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Reciprocity theorems for one-way wavefields

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SUMMARY

Acoustic reciprocity theorems have proved their usefulness in the study of forward and inverse scattering problems. The reciprocity theorems in the literature apply to the *two-way* (i.e. total) wavefield, and are thus not compatible with *one-way* wave theory, which is often applied in seismic exploration. By transforming the two-way wave equation into a coupled system of one-way wave equations for downgoing and upgoing waves it appears to be possible to derive 'one-way reciprocity theorems' along the same lines as the usual derivation of the 'two-way reciprocity theorems'. However, for the one-way reciprocity theorems it is not directly obvious that the 'contrast term' vanishes when the medium parameters in the two different states are identical. By introducing a modal expansion of the Helmholtz operator, its square root can be derived, which appears to have a symmetric kernel. This symmetry property appears to be sufficient to let the contrast term vanish in the above-mentioned situation.

The one-way reciprocity theorem of the convolution type is exact, whereas the one-way reciprocity theorem of the correlation type ignores evanescent wave modes. The extension to the elastodynamic situation is not trivial, but it can be shown relatively easily that similar reciprocity theorems apply if the (non-unique) decomposition of the elastodynamic two-way operator is done in such a way that the elastodynamic one-way operators satisfy similar symmetry properties to the acoustic one-way operators.

Key words: elastic-wave theory, spectral analysis, wave equation.

INTRODUCTION

An acoustic reciprocity theorem relates the sources and wavefields in two admissable acoustic states in the same domain. It can be obtained by inserting the acoustic wave equation into an extended version of Green's theorem (Rayleigh 1878; Morse & Feshbach 1953). In the modern literature, reciprocity theorems also account for possible differences between the medium parameters in the two states (de Hoop 1988).

It is possible to distinguish between convolution-type and correlation-type reciprocity theorems (Bojarski 1983). These two types of reciprocity theorems have proved their usefulness in the study of forward and inverse scattering problems, respectively. An extensive overview of reciprocity and its applications in seismics is given by Fokkema & van den Berg (1993).

In the cited references the reciprocity theorems apply to the *two-way* (i.e. total) wavefield. Therefore in this paper we refer to these theorems as 'two-way reciprocity theorems'. Obviously, these two-way reciprocity theorems are not compatible with *one-way* wave theory. The latter theory is particularly suited for those acoustic disciplines in which there is a clear preferred direction of propagation, such as in seismic exploration, where the vertical direction is the preferred propagation direction. For this situation the coupled one-way wave equations distinguish explicitly between downward and upward propagation. The transformation from downward- to upward-propagating waves is described by a separate reflection operator, which is proportional to the vertical variations of the medium parameters. Note that in the seismic situation these vertical variations (due to layering) are much more pronounced than the horizontal variations. This explains why one-way wave theory is so well suited to seismic applications. For a recent discussion on reflection imaging based on this concept, see Berkhout & Wapenaar (1993).

In this paper we derive reciprocity theorems of the convolution type and of the correlation type for acoustic one-way wavefields. These reciprocity theorems honour the natural separation between propagation and scattering in the one-way wave equations. They form an alternative basis for the study of forward and inverse scattering problems. In particular, they allow new data representations. In a companion paper (Wapenaar 1996; hereafter referred to as Paper B) we will derive, amongst others, a 3-D generalized primary representation, which appears to be very useful in the study of forward and inverse scattering problems in continuous 3-D finely layered media.

THE TWO-WAY WAVE EQUATION

In this section we give the basic equations for an acoustic wavefield in a lossless inhomogeneous fluid medium. The medium parameters are infinitely differentiable functions of position, and time-invariant. The Cartesian position coordinates are denoted by the vector $\mathbf{x}=(x_1, x_2, x_3)$ and the x_3-axis is pointing downwards. The time coordinate is denoted by t. The Fourier transform with respect to time of a real function is defined as

$$U(\omega)=\int_{-\infty}^{\infty} u(t)\exp(-j\omega t)\,dt, \tag{1}$$

and its inverse as

$$u(t)=\frac{1}{\pi}\mathcal{R}e\left[\int_0^{\infty} U(\omega)\exp(j\omega t)\,d\omega\right], \tag{2}$$

where j is the imaginary unit and ω denotes the angular frequency. Note that we consider positive frequencies only. In the remainder of this paper, all functions are in the frequency domain; the ω-dependence is not explicitly denoted.

In the space–frequency domain, the equations that govern linear acoustic wave propagation read

$$\partial_k P + j\omega\varrho V_k = F_k \tag{3}$$

and

$$\partial_k V_k + \frac{j\omega}{K}P = Q, \tag{4}$$

where P is the acoustic pressure, V_k is the particle velocity, ϱ is the volume density of mass, K is the compression modulus, F_k is the volume source density of volume force, and Q is the volume source density of volume injection rate. The Latin subscripts take the values 1 to 3 and the summation convention applies to repeated subscripts.

Throughout this paper the direction of preference is taken along the x_3-axis. It is thus useful to express the vertical variations of the wavefield in terms of the horizontal variations of the same wavefield. To this end we separate the vertical derivatives $\partial_3 P$ and $\partial_3 V_3$ from the horizontal derivatives, and we eliminate V_1 and V_2. The resulting two equations for P and V_3 are combined into one matrix–vector equation, according to

$$\partial_3\mathbf{Q}-\hat{\mathbf{A}}\mathbf{Q}=\mathbf{D}. \tag{5}$$

We refer to this as the *two-way wave equation*. The two-way wave vector $\mathbf{Q}$ and the two-way source vector $\mathbf{D}$ are defined as

$$\mathbf{Q}=\begin{pmatrix}P\\ V_3\end{pmatrix}, \qquad \mathbf{D}=\begin{pmatrix}F_3\\ Q-\frac{1}{j\omega}\partial_\alpha\left(\frac{1}{\varrho}F_\alpha\right)\end{pmatrix}. \tag{6}$$

Greek subscripts take the values 1 and 2. The two-way operator matrix $\hat{\mathbf{A}}$ is defined as

$$\hat{\mathbf{A}}=\begin{pmatrix}0 & -j\omega\varrho\\ \frac{1}{j\omega\varrho}\hat{H}_2 & 0\end{pmatrix}, \tag{7}$$

where

$$\hat{H}_2=\frac{\omega^2}{c^2}+\varrho\partial_\alpha\left(\frac{1}{\varrho}\partial_\alpha\cdot\right), \tag{8}$$

and where $c=(K/\varrho)^{1/2}$ denotes the acoustic propagation velocity. The circumflex denotes an *operator* containing the horizontal differentiation operators ∂_1 and ∂_2. For later convenience we introduce a modified operator $\hat{\mathcal{H}}_2$, according to

$$\hat{\mathcal{H}}_2=\varrho^{-1/2}(\hat{H}_2\varrho^{1/2}\cdot), \tag{9}$$

or, using eq. (8),

$$\hat{\mathcal{H}}_2=\left(\frac{\omega}{c'}\right)^2+\partial_\alpha\partial_\alpha, \tag{10}$$

where

$$\left(\frac{\omega}{c'}\right)^2=\left(\frac{\omega}{c}\right)^2-\frac{3(\partial_\alpha\varrho)(\partial_\alpha\varrho)}{4\varrho^2}+\frac{(\partial_\alpha\partial_\alpha\varrho)}{2\varrho}, \tag{11}$$

(Brekhovskikh 1960; Wapenaar & Berkhout 1989; de Hoop 1992). The structure of the Helmholtz operator $\hat{\mathcal{H}}_2$ in eq. (10) is clearly simpler than the structure of $\hat{H}_2$ in eq. (8). With the Helmholtz operator $\hat{\mathcal{H}}_2$ we can write, instead of eq. (7),

$$\hat{\mathbf{A}}=\begin{pmatrix}0 & -j\omega\varrho\\ \frac{1}{j\omega\varrho^{1/2}}(\hat{\mathcal{H}}_2\varrho^{-1/2}\cdot) & 0\end{pmatrix}. \tag{12}$$

DECOMPOSITION OF THE TWO-WAY OPERATOR

Analogous to the decomposition approach in horizontally layered media (see Ursin 1983 for an overview), we introduce operator matrices $\hat{\mathbf{\Lambda}}$, $\hat{\mathbf{L}}$ and $\hat{\mathbf{L}}^{-1}$, which satisfy the relation

$$\hat{\mathbf{A}}=-j\omega\hat{\mathbf{L}}\hat{\mathbf{\Lambda}}\hat{\mathbf{L}}^{-1} \tag{13}$$

in such a way that $\hat{\mathbf{\Lambda}}$ is diagonal. For an extensive list of references on the theoretical and numerical aspects of these operators, see Fishman, McCoy & Wales (1987). Some recent references in the seismic context were given in the previous section. Due to the anti-diagonal structure of $\hat{\mathbf{A}}$, as defined in eq. (12), the vertical slowness operator $\hat{\mathbf{\Lambda}}$ and the composition and decomposition operators $\hat{\mathbf{L}}$ and $\hat{\mathbf{L}}^{-1}$ have the following structures:

$$\hat{\mathbf{\Lambda}}=\begin{pmatrix}\hat{\Lambda} & 0\\ 0 & -\hat{\Lambda}\end{pmatrix} \tag{14}$$

and

$$\hat{\mathbf{L}}=\begin{pmatrix}\hat{L}_1 & \hat{L}_1\\ \hat{L}_2 & -\hat{L}_2\end{pmatrix}, \qquad \hat{\mathbf{L}}^{-1}=\frac{1}{2}\begin{pmatrix}\hat{L}_1^{-1} & \hat{L}_2^{-1}\\ \hat{L}_1^{-1} & -\hat{L}_2^{-1}\end{pmatrix}, \tag{15}$$

where the operators $\hat{\Lambda}$, $\hat{L}_1$ and $\hat{L}_2$ satisfy the relations

$$-j\omega\varrho=-j\omega\hat{L}_1\hat{\Lambda}\hat{L}_2^{-1} \tag{16}$$

and

$$\frac{1}{j\omega\varrho^{1/2}}(\hat{\mathcal{H}}_2\varrho^{-1/2}\cdot)=-j\omega\hat{L}_2\hat{\Lambda}\hat{L}_1^{-1}. \tag{17}$$

It can be verified by substitution that for $\hat{\Lambda}$, $\hat{L}_1$, $\hat{L}_1^{-1}$, $\hat{L}_2$ and

$\hat{L}_2^{-1}$ we can write

$$\hat{A} = \omega^{-1}\hat{\mathscr{H}}_1, \tag{18}$$

$$\hat{L}_1 = \left(\frac{\omega\varrho}{2}\right)^{1/2}\hat{\mathscr{H}}_1^{-1/2}, \qquad \hat{L}_1^{-1} = \left(\frac{\omega}{2}\right)^{-1/2}(\hat{\mathscr{H}}_1^{1/2}\varrho^{-1/2}\cdot), \tag{19}$$

$$\hat{L}_2 = (2\omega\varrho)^{-1/2}\hat{\mathscr{H}}_1^{1/2}, \qquad \hat{L}_2^{-1} = (2\omega)^{1/2}(\hat{\mathscr{H}}_1^{-1/2}\varrho^{1/2}\cdot), \tag{20}$$

where the square-root operator $\hat{\mathscr{H}}_1$ is related to the Helmholtz operator $\hat{\mathscr{H}}_2$ according to

$$\hat{\mathscr{H}}_2 = \hat{\mathscr{H}}_1\hat{\mathscr{H}}_1. \tag{21}$$

The normalization of this decomposition has been chosen such that it is consistent with the usual flux normalization in horizontally layered media (de Hoop 1992). This could only be accomplished as a result of the introduction of the modified operator $\hat{\mathscr{H}}_2$ in eq. (9).

Unlike $\hat{\mathscr{H}}_2$, the square-root operator $\hat{\mathscr{H}}_1$ cannot be written as a polynomial in ∂_α. Therefore $\hat{\mathscr{H}}_1$ is a so-called pseudo-differential operator (Kumano-go 1974). Eq. (21) is further evaluated in a later section.

THE ONE-WAY WAVE EQUATION

We introduce a one-way wave vector $\mathbf{P}$ and a one-way source vector $\mathbf{S}$, according to

$$\mathbf{P} = \begin{pmatrix} P^+ \\ P^- \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}, \tag{22}$$

where P^+ and P^- represent downgoing and upgoing waves, respectively, and where S^+ and S^- represent source functions for downgoing and upgoing waves, respectively. We relate the one-way wave and source vectors to the two-way vectors, according to

$$\mathbf{Q} = \hat{\mathbf{L}}\mathbf{P} \quad \text{and} \quad \mathbf{D} = \hat{\mathbf{L}}\mathbf{S}. \tag{23}$$

Substitution of eqs (13) and (23) into the two-way wave equation (5) yields the one-way wave equation

$$\partial_3\mathbf{P} - \hat{\mathbf{B}}\mathbf{P} = \mathbf{S}, \tag{24}$$

where the one-way operator matrix $\hat{\mathbf{B}}$ is defined as

$$\hat{\mathbf{B}} = -j\omega\hat{\mathbf{\Lambda}} + \hat{\mathbf{\Theta}}, \tag{25}$$

with

$$\hat{\mathbf{\Theta}} = -\hat{\mathbf{L}}^{-1}\partial_3\hat{\mathbf{L}}. \tag{26}$$

From the structure of eqs (14), (22) and (24) to (26) it follows that $-j\omega\hat{\mathbf{\Lambda}}$ accounts for (downward/upward) propagation and $\hat{\mathbf{\Theta}}$ for scattering due to the vertical variations of the medium parameters (see Table 1). Note that both $\hat{\mathbf{\Lambda}}$ and $\hat{\mathbf{\Theta}}$ also account implicitly for scattering due to the horizontal variations of the medium parameters. In the Earth's subsurface, the vertical variations are much more important than the horizontal variations. Therefore, in the remainder of this paper 'scattering' denotes 'scattering due to the vertical variations'.

Table 1. Propagation and scattering in the one-way operators.

	$-j\omega\hat{\mathbf{\Lambda}}$	$\hat{\mathbf{\Theta}}$
Propagation	×	
Vertical scattering		×
Horizontal scattering	×	×

The explicit distinction between propagation and scattering is an important advantage of the one-way wave equation (24) over the two-way wave equation (5). This property is exploited in Paper B in the derivation of the one-way primary representation and the one-way generalized primary representation.

RECIPROCITY THEOREM OF THE CONVOLUTION TYPE

The aim of this section is to derive a reciprocity theorem of the convolution type for one-way wavefields. In principle, two approaches can be followed. One can start with the reciprocity theorem for two-way wavefields and apply a decomposition to the wavefields appearing in this theorem. Alternatively, one can start with the one-way wave equation and derive the one-way reciprocity theorem along the same lines as de Hoop (1988) and Fokkema & van den Berg (1993) use in their derivation of the two-way reciprocity theorem. We used the former approach previously in the derivation of one-way Kirchhoff integrals (Berkhout & Wapenaar 1989; Wapenaar *et al.* 1989). In the present paper we will use the latter approach because it appears to yield more general results.

In Table 2 we introduce two different states that are distinguished by the subscripts A and B. We will consider the interaction between downgoing waves in one state and upgoing waves in the other and vice versa (Fig. 1). To be more specific, we consider the one-way interaction quantity

$$\partial_3\{P_A^+ P_B^- - P_A^- P_B^+\}. \tag{27}$$

[For comparison, de Hoop (1988) and Fokkema & van den Berg (1993) consider the two-way interaction quantity $\partial_k\{P_A V_{k,B} - V_{k,A} P_B\}$]. To simplify the notation, we rewrite the interaction quantity (27) as

$$\partial_3\{\mathbf{P}_A^T\mathbf{N}\mathbf{P}_B\}, \tag{28}$$

Table 2. States in the reciprocity theorems.

	State A	State B
Wavefield	$\mathbf{P}_A(\mathbf{x})$	$\mathbf{P}_B(\mathbf{x})$
Operator	$\hat{\mathbf{B}}_A(\mathbf{x})$	$\hat{\mathbf{B}}_B(\mathbf{x})$
Source	$\mathbf{S}_A(\mathbf{x})$	$\mathbf{S}_B(\mathbf{x})$

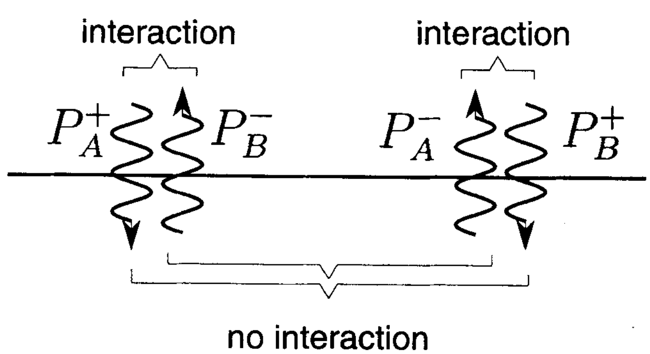

Figure 1. Both terms of the interaction quantity for the reciprocity theorem of the convolution type contain waves that propagate in opposite directions.

where $^{\mathrm{T}}$ denotes transposition and

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{29}$$

Using the one-way wave equation (24), this interaction quantity can be rewritten as

$$\partial_3\{\mathbf{P}_A^{\mathrm{T}}\mathbf{N}\mathbf{P}_B\} = \mathbf{P}_A^{\mathrm{T}}\{\hat{\mathbf{B}}_A^{\mathcal{T}}\mathbf{N} + \mathbf{N}\hat{\mathbf{B}}_B\}\mathbf{P}_B + \mathbf{P}_A^{\mathrm{T}}\mathbf{N}\mathbf{S}_B + \mathbf{S}_A^{\mathrm{T}}\mathbf{N}\mathbf{P}_B. \tag{30}$$

In this notation the transposed operator $\hat{\mathbf{B}}_A^{\mathcal{T}}$ acts upon the quantity left of it, i.e., on $\mathbf{P}_A^{\mathrm{T}}$. In analogy with de Hoop (1988), we call this a *local* reciprocity theorem. Next we integrate both sides of this equation over some volume to obtain a *global* reciprocity theorem. Since we have defined the direction of preference along the x_3-axis, a natural choice is a volume $\mathcal{V}$, enclosed by two infinite parallel surfaces perpendicular to the x_3-axis, see Fig. 2. These surfaces need not be physical boundaries. The combination of these surfaces will be denoted by $\partial\mathcal{V}$ and the outward-pointing normal vector by $\mathbf{n} = (0, 0, n_3)$, with $n_3 = -1$ at the upper surface and $n_3 = +1$ at the lower surface. Carrying out the volume integrations and applying Gauss's theorem yields

$$\int_{\partial\mathcal{V}} \mathbf{P}_A^{\mathrm{T}}\mathbf{N}\mathbf{P}_B n_3 d^2\mathbf{x}_{\mathrm{H}} = \int_{\mathcal{V}} \mathbf{P}_A^{\mathrm{T}}\mathbf{N}\hat{\mathbf{\Delta}}\mathbf{P}_B d^3\mathbf{x} + \int_{\mathcal{V}} \{\mathbf{P}_A^{\mathrm{T}}\mathbf{N}\mathbf{S}_B + \mathbf{S}_A^{\mathrm{T}}\mathbf{N}\mathbf{P}_B\}\, d^3\mathbf{x}, \tag{31}$$

where the contrast function $\hat{\mathbf{\Delta}}$ is given by

$$\hat{\mathbf{\Delta}} = \hat{\mathbf{B}}_B - (-\mathbf{N}^{-1}\hat{\mathbf{B}}_A^{\mathcal{T}}\mathbf{N}) \tag{32}$$

and $\mathbf{x}_{\mathrm{H}} = (x_1, x_2)$ denotes the horizontal coordinates. Note that this global reciprocity theorem has the usual form of a boundary integral over the interaction quantity on one side and volume integrals containing a contrast function and sources on the other side. The main difference with respect to two-way reciprocity theorems (de Hoop 1988; Fokkema & van den Berg 1993) is that here the wavefields are one-way wavefields and that the contrast function is actually a contrast *operator*. Moreover, it is not directly obvious that the volume integral over this contrast operator vanishes when the medium parameters in state A and state B are identical. In the following sections we will show step by step that the contrast operator $\hat{\mathbf{\Delta}}$ is equivalent to $\hat{\mathbf{\Delta}} = \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A$. It is clear that in this form the first volume integral in eq. (31) indeed vanishes when the medium parameters in state A and state B are identical.

SYMMETRY PROPERTIES OF THE BASIC OPERATORS

In order to modify the contrast operator $\hat{\mathbf{\Delta}}$ it is necessary to derive symmetry properties of the one-way operator $\hat{\mathbf{B}}$. All

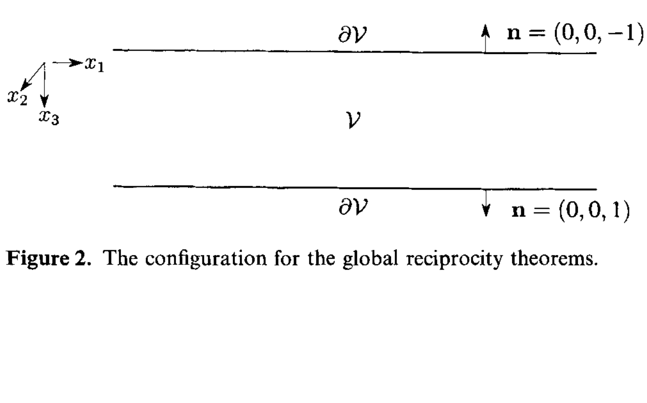

Figure 2. The configuration for the global reciprocity theorems.

entries in $\hat{\mathbf{B}}$ are related in one way or other to the square-root operator $\hat{\mathscr{H}}_1$ which, in turn, is related to the Helmholtz operator $\hat{\mathscr{H}}_2$, according to $\hat{\mathscr{H}}_2 = \hat{\mathscr{H}}_1\hat{\mathscr{H}}_1$. In this section we analyse the symmetry properties of the kernels of these scalar operators. In the next section we derive the symmetry properties of the kernel of the operator matrix $\hat{\mathbf{B}}$.

We start our analysis by deriving a modal expansion of the Helmholtz operator $\hat{\mathscr{H}}_2$ as defined in eq. (10). We introduce the wavenumber $k(\mathbf{x}) = \omega/c'(\mathbf{x})$ and we assume that $k^2(\mathbf{x}) - k_0^2(x_3)$ [where $k_0(x_3)$ is the wavenumber for the homogeneous embedding at x_3] has finite lateral support at depth level x_3. We can thus rewrite the Helmholtz operator as

$$\hat{\mathscr{H}}_2(\mathbf{x}) = k_0^2(x_3) - \underbrace{[\overbrace{\{k_0^2(x_3) - k^2(\mathbf{x}_{\mathrm{H}}, x_3)\}}^{\text{potential } U(\mathbf{x}_{\mathrm{H}})} - \partial_\alpha\partial_\alpha]}_{\text{2-D Hamiltonian, } (x_3 \text{ fixed})}. \tag{33}$$

The term between the rectangular brackets has the form of a 2-D Hamiltonian, which plays an important role in non-relativistic quantum mechanics. In the following analysis of $\hat{\mathscr{H}}_2$ we will lean upon the well-developed theory of the Hamiltonian. The x_3-dependence is irrelevant for this analysis. For notational convenience we omit x_3 in the remainder of this section.

We introduce an eigenvalue λ and the corresponding 2-D eigenfunction ϕ, according to

$$\hat{\mathscr{H}}_2\phi = \lambda\phi. \tag{34}$$

If $\hat{\mathscr{H}}_2$ is examined as an operator acting on a properly chosen subspace of the Hilbert space, it turns out to be self-adjoint, implying that all λs in the spectrum $\sigma(\hat{\mathscr{H}}_2)$ are real-valued (Weidman 1980). Similar to the spectrum of the Hamiltonian, $\sigma(\hat{\mathscr{H}}_2)$ generally consists of a discrete and a continuous part:

$$\sigma(\hat{\mathscr{H}}_2) = \sigma_{\mathrm{discr}}(\hat{\mathscr{H}}_2) \cup \sigma_{\mathrm{cont}}(\hat{\mathscr{H}}_2). \tag{35}$$

All that actually matters for our analysis is that this spectrum is real-valued. Therefore we skip its derivation. We only remark that it can be obtained by mirroring the spectrum of the Hamiltonian around the origin and shifting it to the right over a distance k_0^2. As a result we find that $\sigma_{\mathrm{discr}}(\hat{\mathscr{H}}_2)$ contains a finite number of isolated eigenvalues on the positive real axis between k_0^2 and $\max\{k^2(\mathbf{x}_{\mathrm{H}})\}$, and $\sigma_{\mathrm{cont}}(\hat{\mathscr{H}}_2)$ covers the interval $(-\infty, k_0^2]$, see Fig. 3. The positive and negative eigenvalues

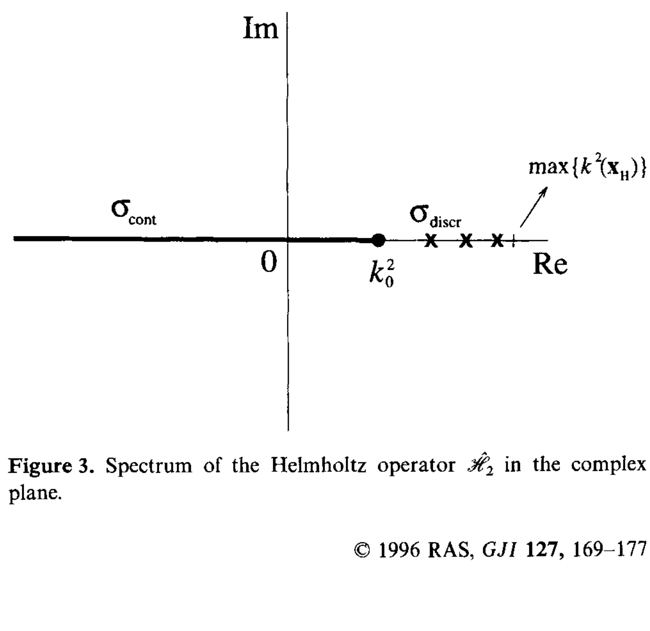

Figure 3. Spectrum of the Helmholtz operator $\hat{\mathscr{H}}_2$ in the complex plane.

correspond to propagating and evanescent wave modes, respectively. The discrete eigenvalues correspond to guided modes; they vanish in a laterally homogeneous medium. For a more elaborate discussion on the modal decomposition of wavefields the reader is referred to Blok (1995) and Grimbergen, Wapenaar & Dessing (1996).

As a result of the spectral theorem for self-adjoint operators (Reed & Simon 1972; Weidman 1980), the eigenfunctions ϕ of $\hat{\mathscr{H}}_2$ constitute a complete orthonormal basis in 2-D space. This means that we can expand an arbitrary function $F(\mathbf{x}_H)$ in the chosen subspace in terms of these eigenfunctions, according to (Reed & Simon 1979; Section XI.6):

$$F(\mathbf{x}_H) = \int_{\mathbb{R}^2} \phi(\mathbf{x}_H, \boldsymbol{\kappa}) \bar{F}(\boldsymbol{\kappa})\, d^2\boldsymbol{\kappa} + \sum_{\lambda_i \in \sigma_{\mathrm{discr}}} \phi^{(i)}(\mathbf{x}_H) \bar{F}^{(i)}, \tag{36}$$

where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ denotes a vector which relates the eigenfunctions $\phi(\mathbf{x}_H, \boldsymbol{\kappa})$ to the corresponding eigenvalues in the continuous spectrum via

$$\lambda(\boldsymbol{\kappa}) = k_0^2 - \kappa_\nu \kappa_\nu, \qquad \lambda \in \sigma_{\mathrm{cont}}. \tag{37}$$

In eq. (36), the first term on the right-hand side is understood in the distribution sense (analogous to a Fourier transformation). The expansion coefficients $\bar{F}(\boldsymbol{\kappa})$ and $\bar{F}^{(i)}$ can be seen as a representation of $F(\mathbf{x}_H)$ in the modal domain, which is defined as the space constituted by the eigenfunctions of $\hat{\mathscr{H}}_2$. The transformation to this domain is given by

$$\bar{F}(\boldsymbol{\kappa}) = \int_{\mathbb{R}^2} F(\mathbf{x}_H) \phi^*(\mathbf{x}_H, \boldsymbol{\kappa}) d^2\mathbf{x}_H, \tag{38}$$

$$\bar{F}^{(i)} = \int_{\mathbb{R}^2} F(\mathbf{x}_H) \phi^{(i)}(\mathbf{x}_H) d^2\mathbf{x}_H, \tag{39}$$

where * denotes complex conjugation [the eigenfunctions $\phi^{(i)}(\mathbf{x}_H)$ corresponding to the discrete eigenvalues λ_i are real-valued].

Since $\hat{\mathscr{H}}_2$ and λ are real-valued, the eigenfunctions $\phi(\mathbf{x}_H, \boldsymbol{\kappa})$ can be chosen to be either complex- or real-valued. We discuss these choices for a laterally invariant medium. For this situation we obtain

$$\phi(\mathbf{x}_H, \boldsymbol{\kappa}) = (2\pi)^{-1} \exp(-j\kappa_\nu x_\nu) \tag{40}$$

or

$$\phi(\mathbf{x}_H, \boldsymbol{\kappa}) = (\pi\sqrt{2})^{-1} \cos(\kappa_\nu x_\nu - \pi/4), \tag{41}$$

respectively. With the former choice, eqs (36) and (38) represent, respectively, inverse and forward 2-D spatial Fourier transformations. With the latter choice they represent inverse and forward 2-D spatial Hartley transformations (Bracewell 1986).

We return to the situation of laterally variant media. The modal expansion of $\hat{\mathscr{H}}_2$ is now obtained by applying $\hat{\mathscr{H}}_2$ to the left- and right-hand sides of eq. (36), which yields (using eq. 34)

$$\hat{\mathscr{H}}_2(\mathbf{x}_H) F(\mathbf{x}_H) = \int_{\mathbb{R}^2} \lambda(\boldsymbol{\kappa}) \phi(\mathbf{x}_H, \boldsymbol{\kappa}) \bar{F}(\boldsymbol{\kappa})\, d^2\boldsymbol{\kappa} + \sum_{\lambda_i \in \sigma_{\mathrm{discr}}} \lambda_i \phi^{(i)}(\mathbf{x}_H) \bar{F}^{(i)}. \tag{42}$$

Upon substitution of eqs (38) and (39) (with $\mathbf{x}_H$ replaced by $\mathbf{x}'_H$), we obtain

$$\hat{\mathscr{H}}_2(\mathbf{x}_H) F(\mathbf{x}_H) = \int_{\mathbb{R}^2} \mathscr{H}_2(\mathbf{x}_H; \mathbf{x}'_H) F(\mathbf{x}'_H)\, d^2\mathbf{x}'_H, \tag{43}$$

where the kernel $\mathscr{H}_2(\mathbf{x}_H; \mathbf{x}'_H)$ is defined as

$$\mathscr{H}_2(\mathbf{x}_H; \mathbf{x}'_H) = \int_{\mathbb{R}^2} \phi(\mathbf{x}_H, \boldsymbol{\kappa}) \lambda(\boldsymbol{\kappa}) \phi^*(\mathbf{x}'_H, \boldsymbol{\kappa})\, d^2\boldsymbol{\kappa} + \sum_{\lambda_i \in \sigma_{\mathrm{discr}}} \phi^{(i)}(\mathbf{x}_H) \lambda_i \phi^{(i)}(\mathbf{x}'_H). \tag{44}$$

We denote a kernel by omitting the circumflex and by replacing $(\mathbf{x}_H)$ by $(\mathbf{x}_H; \mathbf{x}'_H)$. Eq. (44) and other expressions for kernels given below should be understood in the sense of generalized functions.

Analogous to eq. (43), we introduce the kernel of the square-root operator $\hat{\mathscr{H}}_1$ via

$$\hat{\mathscr{H}}_1(\mathbf{x}_H) F(\mathbf{x}_H) = \int_{\mathbb{R}^2} \mathscr{H}_1(\mathbf{x}_H; \mathbf{x}'_H) F(\mathbf{x}'_H)\, d^2\mathbf{x}'_H. \tag{45}$$

The kernels $\mathscr{H}_2$ and $\mathscr{H}_1$ are related to each other, according to

$$\mathscr{H}_2(\mathbf{x}_H; \mathbf{x}'_H) = \int_{\mathbb{R}^2} \mathscr{H}_1(\mathbf{x}_H; \mathbf{x}''_H) \mathscr{H}_1(\mathbf{x}''_H; \mathbf{x}'_H)\, d^2\mathbf{x}''_H. \tag{46}$$

From eqs (44) and (46) and the orthonormality property of the eigenfunctions ϕ it now follows that for $\mathscr{H}_1(\mathbf{x}_H; \mathbf{x}'_H)$ we can write

$$\mathscr{H}_1(\mathbf{x}_H; \mathbf{x}'_H) = \int_{\mathbb{R}^2} \phi(\mathbf{x}_H, \boldsymbol{\kappa}) \lambda^{1/2}(\boldsymbol{\kappa}) \phi^*(\mathbf{x}'_H, \boldsymbol{\kappa})\, d^2\boldsymbol{\kappa} + \sum_{\lambda_i \in \sigma_{\mathrm{discr}}} \phi^{(i)}(\mathbf{x}_H) \lambda_i^{1/2} \phi^{(i)}(\mathbf{x}'_H). \tag{47}$$

For the sign of the square root of λ we choose (in accordance with the homogeneous situation)

$$\mathscr{Re}(\lambda^{1/2}) \geq 0 \quad \text{for } \lambda \geq 0 \quad \text{(propagating wave modes)}, \tag{48}$$

$$\mathscr{Im}(\lambda^{1/2}) < 0 \quad \text{for } \lambda < 0 \quad \text{(evanescent wave modes)}, \tag{49}$$

where λ can stand for $\lambda(\boldsymbol{\kappa})$ or λ_i. The location of the possible values of $\lambda^{1/2}$ in the complex plane is shown in Fig. 4.

From eqs (44) and (47) we find the following symmetry relations for the kernels of $\hat{\mathscr{H}}_2$ and $\hat{\mathscr{H}}_1$:

$$\mathscr{H}_2(\mathbf{x}'_H; \mathbf{x}_H) = \mathscr{H}_2(\mathbf{x}_H; \mathbf{x}'_H) \tag{50}$$

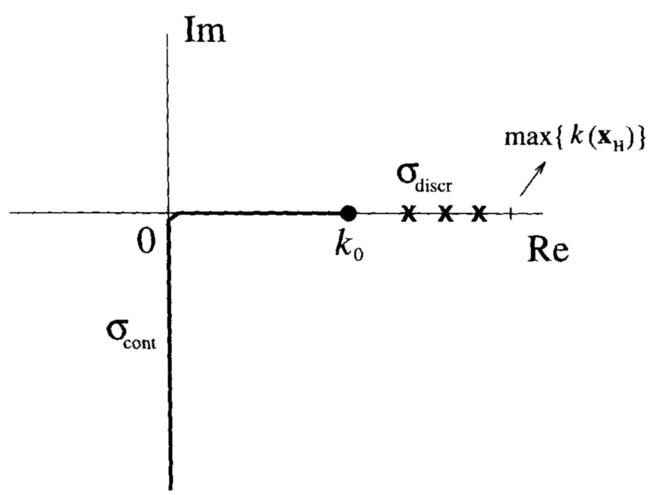

Figure 4. Spectrum of the square-root operator $\hat{\mathscr{H}}_1$ in the complex plane.

and

$$\mathscr{H}_1(\mathbf{x}'_H; \mathbf{x}_H) = \mathscr{H}_1(\mathbf{x}_H; \mathbf{x}'_H). \tag{51}$$

This is most easily seen if we choose the eigenfunctions $\phi(\mathbf{x}_H, \boldsymbol{\kappa})$ real-valued [if we choose complex-valued eigenfunctions, we need to make use of the symmetry properties $\phi(\mathbf{x}_H, -\boldsymbol{\kappa}) = \phi^*(\mathbf{x}_H, \boldsymbol{\kappa})$ and $\lambda^{1/2}(-\boldsymbol{\kappa}) = \lambda^{1/2}(\boldsymbol{\kappa})$].

Note that the operator $\hat{\mathscr{H}}_1$ is *not* self-adjoint since its spectrum is not real-valued. However, the symmetry property (51) of the kernel $\mathscr{H}_1$ will turn out to be sufficient for the modification of the contrast operator in the reciprocity theorem (31).

Finally we derive symmetry relations for the kernels of $\hat{L}_1$, $\hat{L}_1^{-1}$, $\hat{L}_2$ and $\hat{L}_2^{-1}$ as defined in eqs (19) and (20). These operators are all defined in terms of $\hat{\mathscr{H}}_1^{\pm 1/2}$. In a similar way to the above, we obtain for the kernel of this operator

$$\mathscr{H}_{\pm 1/2}(\mathbf{x}_H; \mathbf{x}'_H) = \int_{\mathbb{R}^2} \phi(\mathbf{x}_H, \boldsymbol{\kappa}) \lambda^{\pm 1/4}(\boldsymbol{\kappa}) \phi^*(\mathbf{x}'_H, \boldsymbol{\kappa})\, d^2\boldsymbol{\kappa} + \sum_{\lambda_i \in \sigma_{\rm discr}} \phi^{(i)}(\mathbf{x}_H) \lambda_i^{\pm 1/4} \phi^{(i)}(\mathbf{x}'_H). \tag{52}$$

Note that the root-singularity that occurs for $\lambda(\boldsymbol{\kappa}) = 0$ is integrable. For the kernels of $\hat{L}_1$, $\hat{L}_1^{-1}$, $\hat{L}_2$ and $\hat{L}_2^{-1}$ we can write

$$L_1(\mathbf{x}_H; \mathbf{x}'_H) = \left(\frac{\omega \varrho(\mathbf{x}_H)}{2}\right)^{1/2} \mathscr{H}_{-1/2}(\mathbf{x}_H; \mathbf{x}'_H), \tag{53}$$

$$L_1^{\rm inv}(\mathbf{x}_H; \mathbf{x}'_H) = \left(\frac{\omega}{2}\right)^{-1/2} \mathscr{H}_{+1/2}(\mathbf{x}_H; \mathbf{x}'_H) \varrho^{-1/2}(\mathbf{x}'_H), \tag{54}$$

$$L_2(\mathbf{x}_H; \mathbf{x}'_H) = [2\omega \varrho(\mathbf{x}_H)]^{-1/2} \mathscr{H}_{+1/2}(\mathbf{x}_H; \mathbf{x}'_H) \tag{55}$$

and

$$L_2^{\rm inv}(\mathbf{x}_H; \mathbf{x}'_H) = (2\omega)^{1/2} \mathscr{H}_{-1/2}(\mathbf{x}_H; \mathbf{x}'_H) \varrho^{1/2}(\mathbf{x}'_H), \tag{56}$$

respectively. Here $L_\nu^{\rm inv}$ denotes the kernel of the inverse operator $\hat{L}_\nu^{-1}$ for $\nu = 1, 2$. It is now easily seen that the symmetry relations for these kernels read

$$L_1(\mathbf{x}'_H; \mathbf{x}_H) = \frac{1}{2} L_2^{\rm inv}(\mathbf{x}_H; \mathbf{x}'_H) \tag{57}$$

and

$$L_2(\mathbf{x}'_H; \mathbf{x}_H) = \frac{1}{2} L_1^{\rm inv}(\mathbf{x}_H; \mathbf{x}'_H). \tag{58}$$

For notational convenience we have dropped x_3 throughout this section. Of course the symmetry relations (50), (51), (57) and (58) apply for any x_3. In the remainder of this paper we include x_3 again in the notation, according to $\mathscr{H}_2(\mathbf{x}'_H, x_3; \mathbf{x}_H) = \mathscr{H}_2(\mathbf{x}_H, x_3; \mathbf{x}'_H)$, etc.

SYMMETRY PROPERTIES OF THE ONE-WAY WAVE EQUATION

We use the results of the previous section to derive the symmetry properties of the acoustic one-way wave equation. To this end, we rewrite eq. (24) as follows:

$$\partial_3 \mathbf{P}(\mathbf{x}) - \int_{\mathbb{R}^2} \mathbf{B}(\mathbf{x}_H, x_3; \mathbf{x}'_H) \mathbf{P}(\mathbf{x}'_H, x_3)\, d^2\mathbf{x}'_H = \mathbf{S}(\mathbf{x}), \tag{59}$$

where

$$\mathbf{B}(\mathbf{x}_H, x_3; \mathbf{x}'_H) = -j\omega \boldsymbol{\Lambda}(\mathbf{x}_H, x_3; \mathbf{x}'_H) + \boldsymbol{\Theta}(\mathbf{x}_H, x_3; \mathbf{x}'_H) \tag{60}$$

and

$$\boldsymbol{\Theta}(\mathbf{x}_H, x_3; \mathbf{x}'_H) = -\int_{\mathbb{R}^2} \mathbf{L}^{\rm inv}(\mathbf{x}_H, x_3; \mathbf{x}''_H) \partial_3 \mathbf{L}(\mathbf{x}''_H, x_3; \mathbf{x}'_H)\, d^2\mathbf{x}''_H. \tag{61}$$

Here $\mathbf{L}^{\rm inv}$ denotes the kernel of the inverse operator $\hat{\mathbf{L}}^{-1}$. $\mathbf{L}^{\rm inv}$ is related to $\mathbf{L}$ according to

$$\int_{\mathbb{R}^2} \mathbf{L}^{\rm inv}(\mathbf{x}_H, x_3; \mathbf{x}''_H) \mathbf{L}(\mathbf{x}''_H, x_3; \mathbf{x}'_H)\, d^2\mathbf{x}''_H = \mathbf{I}\delta(\mathbf{x}_H - \mathbf{x}'_H), \tag{62}$$

where $\delta(\mathbf{x}_H) = \delta(x_1)\delta(x_2)$ and $\mathbf{I}$ is the 2×2 identity matrix. On account of eqs (14), (15), (18), (51), (57) and (58) we find the following symmetry properties for $\boldsymbol{\Lambda}$ and $\mathbf{L}$:

$$\boldsymbol{\Lambda}(\mathbf{x}'_H, x_3; \mathbf{x}_H) = \boldsymbol{\Lambda}(\mathbf{x}_H, x_3; \mathbf{x}'_H) \tag{63}$$

and

$$\mathbf{L}^{\rm T}(\mathbf{x}'_H, x_3; \mathbf{x}_H) = -\mathbf{N}\mathbf{L}^{\rm inv}(\mathbf{x}_H, x_3; \mathbf{x}'_H)\mathbf{N}^{-1}. \tag{64}$$

To derive the symmetry property of $\boldsymbol{\Theta}$ we first transpose both sides of eq. (61), yielding

$$\boldsymbol{\Theta}^{\rm T}(\mathbf{x}_H, x_3; \mathbf{x}'_H) = -\int_{\mathbb{R}^2} \{\partial_3 \mathbf{L}(\mathbf{x}''_H, x_3; \mathbf{x}'_H)\}^{\rm T} \times \{\mathbf{L}^{\rm inv}(\mathbf{x}_H, x_3; \mathbf{x}''_H)\}^{\rm T}\, d^2\mathbf{x}''_H. \tag{65}$$

Interchanging $\mathbf{x}'_H$ with $\mathbf{x}_H$ and substitution of the symmetry property (64) gives

$$\boldsymbol{\Theta}^{\rm T}(\mathbf{x}'_H, x_3; \mathbf{x}_H) = -\mathbf{N}\left[\int_{\mathbb{R}^2} \{\partial_3 \mathbf{L}^{\rm inv}(\mathbf{x}_H, x_3; \mathbf{x}''_H)\} \mathbf{L}(\mathbf{x}''_H, x_3; \mathbf{x}'_H)\, d^2\mathbf{x}''_H\right]\mathbf{N}^{-1}. \tag{66}$$

On account of eq. (62) we can write

$$\int_{\mathbb{R}^2} \partial_3 \{\mathbf{L}^{\rm inv}(\mathbf{x}_H, x_3; \mathbf{x}''_H) \mathbf{L}(\mathbf{x}''_H, x_3; \mathbf{x}'_H)\}\, d^2\mathbf{x}''_H = \mathbf{O}, \tag{67}$$

where $\mathbf{O}$ is the 2×2 null matrix. Applying the product rule for differentiation to eq. (67), using the result in eq. (66) and comparing the resulting expression for $\boldsymbol{\Theta}^{\rm T}(\mathbf{x}'_H, x_3; \mathbf{x}_H)$ with eq. (61) yields

$$\boldsymbol{\Theta}^{\rm T}(\mathbf{x}'_H, x_3; \mathbf{x}_H) = -\mathbf{N}\boldsymbol{\Theta}(\mathbf{x}_H, x_3; \mathbf{x}'_H)\mathbf{N}^{-1}. \tag{68}$$

To derive the symmetry property of $\mathbf{B}$ we first note that, due to the special structure of $\boldsymbol{\Lambda}$, its symmetry property (63) can be replaced by

$$\boldsymbol{\Lambda}^{\rm T}(\mathbf{x}'_H, x_3; \mathbf{x}_H) = -\mathbf{N}\boldsymbol{\Lambda}(\mathbf{x}_H, x_3; \mathbf{x}'_H)\mathbf{N}^{-1}. \tag{69}$$

Hence, on account of eqs (60), (68) and (69) we now easily find

$$\mathbf{B}^{\rm T}(\mathbf{x}'_H, x_3; \mathbf{x}_H) = -\mathbf{N}\mathbf{B}(\mathbf{x}_H, x_3; \mathbf{x}'_H)\mathbf{N}^{-1}. \tag{70}$$

RECIPROCITY THEOREM OF THE CONVOLUTION TYPE: REVISITED

Using the kernel notation introduced in the previous sections we can rewrite the local reciprocity theorem (30) as follows:

$$\begin{aligned}\partial_3\{\mathbf{P}_A^{\mathrm{T}}(\mathbf{x})\mathbf{N}\mathbf{P}_B(\mathbf{x})\} &= \int_{\mathbb{R}^2} \mathbf{P}_A^{\mathrm{T}}(\mathbf{x}'_{\mathrm{H}}, x_3)\mathbf{B}_A^{\mathrm{T}}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})\mathbf{N}\mathbf{P}_B(\mathbf{x}_{\mathrm{H}}, x_3)\, d^2\mathbf{x}'_{\mathrm{H}} \\ &+ \int_{\mathbb{R}^2} \mathbf{P}_A^{\mathrm{T}}(\mathbf{x}_{\mathrm{H}}, x_3)\mathbf{N}\mathbf{B}_B(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})\mathbf{P}_B(\mathbf{x}'_{\mathrm{H}}, x_3)\, d^2\mathbf{x}'_{\mathrm{H}} \\ &+ \mathbf{P}_A^{\mathrm{T}}(\mathbf{x})\mathbf{N}\mathbf{S}_B(\mathbf{x}) + \mathbf{S}_A^{\mathrm{T}}(\mathbf{x})\mathbf{N}\mathbf{P}_B(\mathbf{x}). \qquad (71)\end{aligned}$$

Note that the symmetry relation (70) is not sufficient to let the integrals cancel when the medium parameters in state A and state B are identical. Therefore we integrate eq. (71) again over volume $\mathcal{V}$. We can thus interchange $\mathbf{x}_{\mathrm{H}}$ with $\mathbf{x}'_{\mathrm{H}}$ in the term containing $\mathbf{B}_A^{\mathrm{T}}$. This yields the global reciprocity theorem

$$\begin{aligned}\int_{\partial\mathcal{V}} \mathbf{P}_A^{\mathrm{T}}(\mathbf{x})\mathbf{N}\mathbf{P}_B(\mathbf{x})n_3(\mathbf{x})\, d^2\mathbf{x}_{\mathrm{H}} &= \int_{\mathcal{V}} d^3\mathbf{x} \int_{\mathbb{R}^2} \mathbf{P}_A^{\mathrm{T}}(\mathbf{x}_{\mathrm{H}}, x_3)\mathbf{N}\boldsymbol{\Delta}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})\mathbf{P}_B(\mathbf{x}'_{\mathrm{H}}, x_3)\, d^2\mathbf{x}'_{\mathrm{H}} \\ &+ \int_{\mathcal{V}} \{\mathbf{P}_A^{\mathrm{T}}(\mathbf{x})\mathbf{N}\mathbf{S}_B(\mathbf{x}) + \mathbf{S}_A^{\mathrm{T}}(\mathbf{x})\mathbf{N}\mathbf{P}_B(\mathbf{x})\}\, d^3\mathbf{x}, \qquad (72)\end{aligned}$$

where

$$\boldsymbol{\Delta}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}}) = \mathbf{B}_B(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}}) - [-\mathbf{N}^{-1}\mathbf{B}_A^{\mathrm{T}}(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}})\mathbf{N}]. \qquad (73)$$

Unlike in eqs (30) and (32), where operator $\hat{\mathbf{B}}_A^{\mathcal{T}}$ acts by definition upon the quantity left of it (i.e., on $\mathbf{P}_A^{\mathrm{T}}$), $\mathbf{B}_A^{\mathrm{T}}$ in eq. (73) simply denotes a transposed kernel matrix, with no particular connection to the quantity it acts upon. Substituting symmetry relation (70) into (73) we obtain

$$\boldsymbol{\Delta}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}}) = \mathbf{B}_B(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}}) - \mathbf{B}_A(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}}). \qquad (74)$$

Since $\mathbf{B}_A$ and $\mathbf{B}_B$ in eq. (74) are the kernels of the operators $\hat{\mathbf{B}}_A$ and $\hat{\mathbf{B}}_B$, reciprocity theorem (72) is equivalent to reciprocity theorem (31), with $\hat{\boldsymbol{\Delta}} = \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A$. Hence, we have achieved what we aimed for.

In Paper B we derive several one-way representations from reciprocity theorem (31). Here we consider a special situation, to illustrate a fundamental property of one-way wavefields. Consider Fig. 2 and assume that outside $\mathcal{V}$ the medium is homogeneous and source-free in both states. Then, at the upper surface there are no downgoing waves ($P_A^+ = P_B^+ = 0$) and at the lower surface there are no upgoing waves ($P_A^- = P_B^- = 0$). Hence, the interaction quantity (27) vanishes at $\partial\mathcal{V}$, and, consequently, the boundary integral in eq. (31) vanishes. Furthermore, assume that the medium parameters in $\mathcal{V}$ are identical in both states. Then the first volume integral on the right-hand side of eq. (31) also vanishes. Finally, assume that the one-way sources in both states are point sources in $\mathcal{V}$ at $\mathbf{x}_A$ and $\mathbf{x}_B$, respectively. Hence

$$\mathbf{S}_A(\mathbf{x}) = \mathbf{S}_{A,0}(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_A), \qquad \mathbf{x}_A \in \mathcal{V} \qquad (75)$$

and

$$\mathbf{S}_B(\mathbf{x}) = \mathbf{S}_{B,0}(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_B), \qquad \mathbf{x}_B \in \mathcal{V}, \qquad (76)$$

where $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$. Reciprocity theorem (31) thus yields

$$\mathbf{P}_A^{\mathrm{T}}(\mathbf{x}_B)\mathbf{N}\mathbf{S}_{B,0}(\mathbf{x}_B) = -\mathbf{S}_{A,0}^{\mathrm{T}}(\mathbf{x}_A)\mathbf{N}\mathbf{P}_B(\mathbf{x}_A), \qquad (77)$$

or

$$\begin{aligned}&P_A^+(\mathbf{x}_B)S_{B,0}^-(\mathbf{x}_B) - P_A^-(\mathbf{x}_B)S_{B,0}^+(\mathbf{x}_B) \\ &\quad = -S_{A,0}^+(\mathbf{x}_A)P_B^-(\mathbf{x}_A) + S_{A,0}^-(\mathbf{x}_A)P_B^+(\mathbf{x}_A). \qquad (78)\end{aligned}$$

[For comparison, Fokkema & van den Berg (1993) obtain the following two-way source–receiver reciprocity relation:

$$\begin{aligned}&P_A(\mathbf{x}_B)Q_B(\mathbf{x}_B) - V_{k,A}(\mathbf{x}_B)F_{k,B}(\mathbf{x}_B) \\ &\quad = -F_{k,A}(\mathbf{x}_A)V_{k,B}(\mathbf{x}_A) + Q_A(\mathbf{x}_A)P_B(\mathbf{x}_A).]\end{aligned}$$

For the special situation that the sources for the upgoing waves are zero we obtain

$$P_B^-(\mathbf{x}_A)/S_{B,0}^+(\mathbf{x}_B) = P_A^-(\mathbf{x}_B)/S_{A,0}^+(\mathbf{x}_A), \qquad (79)$$

see Fig. 5. For this situation we conclude that the one-way sources for downgoing waves and the one-way receivers for upgoing waves are interchangeable. Note that three other situations could be considered that lead to comparable conclusions.

RECIPROCITY THEOREM OF THE CORRELATION TYPE

We derive a reciprocity theorem of the correlation type for one-way wavefields. Again we consider the interaction between downgoing waves in one state and upgoing waves in the other and vice versa. However, this time we use the property that complex conjugation of a wavefield changes its propagation direction from downgoing to upgoing and vice versa (Fig. 6). To be more specific, we consider the interaction quantity

$$\partial_3\{(P_A^+)^* P_B^+ - (P_A^-)^* P_B^-\}. \qquad (80)$$

To simplify the notation, we rewrite this interaction quantity

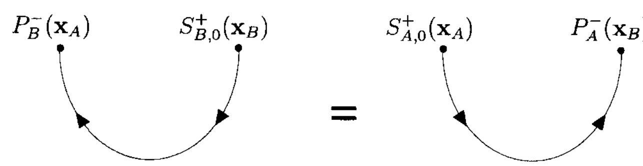

Figure 5. Illustration of reciprocity for one-way sources for downgoing waves and one-way receivers for upgoing waves.

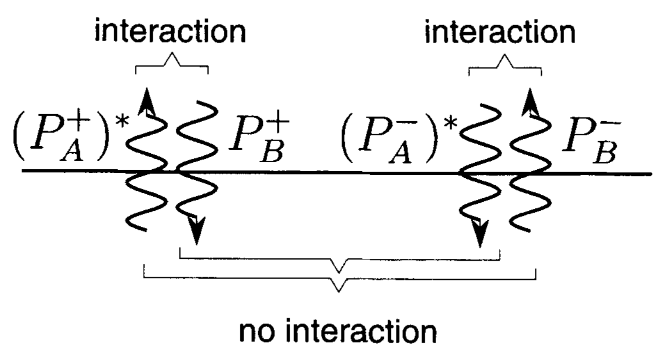

Figure 6. Both terms of the interaction quantity for the reciprocity theorem of the correlation type contain waves that propagate in opposite directions.

as

$$\partial_3\{\mathbf{P}_A^{\mathrm{H}}\mathbf{J}\mathbf{P}_B\}, \tag{81}$$

where $^{\mathrm{H}}$ denotes transposition and complex conjugation and

$$\mathbf{J}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{82}$$

Using the one-way wave equation (24) and integrating the result over the volume $\mathcal{V}$ of Fig. 2 yields

$$\int_{\partial\mathcal{V}} \mathbf{P}_A^{\mathrm{H}}\mathbf{J}\mathbf{P}_B n_3\, d^2\mathbf{x}_{\mathrm{H}} = \int_{\mathcal{V}} \mathbf{P}_A^{\mathrm{H}}\mathbf{J}\hat{\Delta}\mathbf{P}_B\, d^3\mathbf{x} + \int_{\mathcal{V}} \{\mathbf{P}_A^{\mathrm{H}}\mathbf{J}\mathbf{S}_B + \mathbf{S}_A^{\mathrm{H}}\mathbf{J}\mathbf{P}_B\}\, d^3\mathbf{x}, \tag{83}$$

where the contrast operator $\hat{\Delta}$ is given by

$$\hat{\Delta}=\hat{\mathbf{B}}_B-(-\mathbf{J}^{-1}\hat{\mathbf{B}}_A^{\mathscr{H}}\mathbf{J}). \tag{84}$$

Here the complex conjugate transposed operator $\hat{\mathbf{B}}_A^{\mathscr{H}}$ acts upon the quantity to the left of it, i.e. on $\mathbf{P}_A^{\mathrm{H}}$. We could modify this contrast operator in a similar way to that described in the previous sections if the kernel of the square-root operator obeyed the symmetry relation $\mathscr{H}_1^*(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}}) = \mathscr{H}_1(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})$. However, this relation is only approximately valid, since $(\lambda^{1/2})^*$ is equal to $\lambda^{1/2}$ only for $\lambda \geq 0$ (i.e., for propagating wave modes). Using

$$\mathscr{H}_1^*(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}}) \approx \mathscr{H}_1(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}}), \tag{85}$$

we find

$$\mathbf{\Lambda}^{\mathrm{H}}(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}}) \approx \mathbf{J}\mathbf{\Lambda}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})\mathbf{J}^{-1}, \tag{86}$$

$$\mathbf{L}^{\mathrm{H}}(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}}) \approx \mathbf{J}\mathbf{L}^{\mathrm{inv}}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})(\mathbf{J}\mathbf{N})^{-1} \tag{87}$$

and

$$\mathbf{B}^{\mathrm{H}}(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}}) \approx -\mathbf{J}\mathbf{B}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})\mathbf{J}^{-1}. \tag{88}$$

Using the same reasoning as in the previous section we finally obtain

$$\hat{\Delta} \approx \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A. \tag{89}$$

In Paper B we use reciprocity theorem (83), with the contrast function defined in eq. (89), to derive approximate but *stable* inverse propagators for one-way wavefields in arbitrary inhomogeneous media. The stability of these inverse propagators arises from the fact that the 'erroneously handled' evanescent wave modes are suppressed instead of amplified.

EXTENSION TO THE ELASTODYNAMIC SITUATION

For the elastodynamic situation, the derivation of one-way reciprocity theorems (with vanishing contrast operators when the medium parameters in both states are identical) is quite involved. Therefore we only derive the conditions that should be posed to the (non-unique) decomposition of the elastodynamic two-way operator such that reciprocity theorems of the form of eqs (31) and (83) hold true for the elastodynamic situation.

Starting with the basic equations, it is possible to derive a two-way wave equation of the form of eq. (5) for arbitrary inhomogeneous anisotropic media, in which $\mathbf{Q}$ is a 6×1 two-way wave vector containing the 3×1 traction and particle velocity vectors, $\mathbf{D}$ is a 6×1 two-way source vector and $\hat{\mathbf{A}}$ a 6×6 two-way operator matrix (Woodhouse 1974; Wapenaar & Berkhout 1989). We assume that $\hat{\mathbf{A}}$ can be decomposed analogously to eq. (13), hence $\hat{\mathbf{A}} = -j\omega\hat{\mathbf{L}}\hat{\mathbf{\Lambda}}\hat{\mathbf{L}}^{-1}$. In general the 6×6 vertical slowness operator $\hat{\mathbf{\Lambda}}$ will not be purely diagonal, but it will have a block-diagonal structure, according to

$$\hat{\mathbf{\Lambda}}=\begin{pmatrix}\hat{\mathbf{\Lambda}}^+ & \mathbf{O}\\ \mathbf{O} & -\hat{\mathbf{\Lambda}}^-\end{pmatrix}, \tag{90}$$

where $\mathbf{O}$ is a 3×3 null matrix. The off-diagonal elements in the 3×3 submatrices $\hat{\mathbf{\Lambda}}^+$ and $-\hat{\mathbf{\Lambda}}^-$ account for wave conversion due to the lateral variations of the medium parameters. Moreover, in general $\hat{\mathbf{\Lambda}}^+$ will be different from $\hat{\mathbf{\Lambda}}^-$ due to the absence of up/down symmetry in arbitrary anisotropic media.

With this decomposition, a one-way wave equation of the form of eq. (24) is obtained, in which $\hat{\mathbf{B}}$ is a 6×6 one-way operator matrix and $\mathbf{P}$ is a 6×1 one-way wave vector containing 3×1 downgoing and upgoing wave vectors, according to

$$\mathbf{P}=\begin{pmatrix}\mathbf{P}^+\\ \mathbf{P}^-\end{pmatrix} \quad \text{with} \quad \mathbf{P}^{\pm}=\begin{pmatrix}\Phi^{\pm}\\ \Psi^{\pm}\\ \Upsilon^{\pm}\end{pmatrix}. \tag{91}$$

Here $\Phi^{\pm}$, $\Psi^{\pm}$ and $\Upsilon^{\pm}$ represent the (flux-normalized) downgoing and upgoing quasi-*P*, quasi-*S*1 and quasi-*S*2 waves, respectively. A similar subdivision applies for the 6×1 one-way source vector $\mathbf{S}$.

We introduce an interaction quantity of the form of eq. (28), with $\mathbf{N}$ a 6×6 matrix containing two 3×3 null matrices and two 3×3 identity matrices, analogous to eq. (29). Local and global reciprocity theorems of the form of eqs (30) and (31) are thus obtained, with the contrast operator $\hat{\Delta}$, as defined in eq. (32). We derive the conditions for the modification of this contrast operator into the more suitable form $\hat{\Delta} = \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A$. To this end, the kernel $\mathbf{A}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})$ of operator $\hat{\mathbf{A}}(\mathbf{x})$ is introduced via

$$\hat{\mathbf{A}}(\mathbf{x})\mathbf{F}(\mathbf{x}_{\mathrm{H}}) = \int_{\mathbb{R}^2} \mathbf{A}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})\, \mathbf{F}(\mathbf{x}'_{\mathrm{H}}) d^2\mathbf{x}'_{\mathrm{H}}, \tag{92}$$

where

$$\mathbf{A}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}}) = \hat{\mathbf{A}}(\mathbf{x})\delta(\mathbf{x}_{\mathrm{H}} - \mathbf{x}'_{\mathrm{H}}). \tag{93}$$

From the latter equation and the definition of $\hat{\mathbf{A}}$ it can be shown that this kernel obeys the symmetry relation

$$\mathbf{A}^{\mathrm{T}}(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}}) = -\mathbf{N}\mathbf{A}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})\mathbf{N}^{-1}. \tag{94}$$

This kernel is related to the kernels of $\hat{\mathbf{L}}$, $\hat{\mathbf{\Lambda}}$ and $\hat{\mathbf{L}}^{-1}$, according to

$$\mathbf{A}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}}) = -j\omega \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbf{L}(\mathbf{x}_{\mathrm{H}}, x_3; \mathbf{x}''_{\mathrm{H}}) \times \mathbf{\Lambda}(\mathbf{x}''_{\mathrm{H}}, x_3; \mathbf{x}'''_{\mathrm{H}})\mathbf{L}^{\mathrm{inv}}(\mathbf{x}'''_{\mathrm{H}}, x_3; \mathbf{x}'_{\mathrm{H}})\, d^2\mathbf{x}''_{\mathrm{H}}\, d^2\mathbf{x}'''_{\mathrm{H}}. \tag{95}$$

Transposition of both sides of this equation, and interchanging $\mathbf{x}'_{\mathrm{H}}$ with $\mathbf{x}_{\mathrm{H}}$ and $\mathbf{x}'''_{\mathrm{H}}$ with $\mathbf{x}''_{\mathrm{H}}$ in the result, yields

$$\mathbf{A}^{\mathrm{T}}(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}}) = -j\omega \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \{\mathbf{L}^{\mathrm{inv}}(\mathbf{x}''_{\mathrm{H}}, x_3; \mathbf{x}_{\mathrm{H}})\}^{\mathrm{T}} \times \mathbf{\Lambda}^{\mathrm{T}}(\mathbf{x}'''_{\mathrm{H}}, x_3; \mathbf{x}''_{\mathrm{H}})\mathbf{L}^{\mathrm{T}}(\mathbf{x}'_{\mathrm{H}}, x_3; \mathbf{x}'''_{\mathrm{H}})\, d^2\mathbf{x}''_{\mathrm{H}}\, d^2\mathbf{x}'''_{\mathrm{H}}. \tag{96}$$

This result is consistent with eqs (94) and (95) if $\mathbf{L}$ and $\boldsymbol{\Lambda}$ satisfy symmetry relations of the form of eqs (64) and (69). Finally, if these symmetry relations apply, then in a similar way to the acoustic situation, we find that the contrast operator $\hat{\boldsymbol{\Delta}}$ obtains the desired form $\hat{\boldsymbol{\Delta}} = \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A$.

Similarly, if $\mathbf{L}$ and $\boldsymbol{\Lambda}$ satisfy the symmetry relations of the form of eqs (87) and (86) as well, then the reciprocity theorem of the correlation type (83) also holds true, with $\hat{\boldsymbol{\Delta}} \approx \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A$.

We conclude this section by demonstrating the validity of the symmetry relation (69) for $\boldsymbol{\Lambda}$ for the special case of a laterally invariant arbitrary anisotropic medium. In this case we can express $\boldsymbol{\Lambda}(\mathbf{x}_H, x_3; \mathbf{x}'_H)$ via an inverse Fourier transformation as

$$\boldsymbol{\Lambda}(\mathbf{x}_H, x_3; \mathbf{x}'_H) = \left(\frac{\omega}{2\pi}\right)^2 \int_{\mathbb{R}^2} \tilde{\boldsymbol{\Lambda}}(\mathbf{p}, x_3) \exp\{-j\omega p_v(x_v - x'_v)\}\, d^2\mathbf{p}, \tag{97}$$

where $\mathbf{p} = (p_1, p_2)$, p_1 and p_2 being the horizontal slownesses, and where

$$\tilde{\boldsymbol{\Lambda}}(\mathbf{p}, x_3) = \mathrm{diag}(q_P^+, q_{S1}^+, q_{S2}^+, -q_P^-, -q_{S1}^-, -q_{S2}^-)(\mathbf{p}, x_3), \tag{98}$$

$q^\pm$ being the vertical slownesses for the different wave types. Even in a medium without up/down symmetry [i.e. in a medium where $q^\pm(\mathbf{p}) \neq q^\mp(\mathbf{p})$], we still have $q^\pm(\mathbf{p}) = q^\mp(-\mathbf{p})$ for each wave type (Garmany 1983; Fryer & Frazer 1984). Hence,

$$\tilde{\boldsymbol{\Lambda}}^T(-\mathbf{p}, x_3) = -\mathbf{N}\tilde{\boldsymbol{\Lambda}}(\mathbf{p}, x_3)\mathbf{N}^{-1}, \tag{99}$$

or, using eq. (97),

$$\boldsymbol{\Lambda}^T(\mathbf{x}'_H, x_3; \mathbf{x}_H) = -\mathbf{N}\boldsymbol{\Lambda}(\mathbf{x}_H, x_3; \mathbf{x}'_H)\mathbf{N}^{-1}. \tag{100}$$

CONCLUSIONS

We have derived reciprocity theorems of the *convolution* type (eq. 31) and of the *correlation* type (eq. 83), both for acoustic one-way wavefields. The former theorem is exact, the latter ignores evanescent waves. It has been shown that theorems of the same form also apply to the elastodynamic situation, provided that the decomposition of the elastodynamic two-way operator matrix can be done in such a way that the elastodynamic one-way operator kernels satisfy symmetry relations that have the same form as those for the acoustic one-way operator kernels (eqs 64, 69, 86 and 87).

The contrast function in both reciprocity theorems is defined in terms of the one-way operator matrices $\hat{\mathbf{B}}_A$ and $\hat{\mathbf{B}}_B$. These operators distinguish explicitly between *propagation* (downward/upward) and *scattering* (reflection/transmission). This property can be exploited in the analysis of forward and inverse scattering problems for which there is a clear preferred direction of propagation. In particular, the reciprocity theorem of the convolution type (31) will be used in Paper B as the basis for the derivation of several one-way representations of seismic data. In these representations, propagation and scattering are naturally separated. The reciprocity theorem of the correlation type (83) will be used in Paper B as the basis for the derivation of stable inverse propagators for downgoing and upgoing waves. These inverse propagators are the basis for a true amplitude seismic reflection imaging technique that fully accounts for the scattering losses due to fine-layering.

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