout (1982) proposed the following *recursive* extrapolation scheme for propagating waves:

1. Subdivide the subsurface into thin layers. In the i th layer, $z_{i-1} \leq z < z_i$, the medium properties are approximated by

$$c_i = c_i(x, y), \quad \rho_i = \rho_i(x, y), \quad (5.1)$$

as shown in fig. 1.

2. From exact one-way wave equations (4.6a) and (4.6b), with $H_1^+ = H_1^- = H_1$, wave field extrapolation operators can be derived for each layer, such that

$$P^+(z_{i+1}) = W^+(z_{i+1}, z_i) * P^+(z_i) \quad (5.2a)$$

and

$$P^-(z_i) = W^-(z_i, z_{i+1}) * P^-(z_{i+1}). \quad (5.2b)$$

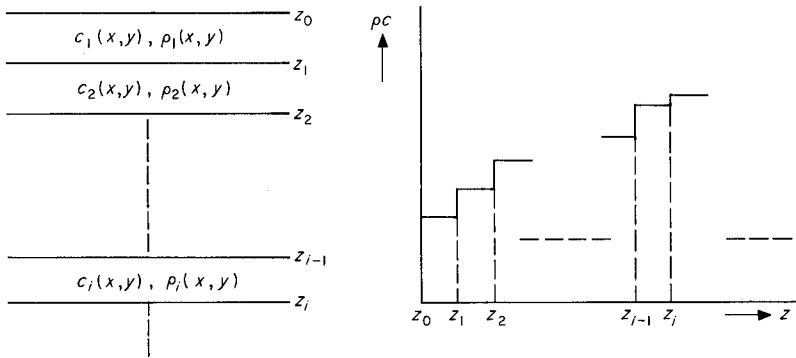


Fig. 1. Acoustic model of the subsurface for recursive wave field extrapolation based on the one-way wave equations.

3. At each layer boundary, transmission operators can be derived from the boundary conditions, such that

$$\lim_{\varepsilon \rightarrow 0} P^+(z_i + \varepsilon) = T^+(z_i) * \left[\lim_{\varepsilon \rightarrow 0} P^+(z_i - \varepsilon) \right], \tag{5.3a}$$

$$\lim_{\varepsilon \rightarrow 0} P^-(z_i - \varepsilon) = T^-(z_i) * \left[\lim_{\varepsilon \rightarrow 0} P^-(z_i + \varepsilon) \right]. \tag{5.3b}$$

4. Recursive wave field extrapolation of *primary* data can now be defined according to

$$P^+(z_m) = W^+(z_m, z_0) * P^+(z_0), \tag{5.4a}$$

$$P^-(z_0) = W^-(z_0, z_m) * P^-(z_m), \tag{5.4b}$$

with

$$W^+(z_m, z_0) = W^+(z_m, z_{m-1}) * T^+(z_{m-1}) * \cdots * T^+(z_1) * W^+(z_1, z_0) \tag{5.4c}$$

and

$$W^-(z_0, z_m) = W^-(z_0, z_1) * T^-(z_1) * \cdots * T^-(z_{m-1}) * W^-(z_{m-1}, z_m). \tag{5.4d}$$

5. The basic relation for modeling of primary data reads

$$P^-(z_0) = \left[\sum_m W^-(z_0, z_m) * R(z_m) * W^+(z_m, z_0) \right] * P^+(z_0), \tag{5.4e}$$

where $R(z_m)$ represents the reflectivity operator at depth $z = z_m$.

Since multiple reflections are neglected, (5.4a) and (5.4b) are not exact. In order to quantify the approximation, we consider the relations in the wavenumber-frequency domain

$$\tilde{P}^+(z_m) = \tilde{W}^+(z_m, z_0) \tilde{P}^+(z_0) \tag{5.5a}$$

and

$$\tilde{P}^-(z_0) = \tilde{W}^-(z_0, z_m) \tilde{P}^-(z_m), \quad (5.5b)$$

with

$$\tilde{W}^+(z_m, z_0) = \tilde{W}^+(z_m, z_{m-1}) \tilde{T}^+(z_{m-1}) \cdots \tilde{T}^+(z_1) \tilde{W}^+(z_1, z_0) \quad (5.5c)$$

and

$$\tilde{W}^-(z_0, z_m) = \tilde{W}^-(z_0, z_1) \tilde{T}^-(z_1) \cdots \tilde{T}^-(z_{m-1}) \tilde{W}^-(z_{m-1}, z_m). \quad (5.5d)$$

If one assumes a continuously layered medium—which is always possible for band limited data—then according to appendix A these operators can be written as

$$\tilde{W}^+(z_m, z_0) = \sqrt{\frac{\rho(z_m) \tilde{H}_1(z_0)}{\rho(z_0) \tilde{H}_1(z_m)}} \exp \int_{z_0}^{z_m} -j \tilde{H}_1(z) dz \quad (5.6a)$$

and

$$\tilde{W}^-(z_0, z_m) = \sqrt{\frac{\rho(z_0) \tilde{H}_1(z_m)}{\rho(z_m) \tilde{H}_1(z_0)}} \exp \int_{z_m}^{z_0} +j \tilde{H}_1(z) dz, \quad (5.6b)$$

with

$$\tilde{H}_1^2(z) = [\omega^2/c^2(z)] - k_x^2 - k_y^2 > 0, \quad z_0 < z < z_m. \quad (5.6c)$$

Notice that (5.5a) and (5.5b), with operators $\tilde{W}^+(z_m, z_0)$ and $\tilde{W}^-(z_0, z_m)$ defined by (5.6a) and (5.6b), describe wave field extrapolation of *primary* propagating waves in continuously layered media. From these relations it follows that primary downgoing and upgoing waves in continuously layered media may be defined for variable z according to

$$\tilde{P}^+(z) = \tilde{C}^+ \left(\frac{\tilde{H}_1(z)}{\rho(z)} \right)^{-1/2} \exp \int_{z_0}^z -j \tilde{H}_1(z') dz' \quad (5.7a)$$

and

$$\tilde{P}^-(z) = \tilde{C}^- \left(\frac{\tilde{H}_1(z)}{\rho(z)} \right)^{-1/2} \exp \int_{z_1}^z +j \tilde{H}_1(z') dz', \quad (5.7b)$$

where \tilde{C}^+ , \tilde{C}^- , z_0 , z_1 are arbitrary constants. Differentiation with respect to z yields

$$\frac{\partial \tilde{P}^+}{\partial z} = \left[-j \tilde{H}_1 - \frac{1}{2} \left(\frac{\tilde{H}_1}{\rho} \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\tilde{H}_1}{\rho} \right) \right] \tilde{P}^+ \quad (5.8a)$$

and

$$\frac{\partial \tilde{P}^-}{\partial z} = \left[+j \tilde{H}_1 - \frac{1}{2} \left(\frac{\tilde{H}_1}{\rho} \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\tilde{H}_1}{\rho} \right) \right] \tilde{P}^-. \quad (5.8b)$$

Notice that these relations represent the spatial Fourier transforms of one-way wave equations (4.6a) and (4.6b) without the coupling operators $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ (case B).

Multiplication by $1/\rho$ and again differentiation with respect to z yields

$$-\rho \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial z} \right) = [\tilde{H}_2 - \tilde{E}_c - \tilde{E}_\rho] \tilde{P}, \quad (5.9a)$$

with

$$\tilde{E}_c = -\frac{1}{4\tilde{H}_2} \frac{\partial^2 \tilde{H}_2}{\partial z^2} + \frac{5}{16} \left(\frac{1}{\tilde{H}_2} \frac{\partial \tilde{H}_2}{\partial z} \right)^2 \quad (5.9b)$$

and

$$\tilde{E}_\rho = \frac{1}{2\rho} \frac{\partial^2 \rho}{\partial z^2} - \frac{3}{4} \left(\frac{1}{\rho} \frac{\partial \rho}{\partial z} \right)^2, \quad (5.9c)$$

where \tilde{P} may represent either \tilde{P}^+ or \tilde{P}^- . Notice that this relation represents the spatial Fourier transform of two-way wave equation (2.2a), assuming

$$|\tilde{E}_c + \tilde{E}_\rho| \ll |\tilde{H}_2|. \quad (5.9d)$$

We may conclude that *recursive* wave field extrapolation of primary waves, as described by (5.4), is not justified when (5.9d) is violated. This occurs when $\tilde{H}_2 \rightarrow 0$ ($\tilde{E}_c \rightarrow \infty$). Since $\tilde{H}_2 = \tilde{H}_1^2 = k^2 \cos^2 \alpha$, recursive relations (5.4a) and (5.4b) are invalid for critical angle events ($\alpha \rightarrow 90^\circ$).

Relations (5.7a) and (5.7b) can be found directly from the two-way wave equation following the approach suggested by Liouville (1837) and Green (1837). They assumed a solution $\exp[\Phi(z)]$ and solved a nonlinear differential equation in $\Phi(z)$ assuming $|\partial_z^2 \Phi| \ll |\partial_z \Phi|^2$. Various other authors solved the two-way wave equation iteratively. Amongst others, Bremmer (1951) and Brekhovskikh (1980) showed that (5.7a) and (5.7b) represent the zero-order solution of an iterative procedure, the so-called geometrical optics approach. For a seismological application the reader is referred to Chapman (1976). In the following, (5.7a) and (5.7b) will be called the "LG approximation" (for "Liouville-Green") for sub-critical angle events. In seismic literature, these relations are erroneously known as the WKB or WKBJ approximation, referring to papers by Wentzel (1926), Kramers (1926), Brillouin (1926) and Jeffreys (1924). However, the extra important item introduced by these authors is a technique to connect LG approximations for $\tilde{H}_2 > 0$ with LG approximations for $\tilde{H}_2 < 0$ by means of an approximation valid for $\tilde{H}_2 \rightarrow 0$. Therefore, following Olver (1974), we reserve the name WKBJ approach for methods which properly handle sub-critical as well as critical angle events. This is discussed in section 6.

Finally, we present closed expressions for the LG operators $\tilde{W}^+(z_{i+1}, z_i)$ and $\tilde{W}^-(z_i, z_{i+1})$. Assuming that the operator \tilde{H}_2 may be linearized according to

$$\tilde{H}_2(z) = \tilde{H}_1^2(z) = \tilde{H}_1^2(z_i) + (z - z_i)\chi_{i+1}, \quad \text{for } z_i < z < z_{i+1}, \quad (5.10)$$

then the LG operators for sub-critical angle events are given by

$$\tilde{W}^+(z_{i+1}, z_i) = \sqrt{\frac{\rho(z_{i+1})\tilde{H}_1(z_i)}{\rho(z_i)\tilde{H}_1(z_{i+1})}} \exp \left[-j \frac{2}{3\chi_{i+1}} \{ \tilde{H}_2^{3/2}(z_{i+1}) - \tilde{H}_2^{3/2}(z_i) \} \right] \quad (5.11a)$$

and

$$\tilde{W}^-(z_i, z_{i+1}) = \sqrt{\frac{\rho(z_i)\tilde{H}_1(z_{i+1})}{\rho(z_{i+1})\tilde{H}_1(z_i)}} \exp \left[-j \frac{2}{3\chi_{i+1}} \{ \tilde{H}_2^{3/2}(z_{i+1}) - \tilde{H}_2^{3/2}(z_i) \} \right]. \quad (5.11b)$$

The inverse operators for propagating waves are given by

$$\tilde{F}^+(z_i, z_{i+1}) = [\tilde{W}^+(z_{i+1}, z_i)]^{-1} = [\tilde{W}^-(z_i, z_{i+1})]^* \quad (5.12a)$$

and

$$\tilde{F}^-(z_{i+1}, z_i) = [\tilde{W}^-(z_i, z_{i+1})]^{-1} = [\tilde{W}^+(z_{i+1}, z_i)]^*, \quad (5.12b)$$

where the symbol * refers to complex conjugation.

For small χ_{i+1} the complex exponential function in both (5.11a) and (5.11b) can be written as

$$[1 - \frac{1}{2}j\chi_{i+1}\Delta z^2\tilde{H}_1^{-1}(z_i)] \exp [-j\tilde{H}_1(z_i)\Delta z], \quad (5.13)$$

with $\Delta z = z_{i+1} - z_i$. For $\chi_{i+1} = 0$ this expression corresponds to the phase shift operator for homogeneous media, as discussed in section 3.

Summarizing, we discussed a wave field extrapolation algorithm for propagating waves. Lateral variations of c and ρ can be properly incorporated since the algorithm is based on exact one-way wave equations. Vertical variations of c and ρ can be incorporated by applying the scheme recursively. By taking the step-size sufficiently small, this approach is correct for primary waves within the seismic bandwidth. The solution is not valid for critical angle events since multiple reflections are neglected.

For 1-D inhomogeneous media the recursive approach complies with the LG approximation. The condition for this approximation is quantified by inequality (5.9d). In the seismic literature several approaches have been suggested for the incorporation of multiple reflections. Berkhout (1982) discussed a recursive scheme for discretely layered 3-D inhomogeneous media. A feedback system at each layer interface generates an infinite number of multiple reflections. On the other hand, Chapman (1976) discussed an iterative scheme for continuously layered 1-D inhomogeneous media. In each successive iteration step a new order of multiple reflections is generated. The WKBJ technique, discussed in the next section, takes automatically into account all multiple reflections within the critical region ($\tilde{H}_2 \rightarrow 0$).

6. THE WKBJ TECHNIQUE FOR CONTINUOUSLY LAYERED 1-D INHOMOGENEOUS MEDIA

In the previous section we have seen that the conventional approach to wave field extrapolation fails for critical angle events as multiple reflections are neglected. In order to satisfy the two-way wave equation, an infinite number of multiple reflections should be incorporated, which is not very attractive from a practical point of view. In this section we discuss an alternative solution for 1-D inhomogeneous

media, suggested by Wentzel (1926), Kramers (1926), Brillouin (1926), and Jeffreys (1924).

In the wavenumber-frequency domain the wave equation for the scaled pressure function (Brekhovskikh 1980) reads

$$\frac{\partial^2}{\partial z^2} \left(\frac{\tilde{P}}{\sqrt{\rho}} \right) = -\tilde{H}_2 \left(\frac{\tilde{P}}{\sqrt{\rho}} \right), \tag{6.1a}$$

with

$$\tilde{H}_2 = \tilde{H}_2(z) = \frac{\omega^2}{c^2(z)} - k_x^2 - k_y^2, \tag{6.1b}$$

assuming $|\tilde{E}_\rho| \ll k^2$, with \tilde{E}_ρ defined by (5.9c). For large $|\tilde{H}_2|$, solutions are given by the LG approximations (5.7a) and (5.7b). Problems occur for $|\tilde{H}_2| \rightarrow 0$, where the LG solutions grow out of bounds. An acceptable solution for all \tilde{H}_2 can be found with the WKB technique, which involves the following steps (see also McHugh 1971):

1. Assume $\tilde{H}_2(z)$ to be linearized for small $|z - z_t|$, where z_t is the turning point of (6.1a):

$$\tilde{H}_2(z) \approx (z - z_t)\chi, \tag{6.2a}$$

with

$$\chi = \left. \frac{\partial \tilde{H}_2}{\partial z} \right|_{z_t} = -2 \left(\frac{\omega^2}{c^3(z_t)} \right) \left. \frac{\partial c(z)}{\partial z} \right|_{z_t}. \tag{6.2b}$$

2. Expand the LG approximations (5.7a) and (5.7b) under assumption (6.2).
3. Replace (6.1a) by the Airy equation under assumption (6.2):

$$\frac{\partial^2 \tilde{\psi}}{\partial \zeta^2} - \zeta \tilde{\psi} = 0, \tag{6.3a}$$

where

$$\tilde{\psi} = \tilde{P}/\sqrt{\rho} \text{ is the scaled pressure} \tag{6.3b}$$

and

$$\zeta = -\chi^{1/3}(z - z_t) \text{ is the scaled depth.} \tag{6.3c}$$

Solutions are given by the Airy functions $\text{Ai}(\zeta)$, $\text{Bi}(\zeta)$ —see Abramowitz and Stegun (1970; section 10.4).

4. Choose linear combinations of LG solutions and Airy functions such that the LG expansions match the asymptotic Airy expansions at both sides of the turning point z_t .

We discuss this method with the aid of an example. Consider a continuously layered medium, with velocity $c(z)$ constant for $z < z_0$ and increasing monotonously with depth for $z \geq z_0$. Assume a downgoing plane wave $\tilde{P}^+(z_0)$ incident at $z = z_0$,

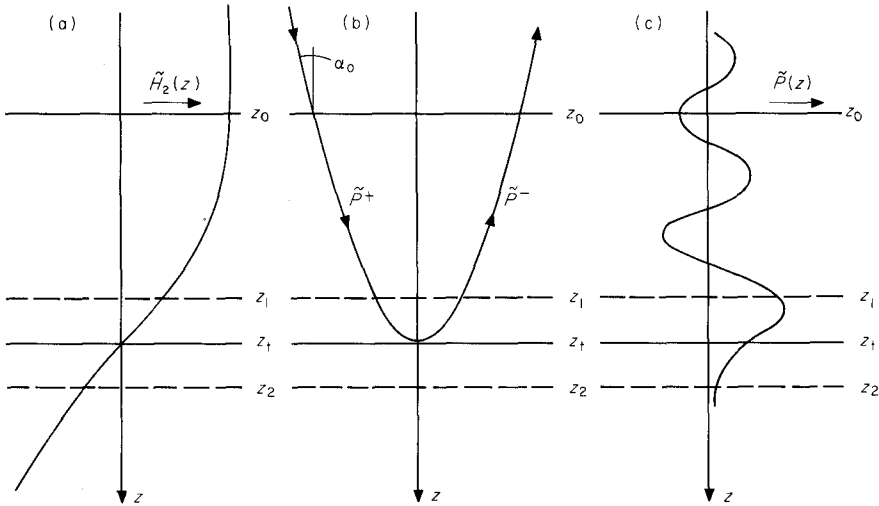


Fig. 2. Waves in continuously layered media, including critical angle events. (a) The operator $\tilde{H}_2(z)$, including a turning point at $z = z_1$. (b) Ray representation. (c) The total wave field $\tilde{P}(z)$.

with propagation angle α_0 , so $\tilde{H}_2(z_0) = \omega^2 c^{-2}(z_0) \cos^2 \alpha_0$. According to (6.1b), $\tilde{H}_2(z)$ decreases monotonously with depth for $z \geq z_0$. We assume that the condition $|\tilde{E}_c + \tilde{E}_\rho| \ll |\tilde{H}_2|$ is violated in the region $z_1 < z < z_2$ and that in this region a turning point is present at $z = z_1$, i.e., $\tilde{H}_2(z_1) = 0$. Furthermore, we assume that $\tilde{H}_2(z)$ may be linearized in the region $z_1 \leq z \leq z_2$, according to $\tilde{H}_2(z) \approx (z - z_1)\chi$. Notice that χ is negative. The operator $\tilde{H}_2(z)$ is shown in fig. 2a.

We assume that for $z_0 \leq z \leq z_1$ the total wave field is given by the superposition of a downgoing and an upgoing wave (see fig. 2b), so that

$$\tilde{P}(z) = \tilde{P}^+(z) + \tilde{P}^-(z),$$

where $\tilde{P}^+(z)$ and $\tilde{P}^-(z)$ are given by the LG solutions (5.7a) and (5.7b).

The amplitudes \tilde{C}^+ and \tilde{C}^- follow from

$$\tilde{C}^+ = (\tilde{H}_1(z_0)/\rho(z_0))^{1/2} \tilde{P}^+(z_0) \tag{6.4a}$$

and

$$\tilde{C}^- = (\tilde{H}_1(z_1)/\rho(z_1))^{1/2} \tilde{P}^-(z_1), \tag{6.4b}$$

where $\tilde{P}^+(z_0)$ is the incident wave. Our aim is to determine $\tilde{P}^-(z_1)$ and to study the total wave field near the turning point $z = z_1$.

Around z_1 the LG solutions can be approximated by

$$\tilde{P}^+(z) = \tilde{D}^+ \left(\frac{\tilde{H}_1(z)}{\rho(z)} \right)^{-1/2} \exp \left[-j \frac{2}{3\chi} \tilde{H}_2^{3/2}(z) \right] \tag{6.5a}$$

and

$$\tilde{P}^-(z) = \tilde{D}^- \left(\frac{\tilde{H}_1(z)}{\rho(z)} \right)^{-1/2} \exp \left[+j \frac{2}{3\chi} \tilde{H}_2^{3/2}(z) \right], \quad (6.5b)$$

with

$$\tilde{D}^+ = [\tilde{H}_1(z_1)/\rho(z_1)]^{1/2} \exp \left[+j \frac{2}{3\chi} \tilde{H}_2^{3/2}(z_1) \right] \tilde{P}^+(z_1) \quad (6.5c)$$

and

$$\tilde{D}^- = [\tilde{H}_1(z_1)/\rho(z_1)]^{1/2} \exp \left[-j \frac{2}{3\chi} \tilde{H}_2^{3/2}(z_1) \right] \tilde{P}^-(z_1), \quad (6.5d)$$

where $\tilde{P}^+(z_1)$ and $\tilde{P}^-(z_1)$ satisfy LG solutions (5.7a) and (5.7b). Around z_1 , expansions (6.5a) and (6.5b) should match a linear combination of the asymptotic expansions ($\zeta \rightarrow -\infty$) of the Airy functions $\text{Ai}(\zeta)$ and $\text{Bi}(\zeta)$, according to

$$\rho^{-1/2}(z)[\tilde{P}^+(z) + \tilde{P}^-(z)] = \tilde{Q}_A \text{Ai}(\zeta) + \tilde{Q}_B \text{Bi}(\zeta), \quad \text{for } z \rightarrow z_1. \quad (6.6)$$

At the other side of the turning point, around z_2 , the same linear combination of Airy functions, expanded for $\zeta \rightarrow \infty$, should again match LG solutions. We shall not discuss this in detail. We only remark that the radiation condition requires that \tilde{Q}_B be zero, since $\text{Bi}(\zeta)$ is exponentially growing for positive ζ . We may conclude that for $z > z_1$ the wave field is proportional to the exponentially decaying function $\text{Ai}(\zeta)$ and its corresponding LG solution away from the turning point. In the following we leave the evanescent field for $z > z_1$ out of consideration.

With the asymptotic expansion of $\text{Ai}(\zeta)$, given by Abramowitz and Stegun (1970; 10.4.60), it follows from (6.6), with $\tilde{Q}_B = 0$, that \tilde{D}^- and \tilde{Q}_A are given by

$$\tilde{D}^- = j\tilde{D}^+, \quad (6.7a)$$

$$\tilde{Q}_A = (1 + j)\sqrt{2\pi}|\chi|^{-1/6}\tilde{D}^+. \quad (6.7b)$$

From (6.5a), (6.5b) and (6.7a) it follows easily that

$$\tilde{P}^-(z_1) = j \exp \left[+2j \frac{2}{3\chi} \tilde{H}_2^{3/2}(z_1) \right] \tilde{P}^+(z_1). \quad (6.8)$$

In the next section we present a more elegant representation of this relation. Apparently, total reflection occurs from the region around the turning point. Since $\tilde{D}^- = j\tilde{D}^+$, the conclusion is sometimes drawn that a phase shift of $\pi/2$ occurs at the turning point. Since (6.5a) and (6.5b) are not valid at the turning point, this conclusion is premature, as will be seen in the next section.

Finally, we conclude that the total wave field in the region $z_1 < z < z_2$ is given by

$$\tilde{P}(z) = \sqrt{\rho(z)} \tilde{Q}_A \text{Ai}(\zeta). \quad (6.9)$$

The total wave field for all z is shown schematically in fig. 2c. We observe a standing wave before the turning point ($z \leq z_1$), and an evanescent wave beyond the turning

point ($z > z_t$). In the above example we assumed a simplified earth model with a continuous velocity function $c(z)$ that is monotonous for all depths, which means that there are no reflecting interfaces, while only one turning point is present in the whole depth range. For seismic applications we assume that the above-described WKBJ technique may be applied *locally* when the velocity function $c(z)$ is monotonously continuous at least in the critical region around the turning point.

Summarizing, the WKBJ approach is suitable for modeling applications of critical angle events. In its present formulation, the WKBJ approach is not suitable for migration applications of critical angle events since in the critical region the *total* wave field is considered. Therefore we will propose a different way of decomposition in the critical region which properly includes critical angle events.

7. DECOUPLED ONE-WAY WAVE EQUATIONS FOR CRITICAL EVENTS

In the previous section we have seen that in the vicinity of a turning point the total wave field may be described in terms of the Airy function $\text{Ai}(\zeta)$. For propagating waves, $\zeta \leq 0$, we formally define a choice of downgoing and upgoing wave functions according to

$$\tilde{\mathcal{P}}^+(z) = \frac{1}{2}\sqrt{\rho(z)} \tilde{Q}_A[\text{Ai}(\zeta) + j \frac{\chi}{|\chi|} \text{Bi}(\zeta)], \quad (7.1a)$$

$$\tilde{\mathcal{P}}^-(z) = \frac{1}{2}\sqrt{\rho(z)} \tilde{Q}_A[\text{Ai}(\zeta) - j \frac{\chi}{|\chi|} \text{Bi}(\zeta)]. \quad (7.1b)$$

These wave functions have also been used by Kennett and Illingworth (1981) for *modeling* applications. In this section we discuss wave field extrapolation operators, based on the above wave functions, which can be used for modeling as well as *migration* applications. Notice that the total wave field, given by

$$\tilde{\mathcal{P}}(z) = \tilde{\mathcal{P}}^+(z) + \tilde{\mathcal{P}}^-(z) = \sqrt{\rho(z)} \tilde{Q}_A \text{Ai}(\zeta), \quad (7.2)$$

satisfies the wave equation, so critical angle events are properly incorporated in $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$. In the following, curled symbols refer to the incorporation of critical angle events in primary waves. We shall now prove that $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$ represent suitable choices for downgoing and upgoing waves, respectively. For a wave in the $+/-z$ direction, the phase should decrease/increase with increasing z . We calculate the derivative of the argument of $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$ in order to determine the propagation direction (see also Abramowitz and Stegun 1970, 10.4.10):

$$\frac{\partial}{\partial z} \left[\pm \frac{\chi}{|\chi|} \text{Bi}(\zeta)/\text{Ai}(\zeta) \right] = \mp |\chi|^{1/3} \frac{\pi^{-1}}{A_1^2(\zeta)} \leq 0. \quad (7.3)$$

We may conclude that $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$ indeed represent downgoing and upgoing waves, respectively, for all $\zeta \leq 0$, where $\zeta = 0$ represents the turning point. Substitution of the asymptotic expansions of $\text{Ai}(\zeta)$ and $\text{Bi}(\zeta)$ for $\zeta \rightarrow -\infty$ in relations (7.1a) and (7.1b) with $\chi < 0$, yields LG approximations (6.5a) and (6.5b).

Differentiation of $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$ with respect to z yields

$$\frac{\partial \tilde{\mathcal{P}}^+}{\partial z} = -j \tilde{\mathcal{H}}_1^+ \tilde{\mathcal{P}}^+ \tag{7.4a}$$

and

$$\frac{\partial \tilde{\mathcal{P}}^-}{\partial z} = +j \tilde{\mathcal{H}}_1^- \tilde{\mathcal{P}}^-, \tag{7.4b}$$

with

$$\tilde{\mathcal{H}}_1^+ = \frac{j}{2\rho} \frac{\partial \rho}{\partial z} - \chi^{1/3} \frac{\left\{ -\frac{\chi}{|\chi|} \pi^{-1} + j \text{Ai}'(\zeta) \text{Ai}'(\zeta) + j \text{Bi}'(\zeta) \text{Bi}'(\zeta) \right\}}{\text{Ai}(\zeta) + \text{Bi}(\zeta)} \tag{7.4c}$$

and

$$\tilde{\mathcal{H}}_1^- = \frac{-j}{2\rho} \frac{\partial \rho}{\partial z} - \chi^{1/3} \frac{\left\{ -\frac{\chi}{|\chi|} \pi^{-1} - j \text{Ai}'(\zeta) \text{Ai}'(\zeta) - j \text{Bi}'(\zeta) \text{Bi}'(\zeta) \right\}}{\text{Ai}(\zeta) + \text{Bi}(\zeta)}, \tag{7.4d}$$

where the primes refer to differentiation with respect to ζ . Notice that, unlike \tilde{P}^+ and \tilde{P}^- , $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$ satisfy *decoupled* one-way wave equations without any further approximation. Consequently all multiple reflections of the conventional approach are incorporated in the primary waves $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$. The underlying assumption is that the medium properties be specified in a given depth interval around the turning point.

At the turning point, $z = z_t$ (or $\zeta = 0$), the ratio of $\tilde{\mathcal{P}}^-$ and $\tilde{\mathcal{P}}^+$ is given by (see also Abramowitz and Stegun 1970, 10.4.4)

$$\frac{\tilde{\mathcal{P}}^-(z_t)}{\tilde{\mathcal{P}}^+(z_t)} = \frac{\text{Ai}(0) - j \frac{\chi}{|\chi|} \text{Bi}(0)}{\text{Ai}(0) + j \frac{\chi}{|\chi|} \text{Bi}(0)} = \exp \left(-j \frac{\chi}{|\chi|} \frac{2\pi}{3} \right). \tag{7.5}$$

Notice that total reflection occurs at the turning point with a phase shift of $2\pi/3$.

From expressions (7.1a) and (7.1b) wave field extrapolation operators for $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$ can be defined according to

$$\tilde{\mathcal{P}}^+(z_{i+1}) = \tilde{\mathcal{W}}^+(z_{i+1}, z_i) \tilde{\mathcal{P}}^+(z_i) \tag{7.6a}$$

and

$$\tilde{\mathcal{P}}^-(z_i) = \tilde{\mathcal{W}}^-(z_i, z_{i+1}) \tilde{\mathcal{P}}^-(z_{i+1}), \tag{7.6b}$$

with

$$\tilde{\mathcal{W}}^+(z_{i+1}, z_i) = \sqrt{\frac{\rho(z_{i+1})}{\rho(z_i)}} \left[\frac{\text{Ai}(\zeta_{i+1}) + j \frac{\chi}{|\chi|} \text{Bi}(\zeta_{i+1})}{\text{Ai}(\zeta_i) + j \frac{\chi}{|\chi|} \text{Bi}(\zeta_i)} \right] \tag{7.6c}$$

and

$$\tilde{\mathcal{W}}^-(z_i, z_{i+1}) = \sqrt{\frac{\rho(z_i)}{\rho(z_{i+1})}} \left[\frac{\text{Ai}(\zeta_i) - j \frac{\chi}{|\chi|} \text{Bi}(\zeta_i)}{\text{Ai}(\zeta_{i+1}) - j \frac{\chi}{|\chi|} \text{Bi}(\zeta_{i+1})} \right], \quad (7.6d)$$

$$\zeta_i = -\chi^{1/3}(z_i - z_t), \quad (7.6c)$$

$$\zeta_{i+1} = -\chi^{1/3}(z_{i+1} - z_t), \quad (7.6f)$$

where both z_i and z_{i+1} are in the vicinity of the turning point z_t , with $z_{i+1} > z_i$. Operators (7.6c) and (7.6d) will be called WKBJ operators. Notice that the inverse operators are given by

$$\tilde{\mathcal{F}}^+(z_i, z_{i+1}) = [\tilde{\mathcal{W}}^+(z_{i+1}, z_i)]^{-1} = [\tilde{\mathcal{W}}^-(z_i, z_{i+1})]^*, \quad (7.7a)$$

$$\tilde{\mathcal{F}}^-(z_{i+1}, z_i) = [\tilde{\mathcal{W}}^-(z_i, z_{i+1})]^{-1} = [\tilde{\mathcal{W}}^+(z_{i+1}, z_i)]^*, \quad (7.7b)$$

where the symbol * refers to complex conjugation. Away from the turning point, the WKBJ operators $\tilde{\mathcal{W}}^+(z_{i+1}, z_i)$ and $\tilde{\mathcal{W}}^-(z_i, z_{i+1})$ reduce to the LG operators $\tilde{W}^+(z_{i+1}, z_i)$ and $\tilde{W}^-(z_i, z_{i+1})$ given by relations (5.11a) and (5.11b). With linearization assumption (5.10), these LG operators are approximate solutions of the Airy equation as well. In fact, they represent the leading term of an asymptotic expansion of the WKBJ operators. A method which uses the Airy equation close to as well as away from the turning point is sometimes called a ‘‘uniform asymptotic approach’’ to the turning point problem. A rigorous *mathematical* treatment of the uniform solution is given by amongst others, Wasow (1965).

Notice that for $\chi = 0$ critical angle events do not occur; the WKBJ operators $\tilde{\mathcal{W}}^+(z_{i+1}, z_i)$ and $\tilde{\mathcal{W}}^-(z_i, z_{i+1})$ then reduce to the phase shift operators for homogeneous layers, as can be easily seen from relations (5.11a), (5.11b), and (5.13). For comparison, the phase shift operators can be written in a way similar to relations (7.6c) and (7.6d), according to

$$\tilde{W}^+(z_{i+1}, z_i) = \frac{\cos(\tilde{H}_1 z_{i+1}) - j \sin(\tilde{H}_1 z_{i+1})}{\cos(\tilde{H}_1 z_i) - j \sin(\tilde{H}_1 z_i)} \quad (7.8a)$$

and

$$\tilde{W}^-(z_i, z_{i+1}) = \frac{\cos(\tilde{H}_1 z_i) + j \sin(\tilde{H}_1 z_i)}{\cos(\tilde{H}_1 z_{i+1}) + j \sin(\tilde{H}_1 z_{i+1})}. \quad (7.8b)$$

Notice that $\tilde{W}^+(z_{i+1}, z_i) = \tilde{W}^-(z_i, z_{i+1})$. The strong resemblance between operators $\tilde{\mathcal{W}}^\pm$ and \tilde{W}^\pm is visualized in fig. 3, where the Airy functions Ai and Bi are compared with the goniometric functions cos and sin, respectively.

Summarizing, in this section we introduced an alternative approach to the decomposition of the total wave field near a turning point into downgoing and upgoing waves $\tilde{\mathcal{F}}^+$ and $\tilde{\mathcal{F}}^-$.

In the conventional approach, \tilde{P}^+ and \tilde{P}^- are coupled for all z according to relations (4.4a) and (4.4b). In the vicinity of the turning point this coupling may not

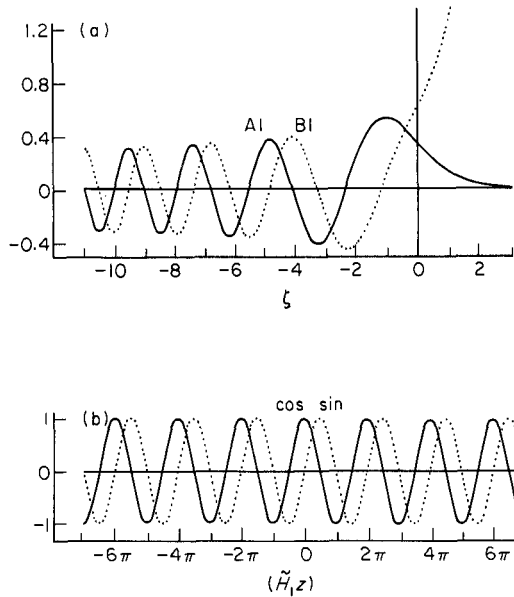


Fig. 3. (a) Graphical representation of the Airy functions $Ai(\zeta)$ and $Bi(\zeta)$. Notice that ζ represents scaled depth: $\zeta = -\chi^{1/3}(z - z_t)$. The turning point depth $z = z_t$ corresponds to $\zeta = 0$. (b) For comparison, graphical representation of the goniometric functions $\cos(\tilde{H}_1 z)$ and $\sin(\tilde{H}_1 z)$.

be neglected. The exact solution requires the incorporation of an infinite number of multiple reflections.

In our alternative approach, $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$ are coupled at the turning point z_t only. This coupling is described by relation (7.5). The exact solution in the critical region is simply given by $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}^+ + \tilde{\mathcal{P}}^-$.

All multiple reflections of the conventional approach are incorporated in the primary waves $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$.

Finally, we consider again the example which was discussed in the previous section. With the decomposition introduced in this section we may write

$$\tilde{\mathcal{P}}^-(z_1) = \tilde{\mathcal{W}}^-(z_1, z_t) \tilde{\mathcal{H}}(z_t) \tilde{\mathcal{W}}^+(z_t, z_1) \tilde{\mathcal{P}}^+(z_1), \tag{7.9a}$$

with

$$\tilde{\mathcal{H}}(z_t) = \exp(j2\pi/3), \tag{7.9b}$$

where we made use of the fact that around z_1 , $\tilde{\mathcal{P}}^+(z)$ and $\tilde{\mathcal{P}}^-(z)$ match the asymptotic expansions of $\tilde{\mathcal{P}}^+(z)$ and $\tilde{\mathcal{P}}^-(z)$, respectively. Relation (7.9) is equivalent to relation (6.8). However, the formulation of (7.9) allows a physical interpretation which links up with the conception of one-way wave propagation. For modeling applications this means that primary waves including critical angle events can be treated independently from interface related multiples ("long period multiples"). For

migration purposes one-way formulation (7.9) can be easily inverted. With the aid of inverse operators (7.7a) and (7.7b) the reflectivity at the turning point can be determined from the downward continued pressure data. Modeling and migration schemes based on the one-way formulation of critical angle events will be discussed in part III of this series.

The operators derived in this section are valid in 1-D inhomogeneous media only. In multi-dimensional linearized inversion schemes, they may describe the propagation of the reference wave field in a 1-D inhomogeneous background medium. Furthermore, it is assumed that no reflecting interfaces are present in the critical region around the turning point. Due to this assumption the applicability of our operators is restricted to simplified earth models with vanishing velocity gradients at layer interfaces.

In part II of this series we will discuss another approach to wave field extrapolation of sub-critical as well as critical angle events, which is based on the two-way wave equation. In the two-way approach arbitrary depth models can be handled, while lateral variations can be incorporated as well.

8. CONCLUSIONS

In principle there are two approaches to modify the wave equation such that wave field extrapolation operators can be derived:

- (i) Decomposition into two first order one-way wave equations for P^+ and P^- , respectively.
- (ii) Reformulation into a first order two-way matrix wave equation for $(P, \rho^{-1} \partial P / \partial z)^T$.

In this part of the paper we discussed methods using the one-way wave equations (methods using the two-way wave equation will be discussed in part II). We have derived two coupled equations for P^+ and P^- which are exact for all propagation angles. We discussed three decoupling approaches:

- A. The *medium is approximated* by a sequence of layers where in each layer the medium properties are functions of the lateral coordinates only. The one-way wave equations decouple exactly in each layer. In principle multiple reflections and transmission effects can be incorporated. When multiple reflections are neglected, then recursive wave field extrapolation of primary waves is valid for sub-critical angle events only.
- B. The *one-way wave equations are approximated* by neglecting multiple reflections ($\Gamma^+ = \Gamma^- = 0$). This decoupling approach is valid for sub-critical angle events only. For 1-D inhomogeneous media, solutions are given by the Liouville–Green approximations. We have shown that approaches A and B are equivalent. The basic relation for modeling and migration of sub-critical angle data reads (section 5)

$$\tilde{P}^- = \left[\sum \tilde{W}^- \tilde{R} \tilde{W}^+ \right] \tilde{P}^+.$$

C. For critical angle events, the operator H_2 is linearized in depth. The total field can be solved according to the WKB approach. We decomposed the total field into a new choice of downgoing and upgoing primary waves $\tilde{\mathcal{P}}^+$ and $\tilde{\mathcal{P}}^-$ which correctly include critical angle events. This alternative decoupling approach is suitable for both *modeling* and *migration* of primary waves, including critical angle events. The basic relation for modeling and migration of sub-critical and critical angle data reads (section 7)

$$\tilde{\mathcal{P}}^- = [\sum \tilde{\mathcal{W}}^- \tilde{\mathcal{R}} \tilde{\mathcal{W}}^+] \tilde{\mathcal{P}}^+,$$

where the operators may describe sub-critical as well as critical angle events. The schemes will be further discussed in part III.

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APPENDIX A

In this appendix we study the relations (5.5c) and (5.5d) for $\Delta z \rightarrow 0$:

$$\lim_{\Delta z \rightarrow 0} \tilde{\mathcal{W}}^+(z_m, z_0) = \lim_{\Delta z \rightarrow 0} \left[\left(\prod_{i=1}^{m-1} \tilde{\mathcal{T}}^+(z_i) \right) \left(\prod_{i=1}^m \tilde{\mathcal{W}}^+(z_i, z_{i-1}) \right) \right] \quad (\text{A.1a})$$

and

$$\lim_{\Delta z \rightarrow 0} \tilde{\mathcal{W}}^-(z_0, z_m) = \lim_{\Delta z \rightarrow 0} \left[\left(\prod_{i=1}^{m-1} \tilde{\mathcal{T}}^-(z_i) \right) \left(\prod_{i=1}^m \tilde{\mathcal{W}}^-(z_{i-1}, z_i) \right) \right], \quad (\text{A.1b})$$

with $\Delta z = z_i - z_{i-1}$.

Notice that z_m is fixed, so $m \rightarrow \infty$ when $\Delta z \rightarrow 0$. Here we consider relation (A.1a) for propagating waves only ($\tilde{\mathcal{H}}_1^2 \geq 0$). According to relation (3.3b), operator $\tilde{\mathcal{W}}^+(z_i, z_{i-1})$ is given by

$$\tilde{\mathcal{W}}^+(z_i, z_{i-1}) = \exp [-j\tilde{\mathcal{H}}_1(z_{i-1})\Delta z]. \quad (\text{A.2})$$

By solving the boundary conditions at $z = z_i$ ($\tilde{\mathcal{P}}$ continuous, $\tilde{\mathcal{V}}_z$ continuous), it follows easily that the transmission operator $\tilde{\mathcal{T}}^+(z_i)$ is given by (Berkhout 1982)

$$\tilde{\mathcal{T}}^+(z_i) = 1 + \tilde{\mathcal{R}}(z_i), \quad (\text{A.3a})$$

with

$$\tilde{\mathcal{R}}(z_i) = \frac{\rho(z_i)\tilde{\mathcal{H}}_1(z_{i-1}) - \rho(z_{i-1})\tilde{\mathcal{H}}_1(z_i)}{\rho(z_i)\tilde{\mathcal{H}}_1(z_{i-1}) + \rho(z_{i-1})\tilde{\mathcal{H}}_1(z_i)}. \quad (\text{A.3b})$$

Assuming a continuously layered medium, we may rewrite $\tilde{T}^+(z_i)$ for $\Delta z = z_i - z_{i-1} \rightarrow 0$ according to

$$\lim_{\Delta z \rightarrow 0} \tilde{T}^+(z_i) = 1 + \left[\frac{1}{2\rho} \frac{\partial \rho}{\partial z} \Big|_{z_i} - \frac{1}{2\tilde{H}_1} \frac{\partial \tilde{H}_1}{\partial z} \Big|_{z_i} \right] \Delta z, \quad (\text{A.4})$$

assuming $\tilde{H}_1(z_i) \neq 0$.

Substitution of (A.2) and (A.4) in relation (A.1a), assuming $\tilde{H}_1^2(z) > 0$ for $z_0 < z < z_m$, yields

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \tilde{W}^+(z_m, z_0) &= \exp \left[\lim_{\Delta z \rightarrow 0} \left\{ \sum_{i=1}^{m-1} \ln \tilde{T}^+(z_i) + \sum_{i=1}^m \ln \tilde{W}^+(z_i, z_{i-1}) \right\} \right] \\ &= \exp \int_{z_0}^{z_m} \left\{ \frac{1}{2\rho} \frac{\partial \rho}{\partial z} - \frac{1}{2\tilde{H}_1} \frac{\partial \tilde{H}_1}{\partial z} - j\tilde{H}_1 \right\} dz \\ &= \exp \int_{z_0}^{z_m} \left[-\frac{\partial}{\partial z} \ln \sqrt{\frac{\tilde{H}_1}{\rho}} \right] dz \exp \int_{z_0}^{z_m} -j\tilde{H}_1 dz, \end{aligned}$$

from which relation (5.6a) follows immediately. Relation (5.6b) can be derived accordingly. Similar relations can be derived for evanescent waves ($\tilde{H}_1^2 < 0$ for $z_0 < z < z_m$). This is beyond the scope of this paper.

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ERRATA

WAVE FIELD EXTRAPOLATION TECHNIQUES FOR INHOMOGENEOUS MEDIA WHICH INCLUDE CRITICAL ANGLE EVENTS, PARTS I-III

by C.P.A. WAPENAAR and A.J. BERKHOUT

Part I: Methods Using the One-Way Wave Equations, vol. 33, no. 8, December 1985, pp. 1138–1159.

Page 1153, relation (7.4b): \tilde{P}^- should read as $\tilde{\mathcal{P}}^-$.

Part II: Methods Using the Two-Way Equation, vol. 34, no. 2, April 1986, pp. 147–179.

Page 156: relations (3.6a) and (3.6b) should read as

$$\tilde{W}(z, z_0) = \mathbf{I} + \tilde{\mathbf{L}}(\tilde{\mathbf{A}} \Delta z)\tilde{\mathbf{L}}^{-1} + \frac{1}{2} \tilde{\mathbf{L}}(\tilde{\mathbf{A}} \Delta z)\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{L}}(\tilde{\mathbf{A}} \Delta z)\tilde{\mathbf{L}}^{-1} + \dots, \quad (3.6a)$$

$$\tilde{W}(z, z_0) = \tilde{\mathbf{L}}[\mathbf{I} + (\tilde{\mathbf{A}} \Delta z) + \frac{1}{2} (\tilde{\mathbf{A}} \Delta z)^2 + \dots]\tilde{\mathbf{L}}^{-1}. \quad (3.6b)$$

Part III: Applications in Modeling, Migration and Inversion, vol. 34, no. 2, April 1986, pp. 180–207.

Page 184, last line: (3.9) should read as (3.12).

Page 190–197: the symbol * should read as a superscript (denoting complex conjugation) in the following relations: (5.4), (5.6), (6.7), (6.8) and (6.9).