Geophys. J. Int. (1993) **115**, 445–455

Kirchhoff–Helmholtz downward extrapolation in a layered medium with curved interfaces

C. P. A. Wapenaar
Laboratory of Seismics and Acoustics, Delft University of Technology, PO Box 5046, 2600 GA Delft, The Netherlands

Accepted 1993 March 22. Received 1993 March 18; in original form 1992 September 29

SUMMARY

In modelling applications, recursive Kirchhoff–Helmholtz extrapolation through a layered medium requires plane-wave decomposition and synthesis at the layer interfaces in order to take transmission effects into account. For inverse applications (downward extrapolation of the measured data into the subsurface), this plane-wave decomposition and synthesis is superfluous: transmission effects are automatically taken into account, since the extrapolated *two-way* (downgoing and upgoing) wavefield is continuous at layer interfaces. Two-way wavefield extrapolation, however, is very sensitive to errors in the model of the layered medium. Therefore, in practice *one-way* extrapolation schemes are preferred. It appears that for backward extrapolation of the *primary* upgoing wavefield, the recursive Kirchhoff–Helmholtz approach again accounts automatically for transmission effects at the layer interfaces. This approach is particularly attractive for situations where the contrasts at the layer interfaces are high.

Key words: inversion, Kirchhoff–Helmholtz integrals, modelling, wavefield extrapolation.

INTRODUCTION

Consider a layered acoustic medium (Fig. 1) consisting of homogeneous layers separated by curved interfaces. In the geophysical literature, wave propagation in such a layered medium is often described in terms of Kirchhoff–Helmholtz boundary integrals (Hilterman 1970; Frazer & Sen 1985; Hill & Wuenschell 1985; Kampfmann 1988; Wenzel, Stenzel & Zimmerman 1990).

The main problem is the connection at the interfaces of the down- and upgoing waves in the different layers. The procedure followed in general involves: decomposition into plane waves, application of *angle-dependent* plane-wave reflection and transmission coefficients and, finally, synthesis of the reflected and transmitted wavefields from the plane-wave constituents. Since the decomposition and synthesis take place at curved interfaces, generally a high-frequency assumption is made.

In this paper we propose an alternative method using Kirchhoff–Helmholtz integrals for wavefield extrapolation through a layered medium with curved interfaces. Using this method it is possible to extrapolate a wavefield, measured at an acquisition surface, downward from layer interface to layer interface without the need of plane-wave decomposition and synthesis at the layer interfaces. Therefore, this method is very simple. Moreover, the approximations inherent to plane-wave decomposition and synthesis at curved interfaces are avoided.

The proposed method is not suited for modelling applications as the *two-way* (downgoing and upgoing) wavefield is assumed to be known in advance at the acquisition surface (or at any other interface). However, the method is well suited for applications in inverse problems such as migration (Schneider 1978; Berkhout 1985), inverse scattering (Bleistein 1984) or redatuming (Berryhill 1984; Wapenaar & Berkhout 1989). In principle the method takes into account all (internal) multiple reflections. It is very sensitive, however, to errors in the description of the sources, the layer velocities and the interfaces. Therefore, a modification is also proposed that allows robust downward extrapolation of the *primary* upgoing waves.

PRINCIPLE OF RECURSIVE WAVEFIELD EXTRAPOLATION

Consider again the layered acoustic medium of Fig. 1. In the following we assume that the *two-way* acoustic wavefield at S_0 is known (from measurements). This is the key for the simplicity of the method: suppose it is possible to compute the two-way wavefield at interface S_1 from the two-way wavefield at S_0, then it is also possible to compute the two-way wavefield at S_2 from the two-way wavefield at S_1

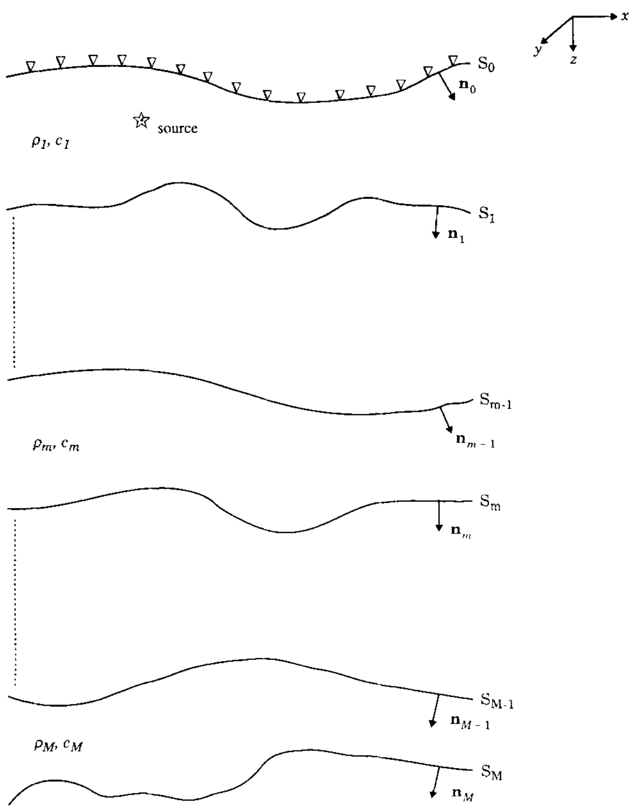

Figure 1. Layered acoustic medium.

and so on *without* any additional effort needed to fulfil the boundary conditions at the layer interfaces. [Bear in mind that the two-way acoustic wavefield is continuous at layer interfaces. This is opposed to one-way (downgoing or upgoing) wavefields, which are discontinuous at interfaces, hence the need for plane-wave decomposition in modelling applications.]

The question is of course how to compute the two-way wavefield at S_m $(m = 1, 2, \ldots, M)$, given the two-way wavefield at S_{m-1}. The Kirchhoff–Helmholtz integral is not directly suited because it pre-supposes knowledge of the wavefield on a *closed* surface.

In the next sections we discuss two modified versions of the Kirchhoff–Helmholtz integral. With these two modified versions the downgoing or upgoing waves, respectively, can be computed at S_1, given the two-way wavefield at S_0. Superposition of the results gives the two-way wavefield at S_1, which is the input for the next recursion.

THE KIRCHHOFF–HELMHOLTZ INTEGRALS WITH CAUSAL AND ANTI-CAUSAL GREEN'S FUNCTIONS

We consider an inhomogeneous lossless fluid, which is described by the space-dependent propagation velocity $c(\mathbf{r})$ and the mass density $\rho(\mathbf{r})$, where $\mathbf{r}$ is a shorthand notation for the Cartesian coordinates (x, y, z). In this fluid we consider a volume V enclosed by a surface S with an outward pointing normal vector $\mathbf{n}$. The space- and frequency-dependent acoustic pressure $P(\mathbf{r}, \omega)$ satisfies in V the following equation:

$$\rho\nabla\cdot\left(\frac{1}{\rho}\nabla P\right) + k^2 P = -\rho Q, \tag{1a}$$

where the wavenumber $k(\mathbf{r}, \omega)$ is defined as

$$k(\mathbf{r}, \omega) = \omega / c(\mathbf{r}). \tag{1b}$$

$Q(\mathbf{r}, \omega)$ represents the source distribution and ω the angular frequency.

We define a Green's wavefield $G(\mathbf{r}, \mathbf{r}_A, \omega)$, which satisfies in V the following equation:

$$\rho\nabla\cdot\left(\frac{1}{\rho}\nabla G\right) + k^2 G = -\rho\delta(\mathbf{r} - \mathbf{r}_A), \tag{2}$$

where $\mathbf{r}_A = (x_A, y_A, z_A)$ denotes the Cartesian coordinates of an internal point A in V. For any point A in V, the acoustic pressure $P(\mathbf{r}_A, \omega)$ may be expressed with the Kirchhoff–Helmholtz (KH) integral according to

$$P(\mathbf{r}_A, \omega) = \oint_S \left[\frac{1}{\rho} G\nabla P - \frac{1}{\rho} P\nabla G\right]\cdot\mathbf{n}\, dS + \int_V GQ\, dV \tag{3}$$

(Morse & Feshbach 1953; Burridge & Knopoff 1964; Aki & Richards 1980; Berkhout 1985). Note that when $G(\mathbf{r}, \mathbf{r}_A, \omega)$ is a solution of (2), then the complex conjugated function $G^*(\mathbf{r}, \mathbf{r}_A, \omega)$ is also a solution of (2). Hence, the KH integral may be alternatively expressed as

$$P(\mathbf{r}_A, \omega) = \oint_S \left[\frac{1}{\rho} G^*\nabla P - \frac{1}{\rho} P\nabla G^*\right]\cdot\mathbf{n}\, dS + \int_V G^* Q\, dV \tag{4}$$

(Bojarski 1983; Wapenaar *et al.* 1989). Throughout this paper, $G(\mathbf{r}, \mathbf{r}_A, \omega)$ is the frequency-domain representation of the *causal* (or forward propagating) Green's wavefield $g(\mathbf{r}, \mathbf{r}_A, t)$, with $g(\mathbf{r}, \mathbf{r}_A, t) = 0$ for $t < 0$. Consequently, $G^*(\mathbf{r}, \mathbf{r}_A, \omega)$ is the frequency-domain representation of the *anti-causal* (or backward propagating) Green's wavefield $g(\mathbf{r}, \mathbf{r}_A, -t)$. Both versions of the KH integral are exact. KH integral (3) will be used to derive an expression for forward extrapolation of downgoing waves; KH integral (4) will be used to derive an expression for backward extrapolation of upgoing waves.

Note

For a homogeneous medium it is more convenient to suppress ρ in the right-hand sides of eqs (1a) and (2). The terms $1/\rho$ in the right-hand sides of eqs (3) and (4) will then also disappear.

FORWARD AND BACKWARD WAVEFIELD EXTRAPOLATION

We apply the KH integrals in the first layer of the acoustic medium depicted in Fig. 1. We construct a closed surface S that consists of the acquisition surface S_0 (with *in*ward pointing normal vector $\mathbf{n}_0$), a horizontal reference surface S_1' (with *out*ward pointing normal vector $\mathbf{n}_1'$), just above the first interface S_1 and a cylindrical surface S_∞ with a vertical axis and an infinite radius R, see Fig. 2. The volume enclosed by this surface will be denoted by V_1. For the moment, our aim is to compute the *downgoing* and *upgoing* waves at any point A in V_1, given the two-way wavefield at the acquisition surface S_0. In the next section we discuss the

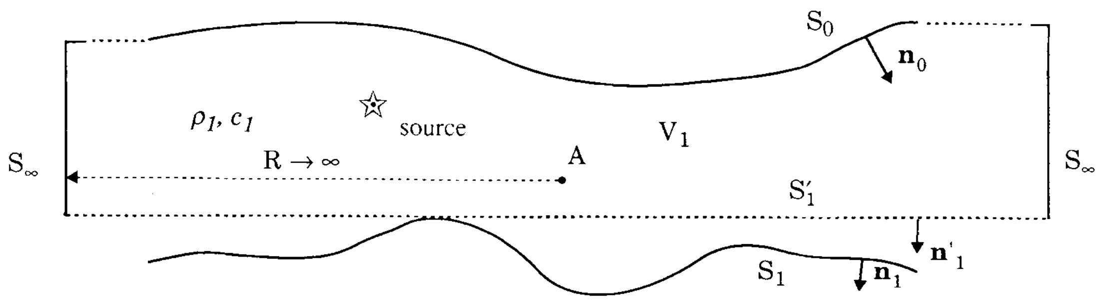

Figure 2. The first layer of the acoustic medium of Fig. 1.

validity of the results when point A is lowered into the 'valleys' of the interface S_1.

Since the volume V_1 is assumed homogeneous (propagation velocity c_1, mass density ρ_1) we may choose for G and G^* the *free-space* solution of eq. (2) (with ρ suppressed in the right-hand side)

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = \frac{1}{4\pi}\frac{e^{-jk_1 \Delta r}}{\Delta r} \tag{5a}$$

and

$$G^*(\mathbf{r}, \mathbf{r}_A, \omega) = \frac{1}{4\pi}\frac{e^{jk_1 \Delta r}}{\Delta r} \tag{5b}$$

with

$$\Delta r = |\mathbf{r} - \mathbf{r}_A| \tag{5c}$$

and

$$k_1 = \omega / c_1. \tag{5d}$$

KH integral (3) may thus be rewritten as

$$P(\mathbf{r}_A, \omega) = P_0(\mathbf{r}_A, \omega) + P_1(\mathbf{r}_A, \omega) + P_s(\mathbf{r}_A, \omega), \tag{6a}$$

where

$$P_0(\mathbf{r}_A, \omega) = \int_{S_0} [P\nabla G - G\nabla P] \cdot \mathbf{n}_0 \, dS_0, \tag{6b}$$

$$P_1(\mathbf{r}_A, \omega) = \int_{S_1'} [G\nabla P - P\nabla G] \cdot \mathbf{n}_1' \, dS_1' \tag{6c}$$

and

$$P_s(\mathbf{r}_A, \omega) = \int_{V_1} GQ \, dV_1, \tag{6d}$$

with G defined by eq. (5a). Note that the contribution over S_∞ vanishes (the area of the cylindrical surface is proportional to the radius R, the integrant is proportional to $1/R^2$). Similarly, KH integral (4) may be rewritten as

$$P(\mathbf{r}_A, \omega) = \hat{P}_0(\mathbf{r}_A, \omega) + \hat{P}_1(\mathbf{r}_A, \omega) + \hat{P}_s(\mathbf{r}_A, \omega), \tag{7a}$$

where

$$\hat{P}_0(\mathbf{r}_A, \omega) = \int_{S_0} [P\nabla G^* - G^*\nabla P] \cdot \mathbf{n}_0 \, dS_0, \tag{7b}$$

$$\hat{P}_1(\mathbf{r}_A, \omega) = \int_{S_1'} [G^*\nabla P - P\nabla G^*] \cdot \mathbf{n}_1' \, dS_1' \tag{7c}$$

and

$$\hat{P}_s(\mathbf{r}_A, \omega) = \int_{V_1} G^*Q \, dV_1, \tag{7d}$$

with G^* defined by eq. (5b). Note that in practice $P_0(\mathbf{r}_A, \omega)$ and $\hat{P}_0(\mathbf{r}_A, \omega)$ can be computed as we assumed that P and $\nabla P \cdot \mathbf{n}_0$ are known on S_0 (for example, at a free surface P is zero and $\nabla P \cdot \mathbf{n}_0$ is proportional to the normal component of the particle velocity, measured by the geophones). Also $P_s(\mathbf{r}_A, \omega)$ and $\hat{P}_s(\mathbf{r}_A, \omega)$ can be computed if we assume that the source distribution $Q(\mathbf{r}, \omega)$ is known. However, $P_1(\mathbf{r}_A, \omega)$ and $\hat{P}_1(\mathbf{r}_A, \omega)$ cannot be computed in practice because no measurements of P and $\nabla P \cdot \mathbf{n}_1'$ are available on the horizontal reference surface S_1'. In Appendix A it is shown that $P_1(\mathbf{r}_A, \omega)$ and $\hat{P}_1(\mathbf{r}_A, \omega)$ may be written as

$$P_1(\mathbf{r}_A, \omega) = P^-(\mathbf{r}_A, \omega) \tag{8a}$$

and

$$\hat{P}_1(\mathbf{r}_A, \omega) \approx P^+(\mathbf{r}_A, \omega), \tag{8b}$$

where P^- and P^+ represent upgoing and downgoing wavefields, respectively, such that

$$P(\mathbf{r}_A, \omega) = P^+(\mathbf{r}_A, \omega) + P^-(\mathbf{r}_A, \omega). \tag{8c}$$

The condition is that the region between z_A (the depth level of point A) and z_1 (the depth level of the horizontal reference surface S_1') is source free (the approximation in (8b) arises from having neglected evanescent waves). Hence, ignoring $P^- = P_1$ in eq. (6a) yields $P^+ = P_0 + P_s$, or

$$P^+(\mathbf{r}_A, \omega) = \int_{S_0} \left[P\frac{\partial G}{\partial n_0} - G\frac{\partial P}{\partial n_0}\right] dS_0 + \int_{V_1} GQ \, dV_1, \tag{9a}$$

where $\partial/\partial n_0$ is a shorthand notation for $\mathbf{n}_0 \cdot \nabla$. Similarly, ignoring $P^+ \approx \hat{P}_1$ in eq. (7a) yields $P^- \approx \hat{P}_0 + \hat{P}_s$, or

$$P^-(\mathbf{r}_A, \omega) \approx \int_{S_0} \left[P\frac{\partial G^*}{\partial n_0} - G^*\frac{\partial P}{\partial n_0}\right] dS_0 + \int_{V_1} G^*Q \, dV_1. \tag{9b}$$

Equations (9a) and (9b) describe forward and backward extrapolation of downgoing and upgoing waves, respectively, to any point A below the sources and above the horizontal reference surface S_1', see Fig. 2. They can be combined, yielding the following expression for the two-way acoustic pressure at A:

$$P(\mathbf{r}_A, \omega) \approx 2\int_{S_0} \left[P\frac{\partial \mathcal{Re}(G)}{\partial n_0} - \mathcal{Re}(G)\frac{\partial P}{\partial n_0}\right] dS_0 + 2\int_{V_1} \mathcal{Re}(G) Q \, dV_1. \tag{10}$$

The only approximation in eqs 8(b), 9(b) and 10 arises from having neglected evanescent waves at A (compare eqs A13 and A14). This imposes a restriction to the maximum obtainable spatial resolution (Berkhout 1984). Ignoring the evanescent waves, however, has the advantage that these equations are unconditionally stable. In the following we will write = instead of $\approx$ when only the evanescent waves are ignored.

THE RAYLEIGH HYPOTHESIS

So far we have assumed that point A lies above the horizontal reference surface S_1', i.e. above the highest point of interface S_1. An interesting question now is: is eq. (9) still valid when point A is lowered into the valleys of interface S_1? Clearly the derivation given above does not apply to the region below S_1'. On the other hand, if there was no contrast at S_1, then S_1' could as well have been chosen just below S_1 so eq. (9) would indeed be valid in the valleys of S_1. Apparently, as long as the interface S_1 does not 'disturb' the downgoing and upgoing waves, then eq. (9) is valid also in the valleys of the interface. Clearly there are situations for which eq. (9) breaks down. When multiple scattering occurs between the irregularities in the interface S_1 (see Fig. 3) then a part of the downgoing (or upgoing) wavefield is not incorporated in eq. (9a) (or 9b). We may also say that a part of the downgoing (or upgoing) wavefield is a scattered (or incident) wavefield.

In his book *The theory of sound* Lord Rayleigh (1878; reprinted in 1965) analyses the scattering by an irregular surface, assuming that everywhere above the surface (i.e. including the valleys) the incident and scattered wavefields may be expressed in terms of, respectively, downgoing and upgoing plane waves. This assumption is commonly known as the Rayleigh hypothesis. Van den Berg & Fokkema (1980) prove that under certain conditions the Rayleigh hypothesis holds rigorously (these conditions set limits to the roughness of the interface; this will not be further discussed in this paper).

Hence, when interface S_1 satisfies the conditions for the validity of the Rayleigh hypothesis, multiple reflections as pictured in Fig. 3 will not occur. Consequently, for this situation $P^+(\mathbf{r}_A, \omega)$ and $P^-(\mathbf{r}_A, \omega)$ as given by eq. (9), describe the 'undisturbed' downgoing and upgoing wavefields, respectively, also in the valleys of interface S_1. Similarly, for the same situation $P(\mathbf{r}_A, \omega)$ as given by eq. (10) describes the *total* two-way wavefield in the valleys of interface S_1.

RECURSIVE TWO-WAY KIRCHHOFF-HELMHOLTZ EXTRAPOLATION

In the following we assume that all interfaces satisfy the conditions for the validity of the Rayleigh hypothesis.

Consider eq. (10), generalized for extrapolation through layer m (see Fig. 1),

$$P(\mathbf{r}_A, \omega) = 2\int_{S_{m-1}} \left\{ P(\mathbf{r}, \omega)\frac{\partial \mathcal{Re}\,[G(\mathbf{r}, \mathbf{r}_A, \omega)]}{\partial n_{m-1}} - \mathcal{Re}\,[G(\mathbf{r}, \mathbf{r}_A, \omega)]\frac{\partial P(\mathbf{r}, \omega)}{\partial n_{m-1}} \right\} dS_{m-1} + 2\int_{V_m} \mathcal{Re}\,[G(\mathbf{r}, \mathbf{r}_A, \omega)] Q(\mathbf{r}, \omega)\, dV_m \tag{11a}$$

where

$$G(\mathbf{r}, \mathbf{r}_A, \omega) = \frac{1}{4\pi}\frac{e^{-jk_m \Delta r}}{\Delta r}, \tag{11b}$$

with

$$\Delta r = |\mathbf{r} - \mathbf{r}_A| \tag{11c}$$

and

$$k_m = \omega / c_m. \tag{11d}$$

Note that the volume integral need only be evaluated in layers containing sources (generally the first layer only).

In the following, point A may be any point just above interface S_m. $P(\mathbf{r}_A, \omega)$ thus represents the total wavefield just above S_m. We can easily find the total wavefield just below S_m by applying the following boundary condition for the acoustic pressure:

$$\lim_{\mathbf{r} \uparrow S_m} P(\mathbf{r}, \omega) = \lim_{\mathbf{r} \downarrow S_m} P(\mathbf{r}, \omega). \tag{12}$$

This total wavefield just below S_m can again be used in eq. (11), (with S_{m-1} and V_m replaced by S_m and V_{m+1}), to

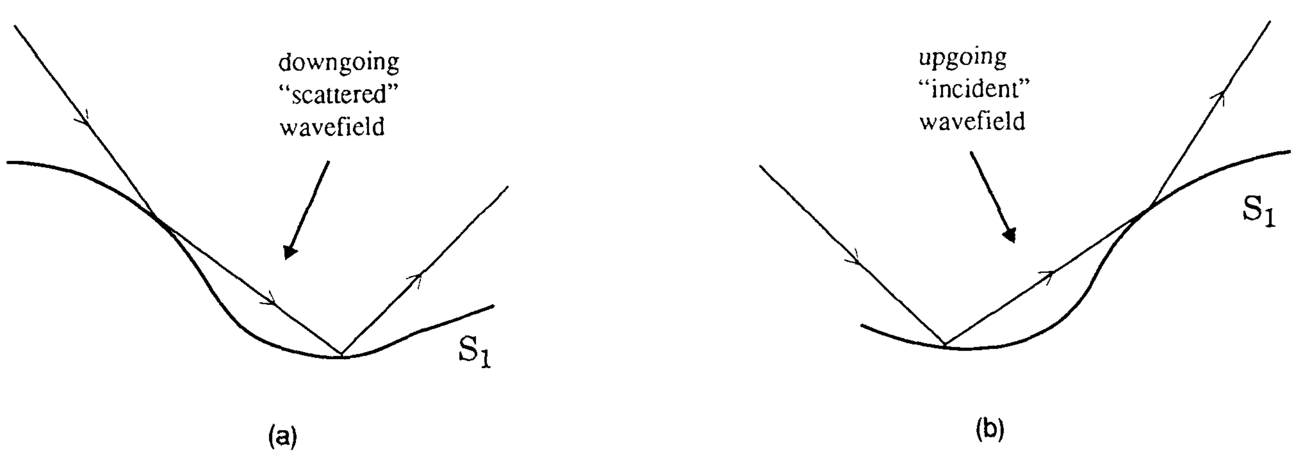

Figure 3. (a) Situation for which eq. (9a) breaks down. (b) Situation for which eq. (9b) breaks down.

compute the total wavefield just above S_{m+1}. For this purpose, however, we also need an expression for $\partial P(\mathbf{r}, \omega)/\partial n_m$. From (11) we obtain

$$\frac{\partial P(\mathbf{r}_A, \omega)}{\partial n_{m,A}} = 2\int_{S_{m-1}} \left\{ P(\mathbf{r}, \omega) \frac{\partial^2 \mathcal{Re}\,[G(\mathbf{r}, \mathbf{r}_A, \omega)]}{\partial n_{m-1}\,\partial n_{m,A}} - \frac{\partial \mathcal{Re}\,[G(\mathbf{r}, \mathbf{r}_A, \omega)]}{\partial n_{m,A}} \frac{\partial P(\mathbf{r}, \omega)}{\partial n_{m-1}} \right\} dS_{m-1} + 2\int_{V_m} \frac{\partial \mathcal{Re}\,[G(\mathbf{r}, \mathbf{r}_A, \omega)]}{\partial n_{m,A}} Q(\mathbf{r}, \omega)\, dV_m, \tag{13}$$

where $\partial/\partial n_{m,A}$ stands for $\mathbf{n}_m \cdot \boldsymbol{\nabla}_A$, $\boldsymbol{\nabla}_A$ being the gradient at $\mathbf{r}_A$.

In analogy with (12), the boundary condition for the particle velocity at S_m reads

$$\lim_{\mathbf{r}\uparrow S_m} \left[\frac{1}{\rho_{m+1}} \frac{\partial P(\mathbf{r}, \omega)}{\partial n_m}\right] = \lim_{\mathbf{r}\downarrow S_m} \left[\frac{1}{\rho_m} \frac{\partial P(\mathbf{r}, \omega)}{\partial n_m}\right]. \tag{14}$$

Equations (11) and (13) together can be used in a recursive mode to extrapolate the two-way wavefield, measured at the surface, downward from layer interface to layer interface.

Note that the boundary conditions at the layer interfaces are extremely simple taken into account (eqs 12 and 14).

RECURSIVE ONE-WAY KIRCHHOFF-HELMHOLTZ EXTRAPOLATION

The recursive scheme discussed in the previous section is very sensitive to errors in the description of the sources, the layer velocities and the interfaces. This is inherent to two-way extrapolation schemes, that take into account primary as well as multiply reflected waves. One-way extrapolation schemes for primary waves are much more robust. The use of one-way extrapolation schemes is validated when the *surface-related* multiple reflections are eliminated and when the *internal* multiple reflections are small (Berkhout 1986). Surface-related multiple elimination does not require any knowledge about the layer velocities and interfaces. It is sensitive, though, for the description of the source distribution, However, this sensitivity can be advantageously used for estimating the source properties (Verschuur, Berkhout & Wapenaar 1992).

In the following we assume that the surface-related multiple reflections as well as the direct source waves have been eliminated. Our aim is to backward extrapolate the *primary* upgoing wave P^- (see Fig. 4) from layer interface to layer interface. To this end we derive a one-way version of the recursive scheme discussed in the previous section (Wapenaar & Berkhout 1989). Consider eq. (9b), generalized for extrapolation through layer m,

$$P^-(\mathbf{r}_A, \omega) = \int_{S_{m-1}} \left[P(\mathbf{r}, \omega) \frac{\partial G^*(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial n_{m-1}} - G^*(\mathbf{r}, \mathbf{r}_A, \omega) \frac{\partial P(\mathbf{r}, \omega)}{\partial n_{m-1}} \right] dS_{m-1}. \tag{15}$$

Note that we omitted the volume integral as we assumed that the source waves were eliminated. $P(\mathbf{r}, \omega)$ represents the two-way acoustic wavefield at $\mathbf{r}$ on S_{m-1}, whereas $P^-(\mathbf{r}_A, \omega)$ represents the *upgoing* acoustic wavefield at $\mathbf{r}_A$ just above S_m. For the *primary* wave, this upgoing term represents the *total* wavefield just above S_m, see Fig. 4. Hence, we can easily find the two-way wavefield just below S_m by applying the following boundary condition for the acoustic pressure:

$$\lim_{\mathbf{r}\uparrow S_m} P(\mathbf{r}, \omega) = \lim_{\mathbf{r}\downarrow S_m} P^-(\mathbf{r}, \omega). \tag{16}$$

(Note the subtle difference with eq. 12.) This total wavefield just below S_m (the upgoing primary wave plus its reflection from S_m, see Fig. 4) can again be used in eq. (15), (with S_{m-1} replaced by S_m), to compute the upgoing wavefield just above S_{m+1}. As in the previous section, for this purpose we also need an expression for $\partial P(\mathbf{r}, \omega)/\partial n_m$. From (15) we obtain

$$\frac{\partial P^-(\mathbf{r}_A, \omega)}{\partial n_{m,A}} = \int_{S_{m-1}} \left[P(\mathbf{r}, \omega) \frac{\partial^2 G^*(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial n_{m-1}\,\partial n_{m,A}} - \frac{\partial G^*(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial n_{m,A}} \frac{\partial P(\mathbf{r}, \omega)}{\partial n_{m-1}} \right] dS_{m-1}. \tag{17}$$

In analogy with (16), the boundary condition for the particle velocity at S_m reads

$$\lim_{\mathbf{r}\uparrow S_m} \left[\frac{1}{\rho_{m+1}} \frac{\partial P(\mathbf{r}, \omega)}{\partial n_m}\right] = \lim_{\mathbf{r}\downarrow S_m} \left[\frac{1}{\rho_m} \frac{\partial P^-(\mathbf{r}, \omega)}{\partial n_m}\right]. \tag{18}$$

Equations (15) and (17) together can be used in a

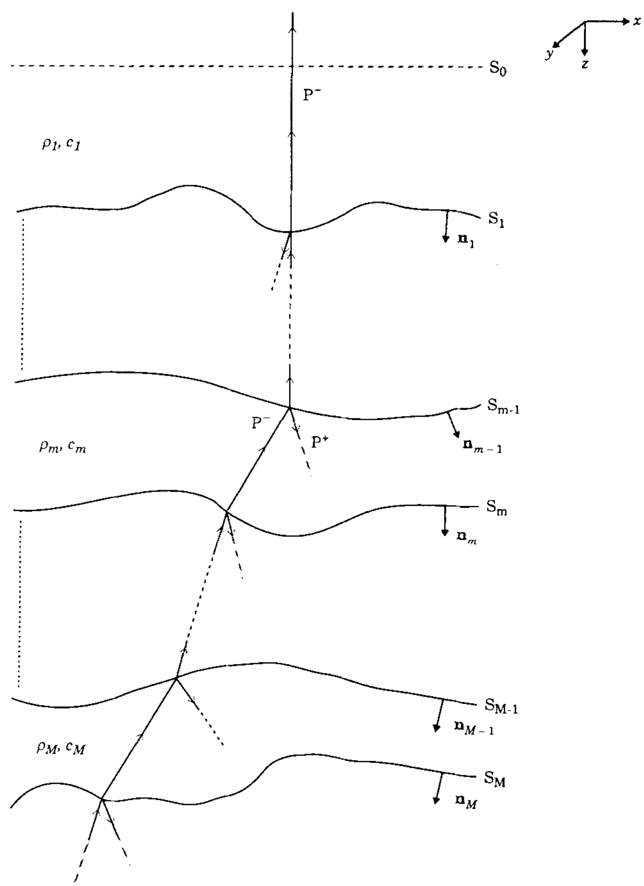

Figure 4. Primary upgoing wave P^- in a layered acoustic medium. After surface-related multiple elimination, surface S_0 may be considered reflection free.

recursive mode to extrapolate the one-way primary upgoing wavefield backward from layer interface to layer interface. The procedure starts at the surface S_0, which is considered reflection free as a result of the surface-related multiple-elimination procedure. The procedure stops when the interface is reached that 'generated' this primary upgoing wavefield.

If surface S_0 is planar, then Kirchhoff–Helmholtz integrals (15) and (17) may be replaced by the following one-way Rayleigh integrals

$$P^-(\mathbf{r}_A, \omega) = 2\int_{S_0}\left[P^-(\mathbf{r}, \omega)\frac{\partial G^*(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial n_0}\right] dS_0 \tag{19}$$

and

$$\frac{\partial P^-(\mathbf{r}_A, \omega)}{\partial n_{1,A}} = 2\int_{S_0}\left[P^-(\mathbf{r}, \omega)\frac{\partial^2 G^*(\mathbf{r}, \mathbf{r}_A, \omega)}{\partial n_0\, \partial n_{1,A}}\right] dS_0. \tag{20}$$

Note that this simplification is useful only at a planar *reflection-free* surface S_0, where the total wavefield is given by the upgoing wavefield only. At interfaces S_m (planar or curved) the use of full Kirchhoff–Helmholtz integrals (15) and (17) is preferred because these account automatically for transmission effects.

EXAMPLE I

In this first example we analyse the *amplitude* handling of the recursive one-way extrapolation scheme, discussed in the previous section. For simplicity we consider two homogeneous half-spaces separated by a horizontal interface at $z_1 = 500$ m, see Fig. 5(a). A 2-D acoustic wavefield is radiated by a line source in the lower half-space at ($x = 0$, $z_3 = 1500$ m) (note that the line source is parallel to the y-axis). The upgoing wavefield at $z_0 = 0$ m is shown in the space–time domain in Fig. 5(b). These are the input data for our experiment. The upgoing wavefield at $z_2 = 1000$ m is shown in Fig. 5(c). These data serve as a reference for the output of our experiment.

The data of Fig. 5(b) are Fourier transformed from the time domain to the frequency domain. For each frequency component the following steps are carried out:

- backward extrapolation from $z_0 = 0$ m to $z_1 = 500$ m, using eqs (19) and (20).
- Application of the boundary conditions (16) and (18) at $z_1 = 500$ m (which is nothing but multiplying $\partial P^-/\partial z$ by $\rho_2/\rho_1 = 2$).
- Backward extrapolation from $z_1 = 500$ m to $z_2 = 1000$ m, using eq. (15).

The result, transformed back to the time domain, is shown in Fig. 6(a). Fig. 6(b) shows amplitude cross-sections of the ideal output data in Fig. 5(c) (solid line) and of the recursive extrapolation result in Fig. 6(a) (dotted line). The perfect match confirms that our recursive procedure properly accounts for the angle-dependent transmission effects at the interface.

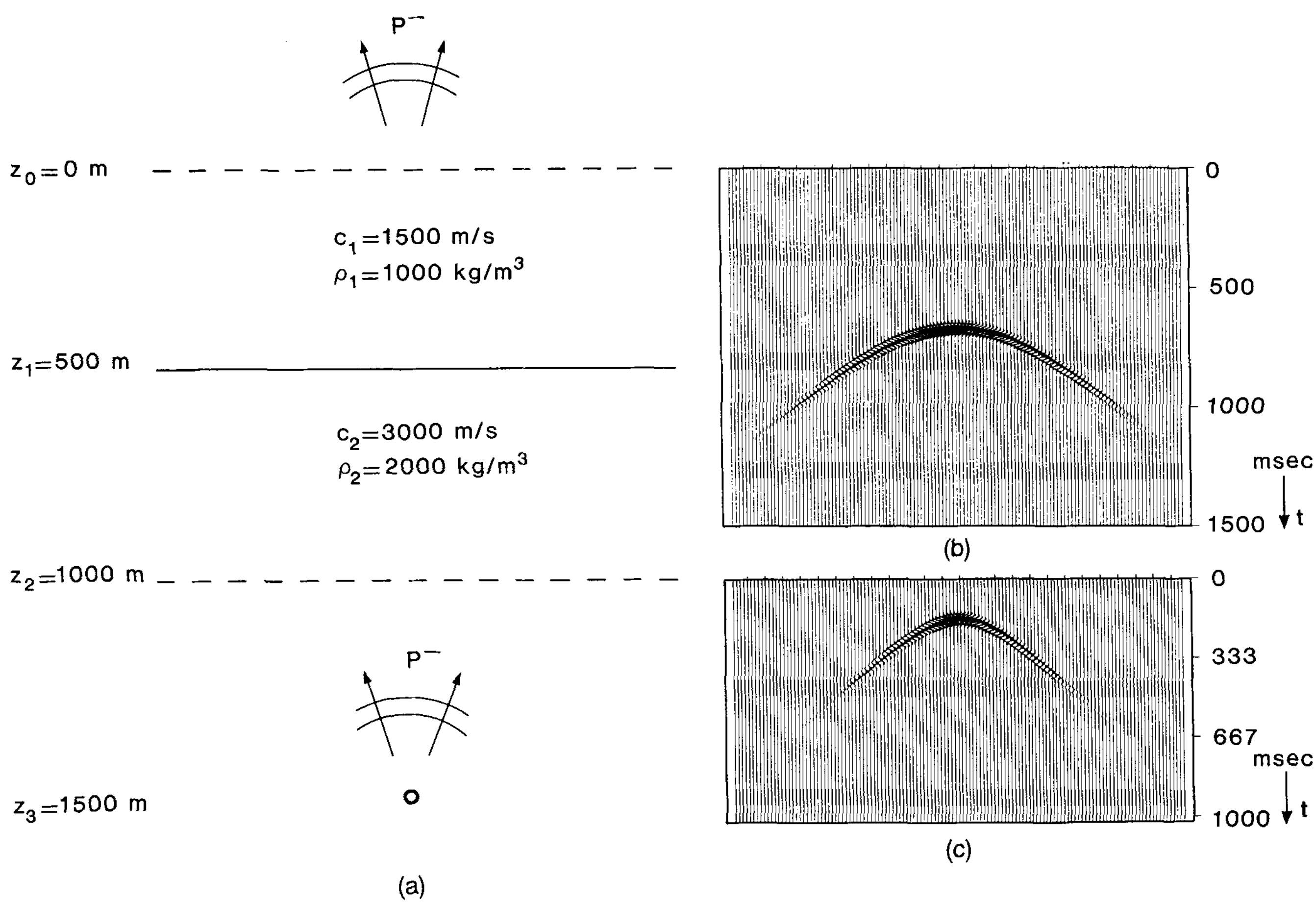

Figure 5. (a) Inhomogeneous acoustic medium with a horizontal interface. (b) Upgoing wavefield at z_0. (c) Upgoing wavefield at z_2.

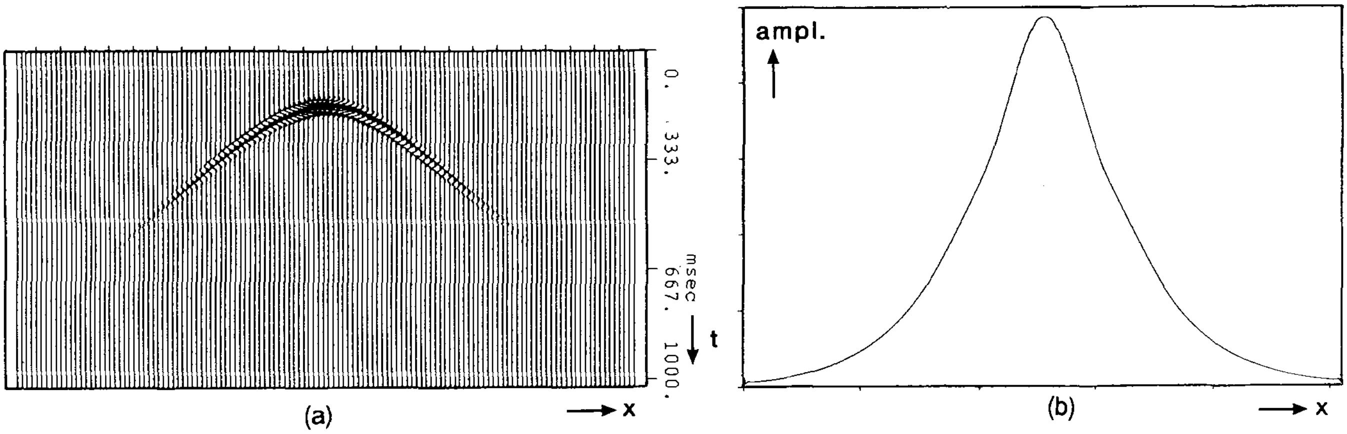

Figure 6. Recursive one-way Kirchhoff-Helmholtz extrapolation result. (a) Upgoing wavefield at z_2. (b) Maximum amplitude per trace (dotted line) compared with the exact result (solid line). (The dotted line is hidden by the solid line.)

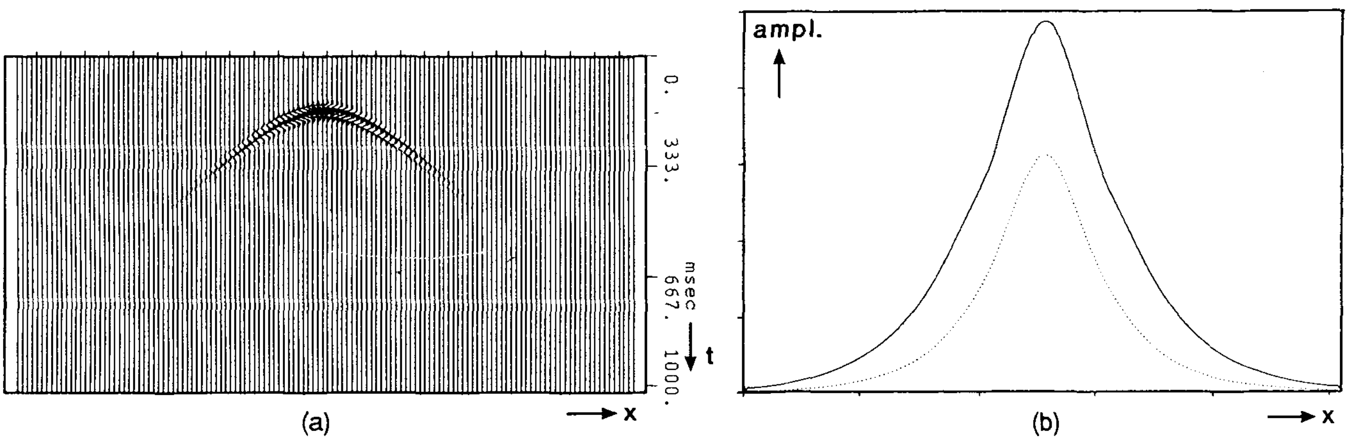

Figure 7. Non-recursive extrapolation result. (a) Upgoing wavefield at z_2. (b) Maximum amplitude per trace (dotted line) compared with the exact result (solid line).

In the practice of seismic exploration, sometimes a single non-recursive extrapolation step is carried out (here: from $z_0 = 0$ m to $z_2 = 1000$ m). Non-recursive extrapolation is described by eq. (4), with P replaced by P^- and with the volume integral omitted. Bear in mind that G in eq. (4) is the solution of eq. (2) for the inhomogeneous medium, hence it includes the transmission effects of the interface at $z_1 = 500$ m. The result of non-recursive extrapolation is shown in Fig. 7(a). Fig. 7(b) shows amplitude cross-sections of the ideal output data in Fig. 5(c) (solid line) and of the non-recursive extrapolation result in Fig. 7(a) (dotted line). Note that there is an overall amplitude loss. In Wapenaar *et al.* (1989) we show that the amplitude loss is proportional to the square of the reflectivity of the interface. For this example the reflection coefficient for normal incidence equals 0.6, hence the amplitude loss at normal incidence is $(0.6)^2 \times 100\% = 36\%$.

Comparing Figs 6 and 7, we may conclude that the amplitude handling of the proposed recursive one-way Kirchhoff–Helmholtz extrapolation scheme is superior.

EXAMPLE II

In this second example we consider a somewhat more complicated configuration. The medium consists of two homogeneous half-spaces separated by an interface at $z_1 = 200$ m, with an anticlinal structure centred around $x = 0$, see Fig. 8. A 2-D acoustic wavefield is radiated by a line source in the lower half-space at ($x = 0$, $z_3 = 1200$ m). The upgoing wavefield at $z_0 = 0$ m is shown in the space–time domain in Fig. 9. These are the input data for our experiment. The second arrival is due to an internal reflection in the anticlinal structure (bear in mind that for the validity of eq. (9) there should be no multiple reflections in the valleys *above* the reflector; the internal reflection *below* the anticline should cause no problems).

The result of recursive one-way Kirchhoff–Helmholtz extrapolation from $z_0 = 0$ m via the interface to $z_2 = 400$ m is shown in Fig. 10. Note that the second arrival vanished almost completely (the remaining artefacts are due to the aperture limitations). This result confirms that our recursive procedure properly accounts for the complicated phenomena related to a curved interface. For comparison, Fig. 11 shows the result of recursively applying the one-way Rayleigh integral (eq. 19), in accordance with the common practice of seismic exploration. The strength of the non-physical arrival is significant.

CONCLUSIONS

In modelling applications, recursive Kirchhoff–Helmholtz extrapolation through a layered acoustic medium requires

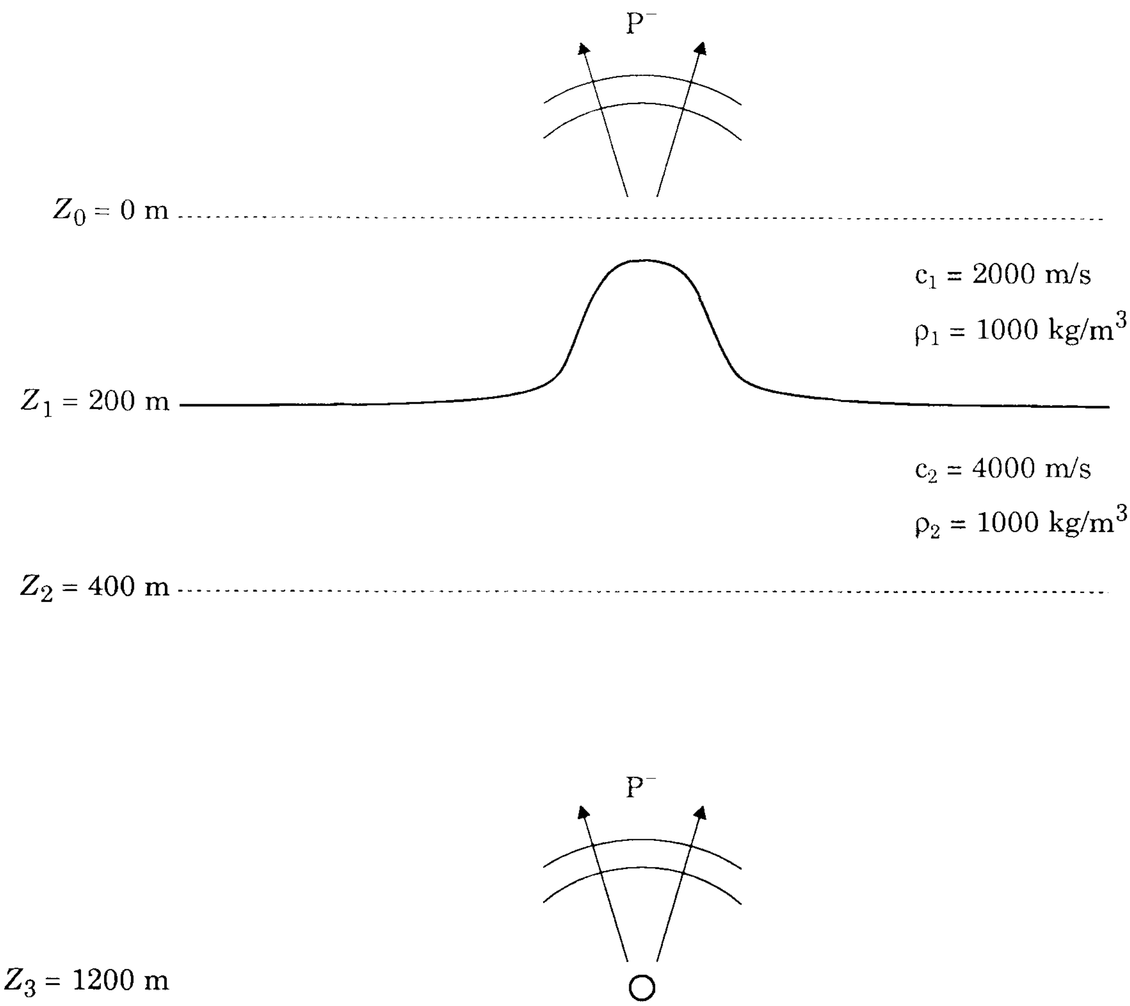

Figure 8. Inhomogeneous medium with an anticlinal structure.

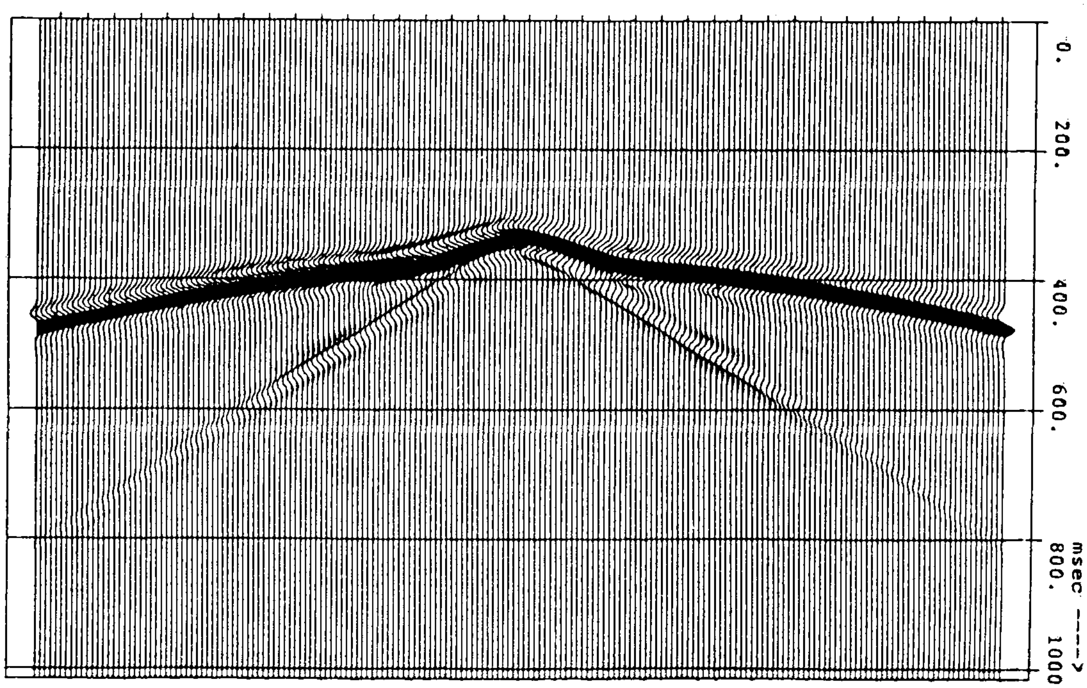

Figure 9. Upgoing wavefield at z_0.

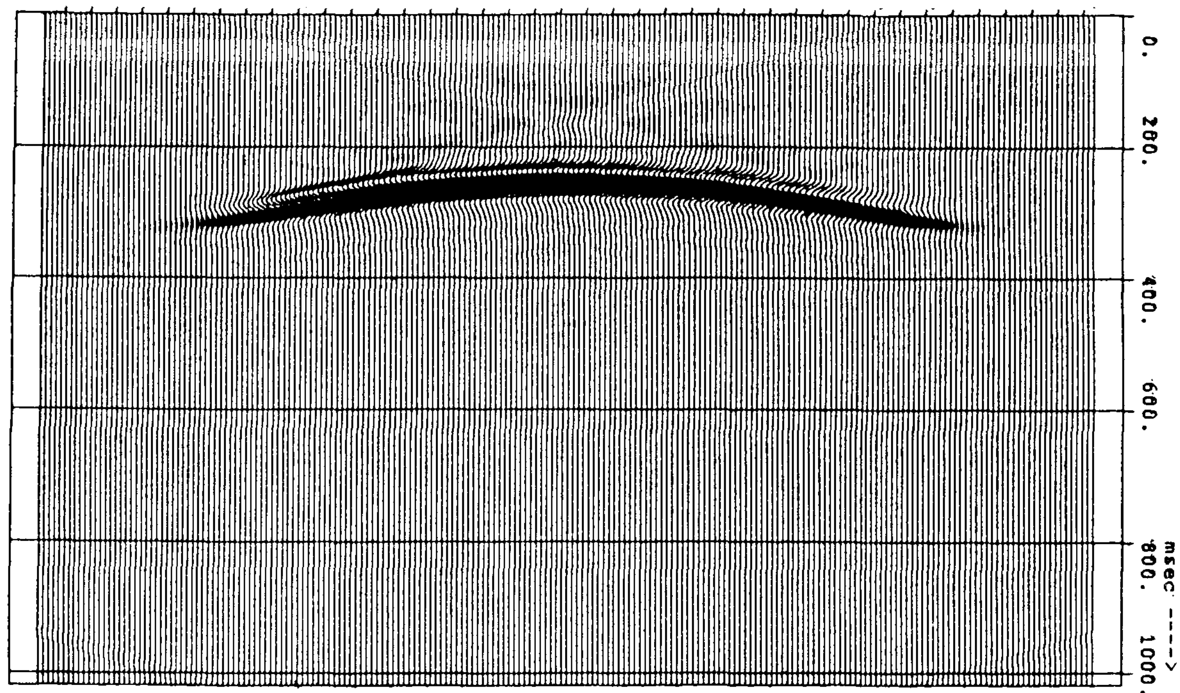

Figure 10. Recursive one-way Kirchhoff–Helmholtz extrapolation result (from Peels 1988).

plane-wave decomposition and synthesis at the layer interfaces in order to take transmission effects into account (Hill & Wuenschel 1985; Wenzel *et al.* 1990).

In principle the same approach can be used for inverse applications, however, the plane-wave decomposition and synthesis is superfluous, as was shown in this paper. The forward and inverse approaches require that the interfaces satisfy the conditions for the validity of the Rayleigh hypothesis (van den Berg & Fokkema 1980). Moreover, both inverse approaches (with or without plane-wave decomposition and synthesis) ignore evanescent waves (which is necessary for stability reasons and which restricts the maximum-obtainable spatial resolution). The advantage of the recursive approaches proposed in this paper is that

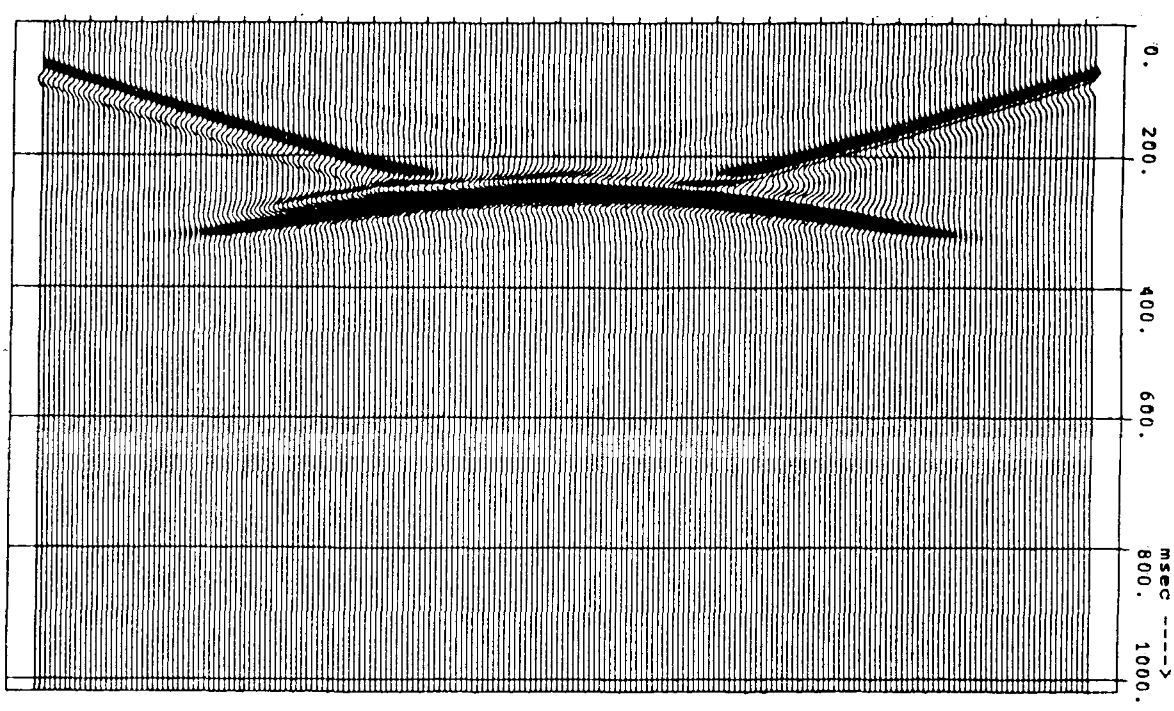

Figure 11. Recursive one-way Rayleigh extrapolation result (from Peels 1988).

the plane-wave decomposition and synthesis at the interfaces is avoided and therefore it is simple and no additional assumptions or approximations are made.

Two different versions of recursive KH extrapolation have been proposed. The *two-way* version (eqs 11 and 13) is applied directly to the measured surface data (e.g. the normal component of the particle velocity at a free surface S_0) and takes into account all primary as well as multiply reflected waves. This two-way version, however, is very sensitive to small errors in the description of the source, the layer velocities and the layer interfaces. (Using this sensitivity for macro model estimation is subject of current research.) The *one-way* version (eqs 15 and 17), on the other hand, is applied to the upgoing wavefield after surface-related multiple elimination. This method ignores internal multiple reflections and is therefore robust with respect to small errors in the description of the layered medium.

Recursive one-way KH extrapolation is proposed as the basis for *true amplitude* migration or redatuming, particularly for situations where the contrasts at the layer interfaces are high.

ACKNOWLEDGMENT

The author thanks Dr J. T. Fokkema for interesting discussions about the Rayleigh hypothesis.

REFERENCES

Aki, K. & Richards, P. G., 1980. *Quantitative seismology: theory and methods*, W. H. Freeman & Company, San Francisco.

Berkhout, A. J., 1984. *Seismic resolution: resolving power of acoustic echo techniques*, Geophysical Press Ltd, London.

Berkhout, A. J., 1985. *Seismic migration: imaging of acoustic energy by wavefield extrapolation A. Theoretical aspects*, Elsevier Science Publishers, The Netherlands, Amsterdam.

Berkhout, A. J., 1986. Seismic inversion in terms of pre-stack migration and multiple elimination, *Proceedings of the IEEE*, **74,** 415–427.

Berkhout, A. J. & Wapenaar, C. P. A., 1989. One-way versions of the Kirchhoff integral, *Geophysics*, **54,** 460–467.

Berryhill, J. R., 1984. Wave equation datuming before stack, *Geophysics*, **49,** 2064–2066.

Bleistein, N., 1984. *Mathematical methods for wave phenomena*, Academic Press, Inc., Orlando.

Bojarski, N. N., 1983. Generalized reaction principles and reciprocity theorems for the wave equations, and the relationship between the time-advanced and time-retarded fields, *J. acoust. Soc. Am.*, **74,** 281–285.

Burridge, R. & Knopoff, L., 1964. Body force equivalents for seismic dislocations, *Bull. seism. Soc. Am.*, **54,** 1875–1888.

Frazer, L. N. & Sen, M. K., 1985. Kirchhoff–Helmholtz reflection seismograms in a laterally inhomogeneous multilayered elastic medium—I. Theory, *Geophys. J. R. astr. Soc.*, **80,** 121–147.

Hill, N. R. & Wuenschel, P. C., 1985. Numerical modelling of refraction arrivals in complex areas, *Geophysics*, **50,** 90–98.

Hilterman, F. J., 1970. Three-dimensional seismic modelling, *Geophysics*, **35,** 1020–1037.

Kampfmann, W., 1988. A study of diffraction-like events on Dekorp 2-S by Kirchhoff theory, *J. Geophys.*, **62,** 163–174.

Morse, P. M. & Feshbach, H., 1953. *Methods of theoretical physics*, McGraw-Hill Book Company, Inc., New York.

Peels, G. L., 1988. True amplitude wave field extrapolation with applications in seismic shot record redatuming, *DSc thesis*, Delft University of Technology.

Rayleigh, J. W. S., 1965. The theory of sound, Vol. II, Section 272 (1st edn printed—1878), Dover Publications, Inc., New York.

Schneider, W. A., 1978. Integral formulation for migration in two and three dimensions, *Geophysics*, **43,** 49–76.

Van den Berg, P. M. & Fokkema, J. T., 1980. The Rayleigh hypothesis in the theory of diffraction by a perturbation in a plane surface, *Radio Sci.*, **15,** 723–732.

Verschuur, D. J., Berkhout, A. J. & Wapenaar, C. P. A., 1992. Adaptive surface related multiple elimination, *Geophysics*, **57,** 1166–1177.

Wapenaar, C. P. A. & Berkhout, A. J., 1989. *Elastic wavefield extrapolation: redatuming of single- and multi-component seismic data*, Elsevier Science Publishers, The Netherlands.

Wapenaar, C. P. A., Peels, G. L., Budejicky, V. & Berkhout, A. J., 1989. Inverse extrapolation of primary seismic waves, *Geophysics*, **54,** 853–863.

Wenzel, F., Stenzel, K. J. & Zimmermann, U., 1990. Wave propagation in laterally heterogeneous layered media, *Geophys. J. Int.*, **103,** 675–684.

APPENDIX A: EVALUATION OF P_1 AND $\hat{P}_1$

We analyse eq. (6c), which may be rewritten as

$$P_1(x_A, y_A, z_A; \omega) = \iint_{-\infty}^{\infty} \left(G\frac{\partial P}{\partial z} - P\frac{\partial G}{\partial z} \right)_{z_1} dx\, dy, \qquad \text{(A1)}$$

where z_1 is the depth of the horizontal reference surface S'_1. Define the 2-D spatial Fourier transform of $P(x, y, z; \omega)$ according to

$$\tilde{P}(k_x, k_y, z; \omega) = \iint_{-\infty}^{\infty} P(x, y, z; \omega) e^{j(k_x x + k_y y)}\, dx\, dy. \qquad \text{(A2a)}$$

Similarly, define the 2-D spatial Fourier transform of $G(x, y, z; x_A, y_A, z_A; \omega)$ according to

$$\tilde{G}(k_x, k_y, z; x_A, y_A, z_A; \omega) = \iint_{-\infty}^{\infty} G(x, y, z; x_A, y_A, z_A; \omega) e^{j(k_x x + k_y y)}\, dx\, dy. \qquad \text{(A2b)}$$

Applying the following version of Parseval's theorem

$$\iint_{-\infty}^{\infty} \mathrm{A}(x, y)\mathrm{B}(x, y)\, dx\, dy = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \tilde{\mathrm{A}}(-k_x, -k_y)\tilde{B}(k_x, k_y)\, dk_x\, dk_y$$

to the integral in the right-hand side of eq. (A1) yields

$$P_1(x_A, y_A, z_A; \omega) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \left(\tilde{G}_0 \frac{\partial \tilde{P}}{\partial z} - \tilde{P}\frac{\partial \tilde{G}_0}{\partial z} \right)_{z_1} dk_x\, dk_y, \qquad \text{(A3)}$$

where

$$\tilde{G}_0 = \tilde{G}(-k_x, -k_y, z; x_A, y_A, z_A; \omega), \qquad \text{(A4a)}$$

or, upon substitution of eq. (5a) into (A2b)

$$\tilde{G}_0 = e^{-j(k_x x_A + k_y y_A)} \frac{e^{-jk_{z,1}|z-z_A|}}{2jk_{z,1}}, \tag{A4b}$$

where

$$k_{z,1} \triangleq +\sqrt{k_1^2 - k_x^2 - k_y^2} \quad \text{for} \quad k_x^2 + k_y^2 \le k_1^2 \tag{A4c}$$

and

$$k_{z,1} \triangleq -j\sqrt{k_x^2 + k_y^2 - k_1^2} \quad \text{for} \quad k_x^2 + k_y^2 \ge k_1^2, \tag{A4d}$$

see Berkhout (1985, Appendix F). Note that

$$\frac{\partial \tilde{G}_0}{\partial z} = \mp jk_{z,1}\tilde{G}_0 \quad \text{for} \quad z \gtrless z_A. \tag{A5}$$

Similarly, if $\tilde{P}$ is defined as a superposition of downgoing and upgoing waves, according to

$$\tilde{P} = \tilde{P}^+ + \tilde{P}^- \tag{A6a}$$

then we may write for $\partial\tilde{P}^\pm/\partial z$ in layer 1 outside the source region

$$\frac{\partial \tilde{P}^\pm}{\partial z} = \mp jk_{z,1}\tilde{P}^\pm. \tag{A6b}$$

Substitution of (A4), (A5) and (A6) into (A3) yields

$$P_1(x_A, y_A, z_A; \omega) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \tilde{P}^-(k_x, k_y, z_A; \omega) \times e^{-j(k_x x_A + k_y y_A)}\, dk_x\, dk_y \tag{A7a}$$

where

$$\tilde{P}^-(k_x, k_y, z_A; \omega) = e^{-jk_{z,1}(z_1 - z_A)}\tilde{P}^-(k_x, k_y, z_1; \omega), \tag{A7b}$$

(Berkhout & Wapenaar 1989). According to eq. (A7a), $P_1(x_A, y_A, z_A; \omega)$ is given by the inverse spatial Fourier transform of $\tilde{P}^-(k_x, k_y, z_A; \omega)$, where, according to (A7b), $\tilde{P}^-(k_x, k_y, z_A; \omega)$ represents the *upgoing* wavefield at depth level z_A (assuming the region between z_A and z_1 is source free). Hence,

$$P_1(x_A, y_A, z_A; \omega) = P^-(x_A, y_A, z_A; \omega). \tag{A8}$$

Next, we analyse eq. (7c), which may be rewritten as

$$\hat{P}_1(x_A, y_A, z_A, \omega) = \iint_{-\infty}^{\infty} \left[G^* \frac{\partial P}{\partial z} - P\frac{\partial G^*}{\partial z}\right]_{z_1} dx\, dy. \tag{A9}$$

Applying the following version of Parseval's theorem

$$\iint_{-\infty}^{\infty} \mathrm{A}^*(x, y)\mathrm{B}(x, y)\, dx\, dy = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \tilde{\mathrm{A}}^*(k_x, k_y)\tilde{\mathrm{B}}(k_x, k_y)\, dk_x\, dk_y$$

to the integral in the right-hand side of eq. (A9) yields

$$\hat{P}_1(x_A, y_A, z_A; \omega) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \left[\tilde{G}^* \frac{\partial \tilde{P}}{\partial z} - \tilde{P}\frac{\partial \tilde{G}^*}{\partial z}\right]_{z_1} dk_x\, dk_y \tag{A10}$$

where

$$\tilde{G}^* = e^{-j(k_x x_A + k_y y_A)} \frac{e^{k_{z,1}^*|z-z_A|}}{-2jk_{z,1}^*}, \tag{A11a}$$

with

$$k_{z,1}^* = k_{z,1} \quad \text{for} \quad k_x^2 + k_y^2 \le k_1^2 \tag{A11b}$$

and

$$k_{z,1}^* = -k_{z,1} \quad \text{for} \quad k_x^2 + k_y^2 > k_1^2, \tag{A11c}$$

see also eqs (A4c) and (A4d). Hence

$$\frac{\partial \tilde{G}^*}{\partial z} = \pm jk_{z,1}\tilde{G}^* \quad \text{for} \quad z \gtrless z_A \quad \text{and} \quad k_x^2 + k_y^2 \le k_1^2 \tag{A12a}$$

whereas

$$\frac{\partial \tilde{G}^*}{\partial z} = \mp jk_{z,1}\tilde{G}^* \quad \text{for} \quad z \gtrless z_A \quad \text{and} \quad k_x^2 + k_y^2 > k_1^2. \tag{A12b}$$

Substitution of (A6), (A11) and (A12) into (A10) yields

$$\hat{P}_1(x_A, y_A, z_A; \omega) = \left(\frac{1}{2\pi}\right)^2 \iint_{k_x^2+k_y^2 \le k_1^2} \tilde{P}^+(k_x, k_y, z_A; \omega) \times e^{-j(k_x x_A + k_y y_A)}\, dk_x\, dk_y + \left(\frac{1}{2\pi}\right)^2 \times \iint_{k_x^2+k_y^2 > k_1^2} \tilde{P}^-(k_x, k_y, z_A; \omega) e^{-j(k_x x_A + k_y y_A)}\, dk_x\, dk_y, \tag{A13a}$$

where

$$\tilde{P}^+(k_x, k_y, z_A; \omega) = e^{jk_{z,1}(z_1 - z_A)}\tilde{P}^+(k_x, k_y, z_1; \omega) \quad \text{for} \quad k_x^2 + k_y^2 \le k_1^2 \tag{A13b}$$

and

$$\tilde{P}^-(k_x, k_y, z_A; \omega) = e^{-jk_{z,1}(z_1 - z_A)}\tilde{P}^-(k_x, k_y, z_1; \omega) \quad \text{for} \quad k_x^2 + k_y^2 > k_1^2 \tag{A13c}$$

(Wapenaar *et al.* 1989). Assuming that evanescent waves may be neglected at A, the second integral in (A13a) may be omitted and the integration area for the first integral may be extended to infinity, yielding

$$\hat{P}_1(x_A, y_A, z_A; \omega) \approx \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \tilde{P}^+(k_x, k_y, z_A; \omega) \times e^{-j(k_x x_A + k_y y_A)}\, dk_x\, dk_y \tag{A14a}$$

where

$$\tilde{P}^+(k_x, k_y, z_A; \omega) = e^{jk_{z,1}(z_1 - z_A)}\tilde{P}^+(k_x, k_y, z_1; \omega) \quad \text{for all} \quad (k_x, k_y). \tag{A14b}$$

Hence, according to (A14a), $\hat{P}_1(x_A, y_A, z_A; \omega)$ is approximately given by the inverse spatial Fourier transform of $\tilde{P}^+(k_x, k_y, z_A; \omega)$, where, according to (A14b), $\tilde{P}^+(k_x, k_y, z_A; \omega)$ represents the downgoing wavefield at depth level z_A (assuming the region between z_A and z_1 is source free). Hence,

$$\hat{P}_1(x_A, y_A, z_A; \omega) \approx P^+(x_A, y_A, z_A; \omega). \tag{A15}$$