Geophys. J. Int. (2005) **163,** 106–125 doi: 10.1111/j.1365-246X.2005.02669.x

Recovering acoustic reflectivity using Dirichlet-to-Neumann maps and left- and right-operating adjoint propagators

M. W. P. Dillen,[1,*] J. T. Fokkema[2] and C. P. A. Wapenaar[2]

[1]*Center for Lithospheric Studies, University of Texas at Dallas,* 2601 *North Floyd Road, Richardson, Texas* 75083-0688, *USA*
[2]*Department of Applied Earth Sciences, Delft University of Technology, Mijnbouwstraat* 120, 2628 *RX Delft, Netherlands*

Accepted 2005 March 24. Received 2005 March 23; in original form 2003 September 24

SUMMARY

Constructing an image of the Earth subsurface from acoustic wave reflections has previously been described as a recursive downward redatuming of sources and receivers. Most of the methods that have been presented involve reflectivity and propagators associated with one-way wavefield components. In this paper, we consider the reflectivity relation between two-way wavefield components, each a solution of a Helmholtz equation. To construct forward and inverse propagators, and a reflection operator, the invariant-embedding technique is followed, using Dirichlet-to-Neumann maps. Employing bilinear and sesquilinear forms, the forward- and inverse-scattering problems, respectively, are treated analogously. Through these mathematical constructs, the relationship between a causality radiation condition and symmetry, with respect to a bilinear form, is associated with the requirement of an anticausality radiation condition with respect to a sesquilinear form. Using reciprocity, sources and receivers are redatumed recursively to the reflector, employing left- and right-operating adjoint propagators. The exposition of the proposed method is formal, that is numerical applications are not derived. The key to applications lies in the explicit representation, characterization and approximation of the relevant operators (symbols) and fundamental solutions (path integrals). Existing constructive work which could be applied to the proposed method are referred to in the text.

Key words: Dirichlet-to-Neumann map, Helmholtz equation, imaging, inverse scattering, migration, reciprocity, reflectivity, wavefield propagator.

1 INTRODUCTION

In Berkhout (1985) acoustic wave propagation and reflection in discretized space is represented by matrix operations. According to this model the reflected wavefield from an acoustic contrast is obtained by multiplying a multisource wavefields matrix by a product of three matrices. Evaluating this product from the right to the left, modelling a surface seismic experiment, the first matrix propagates the source wavefields downward to the reflector, the second matrix reflects these and the third matrix propagates the reflected wavefields upward to the measurement surface. Because the wavefields are considered in the temporal frequency domain, forward wavefields propagation is recursive with respect to depth. This means that propagation can be handled incrementlly through the medium by ordered matrix multiplications. Inversion for the reflection matrix is also represented by a three-matrix product. The measured wavefield matrix, in which each column represents a different common source gather, is then multiplied from the right and the left with an approximate inverse-propagation matrix, which is the Hermitian of the forward-propagation matrix.

In Wapenaar (1996a) Berkhout's model is generalized to $\mathbb{R}^3$ for one-way wavefields employing operators for matrices. Wavefield decomposition into one-way wavefields (Weston 1988; de Hoop 1996; Wapenaar & Grimbergen 1996) follows from a factorization of the two-way Helmholtz wave equation. Propagation in the marching direction, determined by the square-root Helmholtz operator, is then separated from scattering from medium variations in the same direction. The total scattering from an acoustic contrast is represented by an integral over the marching coordinate, which can be expanded in a Bremmer's series of singular scattering events (de Hoop 1996).

In this paper we generalize Berkhout's approach to $\mathbb{R}^3$ for two-way wavefields obeying the Helmholtz equation. The wavefield propagators are derived following the invariant embedding approach of Bellman & Vasudevan (1986), as generalized by Haines & de Hoop (1996). The

*Now at: dGB Earth Sciences, Nijverheidstraat 11-2, 7511 JM, Enschede, Netherlands. E-mail: menno.dillen@dgb-group.com

method yields a Riccati equation for the Dirichlet-to-Neumann (D-t-N) operators, which map the acoustic pressure to the particle velocity component in the marching direction. Wavefield decomposition into an incident and a scattered wavefield, reflection and propagation operators are given in terms of two D-t-N operator solutions (Fishman *et al.* 1997). Each two-way wavefield component obeys an evolution equation, its directionality defined globally. The total scattering from an acoustic contrast is described by a global reflectivity operator, in contrast to the local one-way approach, where the latter method necessitates an integration over the contrast (Fishman 2002, 2004).

Using the semi-group property of the propagators the two-way wavefields solutions of the evolution equations are naturally expressed as a product integral of exponentials (DeWitt-Morette *et al.* 1979; Dollard & Friedman 1979; Fishman *et al.* 1997). The product integral is to the product what the integral is to the sum (Dollard & Friedman 1979), with the difference that the infinitisimal summations are commutative whereas the infinitesimal products are not.

We start with the derivation of the acoustic vector wave equation from its scalar form by identifying a general wavefield or marching direction. A wavefield decomposition into an incident and scattered wavefield is introduced, associated with a medium perturbation superposed on a background medium. The vector wave equations for these wavefield components and Green's vector-valued functions are given. Next, the reciprocity theorems of the time-convolution and time-correlation types are introduced, for example, Fokkema & van den Berg (1993), as a bilinear and a sesquilinear form, respectively (Lang 1993). These latter mathematical constructs, and their symmetry properties, are explained in the appendices. Using the bilinear form for the forward problem enables to derive the inverse problem analogously in terms of sesquilinear forms. Applications of the reciprocity theorems yield wavefield representations for the incident, scattered and total wavefields. Taking limits, we obtain from these representations the D-t-N operators, and subsequently, the reflection operator. Identification of symmetry, related to a radiation boundary condition, yields the forward propagators, that act either on the left or on the right of a wavefield component, propagating either receivers or sources, respectively. The left- and right inverse propagators, which give the kernel of the reflection operator from the data, are approximated by the adjoint of the forward propagators (e.g. Wapenaar 1996a, for one-way wavefields). Finally, the propagators are shown to represent product integrals of complex exponentials, containing the D-t-N operator in the phase term. The generators of the semi-group propagators and the D-t-N operators are shown to be solutions of non-linear operator Riccati equations (Haines & de Hoop 1996; Lu & McLaughlin 1996; Fishman *et al.* 1997, 1998).

Numerical solutions for operations on functions can be obtained by an appropriate approximation and discretization of the operator symbol through the operator kernel. Employing the pseudo-differential operator calculus, the relation between a pseudo-differential D-t-N operator and its symbol can be obtained as a function of the transverse space coordinates and the Fourier dual transverse wavenumber coordinates, together constituting the transverse phase space (Fishman & McCoy 1984a,b; Grubb 1996; Fishman *et al.* 2000). This calculus generalizes previous methods where medium parameters are assumed to be locally constant, enabling a reduction to an ordinary differential equation by Fourier transform over the transverse coordinates (Grubb 1996). Approximations and numerical solutions based on the operator calculus are not discussed in this paper. Constructions of square-root Helmholtz operator symbols can be found in Fishman (1992) and Fishman *et al.* (2000), for the defocussing and focussing quadratic profile, respectively, and in Fishman (2002), for the Dirac delta, discontinuity (two-layer), and three-layer profiles. For the uniform high-frequency asymptotic constructions see Fishman *et al.* (1997), de Hoop & Gautesen (2000, 2003) and Le Rousseau & de Hoop (2001). For the exact constructions of the scattering (reflection and transmissions) and D-t-N operator symbols in the transversely homogeneous limit see Fishman (1994) and Fishman *et al.* (1998).

2 WAVEFIELD EQUATIONS

In the following section the wavefield equations are introduced in their scalar and vectorial forms. In the vectorial form, see e.g. de Hoop (1992) and Wapenaar (1996b), the medium parameters are grouped in a symplectic matrix operator. A standard scattering formalism, in terms of a background medium and a superposed medium perturbation, yields the vectorial wavefield equations for the incident and scattered wavefields. With respect to the background medium, two vectorial Green's functions are introduced, one originating from a monopole volume-injection source and the other from a dipole force source.

2.1 The acoustic scalar wavefield equations

In an inhomogeneous isotropic medium we consider the linearized acoustic wavefield equations, for example, Fokkema & van den Berg (1993),

$$\partial_k p(\mathbf{x}, t) + \rho(\mathbf{x})\partial_t v_k(\mathbf{x}, t) = f_k(\mathbf{x}, t), \tag{1}$$

and

$$\partial_k v_k(\mathbf{x}, t) + \kappa(\mathbf{x})\partial_t p(\mathbf{x}, t) = q(\mathbf{x}, t), \tag{2}$$

where eq. (1) is the equation of motion and eq. (2) constitutes the deformation rate equation. The Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$ define a position in the 3-D Euclidean space $\mathbb{R}^3$. The time coordinate is defined by the real line, $t \in \mathbb{R}$. Einstein's summation convention is assumed for repeated subscripts. Latin subscripts (except t) take the values 1, 2 and 3, whereas Greek subscripts take the values 1 and 2. The symbol ∂_k denotes the partial derivative with respect to x_k, whereas ∂_t denotes the partial derivative with respect to t. The wavefield quantity p constitutes the pressure, v_k constitutes the kth component of the particle velocity, κ is the compressibility and ρ the volume density of mass,

f_k signifies the kth component of the volume source density of volume force, and q represents the volume source density of volume injection rate. The *causal* wavefield quantities, p and v_k, are subject to the *initial* conditions

$$p(\mathbf{x}, t) = 0 \quad \text{for } t < 0, \qquad v_k(\mathbf{x}, t) = 0 \quad \text{for } t < 0. \tag{3}$$

Application of the Fourier transform,

$$\hat{g}(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} \exp(-i\omega t) g(\mathbf{x}, t)\, \mathrm{d}t, \tag{4}$$

to eqs (1) and (2) yields

$$\partial_k \hat{p}(\mathbf{x}, \omega) + i\omega\rho(\mathbf{x})\hat{v}_k(\mathbf{x}, \omega) = \hat{f}_k(\mathbf{x}, \omega), \tag{5}$$

and

$$\partial_k \hat{v}_k(\mathbf{x}, \omega) + i\omega\kappa(\mathbf{x})\hat{p}(\mathbf{x}, \omega) = \hat{q}(\mathbf{x}, \omega), \tag{6}$$

in which ω denotes the angular frequency, i denotes the imaginary unit and the symbol $\hat{}$ is used to denote frequency domain quantities.

2.2 The acoustic vectorial wavefield equation

We regard the vectorial wavefields quantities as a function of $\mathbf{x} = (\mathbf{x}_\mathrm{T}, x_3)$, in terms of the transverse vector coordinate $\mathbf{x}_\mathrm{T} = (x_1, x_2)$ and the longitudinal scalar coordinate x_3. The orientation of the Cartesian reference frame is fixed by choosing the longitudinal coordinate x_3 to coincide with the general wavefield direction (Fishman & McCoy 1984a,b). With surface seismic measurements the longitudinal direction is chosen to be vertical and measures depth, whereas, for example, with crosswell seismic measurements and ocean acoustics, the longitudinal direction is the interwell distance and range direction, respectively. To accommodate such a directional preference the vectorial wavefield equation are derived. To this end the transverse particle velocity components $\hat{v}_1$ and $\hat{v}_2$ are eliminated from the wavefield eqs (5) and (6), obtaining

$$\partial_3 \hat{p} + i\omega\rho\hat{v}_3 = \hat{f}_3, \tag{7}$$

$$\partial_3 \hat{v}_3 + i\omega\hat{\mathcal{K}}\hat{p} = \hat{q} - (i\omega)^{-1}\, \partial_\alpha \left(\rho^{-1} \hat{f}_\alpha\right), \tag{8}$$

in which the operator $\hat{\mathcal{K}}$ is given by,

$$\hat{\mathcal{K}} = \kappa + \omega^{-2}\partial_\alpha \left(\rho^{-1}\partial_\alpha \cdot\right). \tag{9}$$

Combining eqs (7) with (8) results in the first-order ordinary differential equation with respect to x_3,

$$\partial_3 \hat{\mathbf{F}} + \hat{\mathbf{A}}\hat{\mathbf{F}} = \hat{\mathbf{N}}, \tag{10}$$

see e.g. de Hoop (1992) and Wapenaar (1996b). The wavefield vector $\hat{\mathbf{F}}$, the acoustic system operator $\hat{\mathbf{A}}$ and the source vector $\hat{\mathbf{N}}$ are given by

$$\hat{\mathbf{F}} = \begin{pmatrix} \hat{p} \\ \hat{v}_3 \end{pmatrix}, \quad \hat{\mathbf{A}} = \begin{pmatrix} 0 & i\omega\rho \\ i\omega\hat{\mathcal{K}} & 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{N}} = \begin{pmatrix} \hat{f}_3 \\ \hat{q} - (i\omega)^{-1}\, \partial_\alpha \left(\rho^{-1} \hat{f}_\alpha\right) \end{pmatrix}, \tag{11}$$

respectively. The wavefield vector eq. (10) applies to those points where the medium parameters κ and ρ are smooth, that is, infinitely differentiable with respect to the spatial coordinates. At interfaces x_3 is constant, for which the medium parameters are discontinuous with respect to x_3, the wavefield vector equation must be supplemented with the boundary condition,

$\hat{\mathbf{F}}$ is continuous across the interface. (12)

2.3 Scattering formalism

Consider the planar surface $\partial\mathbb{D}^{\mathrm{sct}} = \{(\mathbf{x}_\mathrm{T}, x_3) | \, \mathbf{x}_\mathrm{T} \in \mathbb{R}^2, x_3 = x_3^{\mathrm{sct}}\}$, which divides $\mathbb{R}^3$ into two half spaces: the scattering domain, $\mathbb{D}^{\mathrm{sct}}$, for which $x_3 > x_3^{\mathrm{sct}}$, and the domain $\mathbb{D}^{\mathrm{sct}'} = \mathbb{R}^3 \setminus \{\partial\mathbb{D}^{\mathrm{sct}} \cup \mathbb{D}^{\mathrm{sct}}\}$, for which $x_3 < x_3^{\mathrm{sct}}$ (Fig. 1). In $\mathbb{R}^3$ we consider the following wavefields equation for $\hat{\mathbf{F}}^{\mathrm{tot}} = \hat{\mathbf{F}}^{\mathrm{tot}}(\mathbf{x}; \mathbf{x}^\mathrm{S}, \omega)$,

$$\partial_3 \hat{\mathbf{F}}^{\mathrm{tot}} + \hat{\mathbf{A}}\hat{\mathbf{F}}^{\mathrm{tot}} = \hat{\mathbf{N}}^{\mathrm{tot}}, \tag{13}$$

with source vector

$$\hat{\mathbf{N}}^{\mathrm{tot}}(\mathbf{x}; \mathbf{x}^\mathrm{S}, \omega) = \begin{pmatrix} 0 \\ \hat{q}(\omega)\delta(\mathbf{x} - \mathbf{x}^\mathrm{S}) \end{pmatrix}, \tag{14}$$

in which the source is a Dirac distribution with support at $\mathbf{x}^\mathrm{S} \in \mathbb{D}^{\mathrm{sct}'}$, and with source spectrum $\hat{q}$.

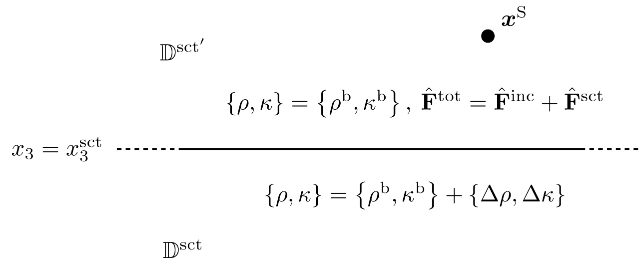

Figure 1. Scattering configuration with the medium parameters defined in terms of background medium parameters and a perturbation. The perturbations are in domain $\mathbb{D}^{\mathrm{sct}}$. Wavefield decomposition and source position $\mathbf{x}^{\mathrm{S}}$ are in domain $\mathbb{D}^{\mathrm{sct}'}$.

Introducing the *perturbations* $\{\Delta\rho, \Delta\kappa\}$, the *background* medium parameters $\{\rho^{\mathrm{b}}, \kappa^{\mathrm{b}}\}$, characterizing the background medium, are defined with respect to the *actual* medium parameters $\{\rho, \kappa\}$, according to

$$\{\rho, \kappa\} = \begin{cases} \{\rho^{\mathrm{b}}, \kappa^{\mathrm{b}}\} & \text{in} \quad \mathbb{D}^{\mathrm{sct}'}, \\ \{\rho^{\mathrm{b}}, \kappa^{\mathrm{b}}\} + \{\Delta\rho, \Delta\kappa\} & \text{in} \quad \mathbb{D}^{\mathrm{sct}}. \end{cases} \tag{15}$$

Accordingly, following standard scattering theory, the actual wavefield $\hat{\mathbf{F}}^{\mathrm{tot}}$ of eq. (13), also denoted as the *total* wavefield, is decomposed into an *incident* wavefield $\hat{\mathbf{F}}^{\mathrm{inc}} = \hat{\mathbf{F}}^{\mathrm{inc}}(\mathbf{x}; \mathbf{x}^{\mathrm{S}}, \omega)$, and a *scattered* wavefield $\hat{\mathbf{F}}^{\mathrm{sct}} = \hat{\mathbf{F}}^{\mathrm{sct}}(\mathbf{x}; \mathbf{x}^{\mathrm{S}}, \omega)$, as

$$\hat{\mathbf{F}}^{\mathrm{tot}} = \hat{\mathbf{F}}^{\mathrm{inc}} + \hat{\mathbf{F}}^{\mathrm{sct}}. \tag{16}$$

The incident wavefield is governed by the background medium, and originates from the same source (eq. 14) as the total wavefield, according to

$$\partial_3 \hat{\mathbf{F}}^{\mathrm{inc}} + \hat{\mathbf{A}}^{\mathrm{b}} \hat{\mathbf{F}}^{\mathrm{inc}} = \hat{\mathbf{N}}^{\mathrm{tot}}. \tag{17}$$

The scattered wavefields is then given by

$$\partial_3 \hat{\mathbf{F}}^{\mathrm{sct}} + \hat{\mathbf{A}}^{\mathrm{b}} \hat{\mathbf{F}}^{\mathrm{sct}} = \hat{\mathbf{N}}^{\mathrm{sct}}, \tag{18}$$

in which the contrast sources are obtained as

$$\hat{\mathbf{N}}^{\mathrm{sct}}(\mathbf{x}; \mathbf{x}^{\mathrm{S}}, \omega) = \begin{cases} \mathbf{O} & \mathbf{x} \in \mathbb{D}^{\mathrm{sct}'} \\ -(\hat{\mathbf{A}} - \hat{\mathbf{A}}^{\mathrm{b}}) \hat{\mathbf{F}}^{\mathrm{tot}}(\mathbf{x}; \mathbf{x}^{\mathrm{S}}, \omega), & \mathbf{x} \in \mathbb{D}^{\mathrm{sct}}, \end{cases} \tag{19}$$

in which $\mathbf{O}$ is the null two-vector.

2.4 Green's functions

Following Fokkema & van den Berg (1993) (for the scalar form) and Haines & de Hoop (1996) (for the vectorial form) we introduce, with respect to the background medium, the volume-injection Green's wavefield $\hat{\mathbf{G}}^{q,\mathrm{b}} = \hat{\mathbf{G}}^{q,\mathrm{b}}(\mathbf{x}; \mathbf{x}', \omega)$, with wavefield equation

$$\partial_3 \hat{\mathbf{G}}^{q,\mathrm{b}} + \hat{\mathbf{A}}^{\mathrm{b}} \hat{\mathbf{G}}^{q,\mathrm{b}} = \hat{\mathbf{N}}^{q,\mathrm{b}}, \tag{20}$$

and with wavefield and source vectors,

$$\hat{\mathbf{G}}^{q,\mathrm{b}} = \begin{pmatrix} \hat{G}^{q,\mathrm{b}} \\ \hat{\Gamma}_3^{q,\mathrm{b}} \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{N}}^{q,\mathrm{b}} = \begin{pmatrix} 0 \\ \delta\,(\mathbf{x} - \mathbf{x}') \end{pmatrix}, \tag{21}$$

respectively. The wavefield vector $\hat{\mathbf{G}}^{q,\mathrm{b}}$ is causally related to a Dirac distribution with support at $\mathbf{x} = \mathbf{x}'$. Similarly, we introduce the force source Green's wavefield equation for $\hat{\mathbf{G}}^{f,\mathrm{b}} = \hat{\mathbf{G}}^{f,\mathrm{b}}(\mathbf{x}; \mathbf{x}', \omega)$,

$$\partial_3 \hat{\mathbf{G}}^{f,\mathrm{b}} + \hat{\mathbf{A}}^{\mathrm{b}} \hat{\mathbf{G}}^{f,\mathrm{b}} = \hat{\mathbf{N}}^{f,\mathrm{b}}, \tag{22}$$

with wavefield and source vectors,

$$\hat{\mathbf{G}}^{f,\mathrm{b}} = \begin{pmatrix} \hat{G}_3^{f,\mathrm{b}} \\ \hat{\Gamma}_{33}^{f,\mathrm{b}} \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{N}}^{f,\mathrm{b}} = \begin{pmatrix} \delta\,(\mathbf{x} - \mathbf{x}') \\ 0 \end{pmatrix}, \tag{23}$$

respectively. The subscripts of the Green's functions denote the longitudinal direction of either the particle velocity vector or the force source vector.

3 ACOUSTIC RECIPROCITY THEOREMS

The acoustic reciprocity theorems of the time-convolution and time-correlation types are introduced (de Hoop 1995). The theorems are given in the frequency domain. Hence, a convolution of two wavefield quantities in the time domain becomes a multiplication in the frequency domain, whereas a correlation transforms to a multiplication of a wavefield quantity with a complex conjugate one. The time-convolution type reciprocity theorem represents a bilinear form with respect to the two wavefield vectors. Bilinear forms and some of their properties are discussed in Appendix A. Accordingly, the time-correlation type reciprocity theorem is formalized as a sesquilinear form and some of its properties are given in Appendix B.

3.1 Time-convolution type reciprocity theorem

Using eqs (A1) and (A2), given two wavefield $\hat{\mathbf{F}}^A$ and $\hat{\mathbf{F}}^B$, associated with acoustic states A and B, respectively, the following bilinear form is defined

$$\langle \hat{\mathbf{F}}^A(\mathbf{x},\omega), \mathbf{J}\hat{\mathbf{F}}^B(\mathbf{x},\omega)\rangle_{\mathrm{b}} \stackrel{\mathrm{def}}{=} \int_{\mathbf{x}_\mathrm{T}\in\mathbb{R}^2} (\hat{\mathbf{F}}^A)^{\mathrm{t}}(\mathbf{x}_\mathrm{T}, x_3, \omega)\mathbf{J}\hat{\mathbf{F}}^B(\mathbf{x}_\mathrm{T}, x_3, \omega)\,\mathrm{d}\mathbf{x}_\mathrm{T}. \tag{24}$$

Both acoustic states are governed by the wavefield vector eq. (10). The *standard alternating* matrix operator $\mathbf{J}$ is given by

$$\mathbf{J} = \begin{pmatrix} \mathcal{O} & \mathcal{I} \\ -\mathcal{I} & \mathcal{O} \end{pmatrix}, \tag{25}$$

with $\mathcal{O}$ and $\mathcal{I}$ representing the scalar null and scalar identity operators, respectively. Comparing the right-hand sides of eqs (24) and (A1), the former equation has the extra parameters x_3 and ω. The left-hand side of eq. (24) employs notation (A2), and the function arguments ($\mathbf{x}_\mathrm{T}$, x_3, ω) are abbreviated to ($\mathbf{x}$, ω), while it is understood that the integration is with respect to $\mathbf{x}_\mathrm{T}$, and x_3 is a parameter of the bilinear form. Taking the derivative of the left-hand side of eq. (24) with respect to the longitudinal coordinate x_3, yields

$$\partial_3\langle \hat{\mathbf{F}}^A, \mathbf{J}\hat{\mathbf{F}}^B\rangle_{\mathrm{b}} = \left\langle \partial_3\hat{\mathbf{F}}^A, \mathbf{J}\hat{\mathbf{F}}^B\right\rangle_{\mathrm{b}} + \left\langle \hat{\mathbf{F}}^A, \mathbf{J}\partial_3\hat{\mathbf{F}}^B\right\rangle_{\mathrm{b}}. \tag{26}$$

Substitution of eq. (10), for both states A and B, into the right-hand side of eq. (26), using bilinearity and the transposition operation (A3), leads to

$$\partial_3\langle \hat{\mathbf{F}}^A, \mathbf{J}\hat{\mathbf{F}}^B\rangle_{\mathrm{b}} = -\langle \hat{\mathbf{F}}^A, [\,(\hat{\mathbf{A}}^A)^{\mathrm{t}}\mathbf{J} + \mathbf{J}\hat{\mathbf{A}}^B\,]\hat{\mathbf{F}}^B\rangle_{\mathrm{b}} + \langle \hat{\mathbf{N}}^A, \mathbf{J}\hat{\mathbf{F}}^B\rangle_{\mathrm{b}} + \langle \hat{\mathbf{F}}^A, \mathbf{J}\hat{\mathbf{N}}^B\rangle_{\mathrm{b}}. \tag{27}$$

The contrast operator in the first term on the right-hand side of this last equation is, using eq. (11), given by

$$(\hat{\mathbf{A}}^A)^{\mathrm{t}}\mathbf{J} + \mathbf{J}\hat{\mathbf{A}}^B = \begin{pmatrix} i\omega[\hat{\mathcal{K}}^B - (\hat{\mathcal{K}}^A)^{\mathrm{t}}] & \mathcal{O} \\ \mathcal{O} & -i\omega(\rho^B - \rho^A)\mathcal{I} \end{pmatrix}. \tag{28}$$

Using eqs (A4) and (A5), using skew-symmetry of ∂_α in eq. (9), one can show that $\hat{\mathcal{K}}$ is symmetric with respect to the bilinear form of scalar-valued functions,

$$\hat{\mathcal{K}} = \hat{\mathcal{K}}^{\mathrm{t}}. \tag{29}$$

Hence, $\hat{\mathbf{A}}$ is symplectic,

$$\hat{\mathbf{A}}^{\mathrm{t}}\mathbf{J} = -\mathbf{J}\hat{\mathbf{A}}. \tag{30}$$

Using the symplectic property we can write

$$(\hat{\mathbf{A}}^A)^{\mathrm{t}}\mathbf{J} + \mathbf{J}\hat{\mathbf{A}}^B = \mathbf{J}(\hat{\mathbf{A}}^B - \hat{\mathbf{A}}^A) = \mathbf{J}\Delta\hat{\mathbf{A}}. \tag{31}$$

Integration of eq. (27) with respect to x_3, from x_3^0 to x_3^1, with $x_3^0 < x_3^1$, using eq. (31), yields the reciprocity theorem of the time-convolution type,

$$\langle \hat{\mathbf{F}}^A, \mathbf{J}\hat{\mathbf{F}}^B\rangle_{\mathrm{b}}\left(x_3 = x_3^1\right) - \langle \hat{\mathbf{F}}^A, \mathbf{J}\hat{\mathbf{F}}^B\rangle_{\mathrm{b}}\left(x_3 = x_3^0\right) = -\int_{x_3=x_3^0}^{x_3^1} \langle \hat{\mathbf{F}}^A, \mathbf{J}\Delta\hat{\mathbf{A}}\hat{\mathbf{F}}^B\rangle_{\mathrm{b}}\,\mathrm{d}x_3 + \int_{x_3=x_3^0}^{x_3^1} \left[\langle \hat{\mathbf{N}}^A, \mathbf{J}\hat{\mathbf{F}}^B\rangle_{\mathrm{b}} + \langle \hat{\mathbf{F}}^A, \mathbf{J}\hat{\mathbf{N}}^B\rangle_{\mathrm{b}}\right]\mathrm{d}x_3; \tag{32}$$

see Wapenaar (1996b) and Haines & de Hoop (1996). In this last equation the left-hand side represents the boundary interaction between the two wavefields, whereas the two integrals on the right-hand side depend explicitly on the media and the sources, respectively.

3.2 Time-correlation type reciprocity theorem

According to eq. (B1), given two wavefields $\hat{\mathbf{F}}^A$ and $\hat{\mathbf{F}}^B$, associated with acoustic states A and B, respectively, the following sesquilinear form is defined

$$\langle \hat{\mathbf{F}}^A(\mathbf{x},\omega), \mathbf{K}\hat{\mathbf{F}}^B(\mathbf{x},\omega)\rangle_{\mathrm{s}} \stackrel{\mathrm{def}}{=} \int_{\mathbf{x}_\mathrm{T}\in\mathbb{R}^2} (\hat{\mathbf{F}}^A)^{\dagger}\,(\mathbf{x}_\mathrm{T}, x_3, \omega)\,\mathbf{K}\hat{\mathbf{F}}^B\,(\mathbf{x}_\mathrm{T}, x_3, \omega)\;\mathrm{d}\mathbf{x}_\mathrm{T}. \tag{33}$$

The self-adjoint matrix operator $\mathbf{K}$ is given by

$$\mathbf{K} = \begin{pmatrix} \mathcal{O} & \mathcal{I} \\ \mathcal{I} & \mathcal{O} \end{pmatrix}. \tag{34}$$

Taking the derivative of the left-hand side of eq. (33) with respect to the longitudinal coordinate x_3, and substituting eq. (10), for both states, using sesquilinearity and the adjoint operation (B5), leads to

$$\partial_3\langle\hat{\mathbf{F}}^A,\mathbf{K}\hat{\mathbf{F}}^B\rangle_{\mathrm{s}} = -\langle\hat{\mathbf{F}}^A,[(\hat{\mathbf{A}}^A)^\dagger\mathbf{K}+\mathbf{K}\hat{\mathbf{A}}^B]\hat{\mathbf{F}}^B\rangle_{\mathrm{s}} + \langle\hat{\mathbf{N}}^A,\mathbf{K}\hat{\mathbf{F}}^B\rangle_{\mathrm{s}} + \langle\hat{\mathbf{F}}^A,\mathbf{K}\hat{\mathbf{N}}^B\rangle_{\mathrm{s}}. \tag{35}$$

The contrast operator in the first term on the right-hand side of this last equation is given by

$$(\hat{\mathbf{A}}^A)^\dagger\mathbf{K}+\mathbf{K}\hat{\mathbf{A}}^B = \begin{pmatrix} i\omega[\hat{\mathcal{K}}^B-(\hat{\mathcal{K}}^A)^\dagger] & \mathcal{O} \\ \mathcal{O} & i\omega(\rho^B-\rho^A)\mathcal{I}\end{pmatrix}. \tag{36}$$

Taking identical states in eq. (35), that is, $A = B$, using the self-adjointness of $\mathbf{K}$ in eqs (B7) and (B8), yields the quadratic form

$$\langle\hat{\mathbf{F}},(\hat{\mathbf{A}}^\dagger\mathbf{K}+\mathbf{K}\hat{\mathbf{A}})\hat{\mathbf{F}}\rangle_{\mathrm{s}} \text{ which is real.} \tag{37}$$

Because for a lossless medium $\hat{\mathcal{K}}$ of eq. (9) is a real operator it is also self-adjoint with respect to a sesquilinear form of scalar-valued functions,

$$\hat{\mathcal{K}} = \hat{\mathcal{K}}^\dagger, \tag{38}$$

and

$$\hat{\mathbf{A}}^\dagger\mathbf{K} = -\mathbf{K}\hat{\mathbf{A}}. \tag{39}$$

This is consistent with Wapenaar (1996b) who showed that, employing a compact domain, $\hat{\mathcal{K}}$ is a self-adjoint operator using boundary conditions at $x_1^2+x_2^2\to\infty$. Using this property we can write

$$(\hat{\mathbf{A}}^A)^\dagger\mathbf{K}+\mathbf{K}\hat{\mathbf{A}}^B = \mathbf{K}(\hat{\mathbf{A}}^B-\hat{\mathbf{A}}^A) = \mathbf{K}\Delta\hat{\mathbf{A}}. \tag{40}$$

Integration of eq. (35) with respect to x_3, from x_3^0 to x_3^1, with $x_3^0 < x_3^1$, using eq. (40), yields the reciprocity theorem of the time-correlation type,

$$\langle\hat{\mathbf{F}}^A,\mathbf{K}\hat{\mathbf{F}}^B\rangle_{\mathrm{s}}\left(x_3=x_3^1\right) - \langle\hat{\mathbf{F}}^A,\mathbf{K}\hat{\mathbf{F}}^B\rangle_{\mathrm{s}}\left(x_3=x_3^0\right) = -\int_{x_3=x_3^0}^{x_3^1}\langle\hat{\mathbf{F}}^A,\mathbf{K}\Delta\hat{\mathbf{A}}\hat{\mathbf{F}}^B\rangle_{\mathrm{s}}\,\mathrm{d}x_3 + \int_{x_3=x_3^0}^{x_3^1}\left[\langle\hat{\mathbf{N}}^A,\mathbf{K}\hat{\mathbf{F}}^B\rangle_{\mathrm{s}}+\langle\hat{\mathbf{F}}^A,\mathbf{K}\hat{\mathbf{N}}^B\rangle_{\mathrm{s}}\right]\mathrm{d}x_3; \tag{41}$$

see Wapenaar (1996b) and Haines & de Hoop (1996).

3.3 Source–receiver reciprocity

The domain of application of the reciprocity theorem of eq. (32) is bounded by the surfaces at x_3^0 and x_3^1. By taking the limits, $x_3^0\to-\infty$ and $x_3^1\to\infty$, the domain of application can be enlarged to cover the whole $\mathbb{R}^3$. Assume that the medium parameters of States A and B are equal in $\mathbb{R}^3$. We have $\rho^A=\rho^B$ and $\kappa^A=\kappa^B$, and hence, $\Delta\hat{\mathbf{A}}$ of eq. (31) becomes the matrix null operator, and consequently, the first integral on the right-hand side of eq. (32) vanishes. Taking the above limits, using time-domain causality, and assuming that for large $|x_3|$ the respective far-field wavefields radiate in an homogeneous medium, the two boundary integrals on the left-hand side of eq. (32) vanish. Hence, eq. (32) becomes

$$\int_{x_3\in\mathbb{R}}\left[\langle\hat{\mathbf{N}}^A,\mathbf{J}\hat{\mathbf{F}}^B\rangle_{\mathrm{b}}+\langle\hat{\mathbf{F}}^A,\mathbf{J}\hat{\mathbf{N}}^B\rangle_{\mathrm{b}}\right]\mathrm{d}x_3 = 0. \tag{42}$$

Let the wavefield of State A be described by eqs (20) and (21) with $\mathbf{x}'=\mathbf{x}^{\mathrm{S}}$, and let the wavefield of State B also be described by these same equations, but with $\mathbf{x}'=\mathbf{x}^{\mathrm{R}}$. Application of these two states to eq. (42) yields,

$$\hat{G}^{q,\mathrm{b}}(\mathbf{x}^{\mathrm{R}};\mathbf{x}^{\mathrm{S}}) = \hat{G}^{q,\mathrm{b}}(\mathbf{x}^{\mathrm{S}};\mathbf{x}^{\mathrm{R}}). \tag{43}$$

Alternatively, keep State A as above and let the wavefield of State B be described by eqs (22) and (23) with $\mathbf{x}'=\mathbf{x}^{\mathrm{R}}$. eq. (42) then yields

$$\hat{\Gamma}_3^{q,\mathrm{b}}(\mathbf{x}^{\mathrm{R}};\mathbf{x}^{\mathrm{S}}) = -\hat{G}_3^{f,\mathrm{b}}(\mathbf{x}^{\mathrm{S}};\mathbf{x}^{\mathrm{R}}). \tag{44}$$

These Source–receiver reciprocities for the Green's functions (Fokkema & van den Berg 1993) will be needed in the following boundary integral representations.

4 WAVEFIELD DECOMPOSITION

Using the reciprocity theorem of the time-convolution type, integral representations are derived for the first component of the wavefield vectors of the incident, scattered and total wavefields. Taking appropriate limits, these representations are used to derive D-t-N operators, which transform the first component of the wavefield vectors to the second component. Composition/decomposition and reflection operators are expressed in terms of these D-t-N operators.

4.1 Boundary integral representations

4.1.1 Incident wavefield

To arrive at a boundary integral representation of the incident wavefield with respect to a single planar surface, the reciprocity theorem of the time-convolution type (32) is applied to the domain $\mathbb{D}^{\rm a} = \{(\mathbf{x}'_{\rm T}, x'_3) | \, \mathbf{x}'_{\rm T} \in \mathbb{R}^2, x_3^0 < x'_3 < x_3^1\}$. Take for State A the volume-injection Green's wavefield of eq. (20), with source at $(\mathbf{x}_{\rm T}, x_3) \in \mathbb{R}^3$, that is, $\hat{\mathbf{G}}^{q,\rm b} = \hat{\mathbf{G}}^{q,\rm b}(\mathbf{x}';\mathbf{x})$. Take for State B the incident wavefield of eq. (17), with source at $(\mathbf{x}_{\rm T}^{\rm S}, x_3^{\rm S})$, i.e. $\hat{\mathbf{F}}^{\rm inc} = \hat{\mathbf{F}}^{\rm inc}(\mathbf{x}';\mathbf{x}^{\rm S})$, and with $x_3^{\rm S} < x_3^0$. Hence, the source of the incident wavefield is outside $\mathbb{D}^{\rm a}$. Employing the causality radiation condition, as in Section 3.3, the contribution of the first bilinear form on the left-hand side of eq. (32) vanishes as $x_3^1 \to \infty$. Because both wavefields radiate through the same background medium the first integral on the right-hand side of eq. (32) also vanishes. In terms of scalar-valued functions eq. (32) becomes

$$H\left(x_3 - x_3^0\right) \hat{p}^{\rm inc}(\mathbf{x};\mathbf{x}^{\rm S}) = \left\langle \hat{G}^{q,\rm b}(\mathbf{x}';\mathbf{x}), \hat{v}_3^{\rm inc}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b} \left(x'_3 = x_3^0\right) - \left\langle \hat{\Gamma}_3^{q,\rm b}(\mathbf{x}';\mathbf{x}), \hat{p}^{\rm inc}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b} \left(x'_3 = x_3^0\right), \tag{45}$$

in which we used the Heaviside function,

$$H(x_3) = \begin{cases} 0, & x_3 < 0, \\ \frac{1}{2} & x_3 = 0, \\ 1, & x_3 > 1. \end{cases} \tag{46}$$

In eq. (45) we used the short-hand notation (24), that is, the integration is over $\mathbf{x}'_{\rm T}$ at $x'_3 = x_3^0$. Because the value of x'_3 is now fixed to one level of integration, and to emphasize that the level of integration is considered a variable, we will replace x_3^0 by x'_3, and write

$$H(x_3 - x'_3)\hat{p}^{\rm inc}(\mathbf{x};\mathbf{x}^{\rm S}) = \left\langle \hat{G}^{q,\rm b}(\mathbf{x}';\mathbf{x}), \hat{v}_3^{\rm inc}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b} - \left\langle \hat{\Gamma}_3^{q,\rm b}(\mathbf{x}';\mathbf{x}), \hat{p}^{\rm inc}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b}, \tag{47}$$

for $x_3^{\rm S} < x'_3$. Observe that the arguments of $\hat{p}^{\rm inc}$ on the left-hand side are the source positions of the wavefields on the right-hand side. The particular value of $H(x_3 - x'_3)$ depends on the source level x_3 of the Green's wavefield $\hat{\mathbf{G}}^{q,\rm b}$ with respect to the integration level x'_3. At the limiting value $x_3 = x'_3$, the integral is a Cauchy principal value integral, in which the integration is over the pertaining boundary with the symmetric exclusion of the singular point (Colton & Kress 1983; Fokkema & van den Berg 1993). Invoking the Source–receiver reciprocity relations of eqs (43) and (44) in eq. (47), gives

$$H(x_3 - x'_3)\hat{p}^{\rm inc}(\mathbf{x};\mathbf{x}^{\rm S}) = \left\langle \hat{G}^{q,\rm b}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\rm inc}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b} + \left\langle \hat{G}_3^{f,\rm b}(\mathbf{x};\mathbf{x}'), \hat{p}^{\rm inc}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b}, \tag{48}$$

In the latter representation of the incident wavefield the Green's functions radiate from impulsive monopole and dipole sources located at the boundary surface (Fokkema & van den Berg 1993).

4.1.2 Scattered wavefield

For the derivation of the boundary integral representation of the scattered wavefield eq. (32) is applied to the domain $\mathbb{D}^{\rm a}$ of Section 4.1.1. Take for State A the same volume-injection Green's wavefield $\hat{\mathbf{G}}^{q,\rm b}$ as in Section 4.1.1 Take for State B the scattered wavefield of eq. (18), i.e. $\hat{\mathbf{F}}^{\rm sct} = \hat{\mathbf{F}}^{\rm sct}(\mathbf{x}';\mathbf{x}^{\rm S})$, with $x_3^{\rm sct} > x_3^1$. Hence the virtual contrast sources of the scattered wavefield are outside $\mathbb{D}^{\rm a}$. Using the causality radiation condition the contribution of the second bilinear form on the left-hand side of eq. (32) vanishes as $x_3^0 \to -\infty$. Taking into account that both wavefields radiate through the background medium, using relations (43) and (44), yields, following similar steps as in Section 4.1.1,

$$H\left(x'_3 - x_3\right) \hat{p}^{\rm sct}(\mathbf{x};\mathbf{x}^{\rm S}) = -\left\langle \hat{G}^{q,\rm b}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\rm sct}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b} - \left\langle \hat{G}_3^{f,\rm b}(\mathbf{x};\mathbf{x}'), \hat{p}^{\rm sct}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b}, \tag{49}$$

for $x'_3 < x_3^{\rm sct}$. Because $x'_3 = x_3^1$, we replaced x_3^1 by x'_3 in this last equation. At the limiting value $x_3 = x'_3$, the representation of eq. (49) is a Cauchy principal value integral.

4.1.3 Total wavefield

For the derivation of the boundary integral representation of the total wavefield we use the domain of application $\mathbb{D}^{\rm a}$ of Section 4.1.1 for eq. (32). State A is taken to be the volume-injection Green's wavefield $\hat{\mathbf{G}}^q$, of eq. (20), with source $\mathbf{x} \in \mathbb{R}^3$, that is $\hat{\mathbf{G}}^q = \hat{\mathbf{G}}^q(\mathbf{x}';\mathbf{x})$. The extra superscript $^{\rm b}$ is omitted because the Green's wavefield is taken with respect to the actual medium, as defined in eq. (15). We take for State B the total wavefield of eq. (13), with $x_3^{\rm S} < x_3^0$, that is, $\hat{\mathbf{F}}^{\rm tot} = \hat{\mathbf{F}}^{\rm tot}(\mathbf{x}';\mathbf{x}^{\rm S})$. Hence the source of the total wavefield is outside $\mathbb{D}^{\rm a}$. Application of eq. (32), following the same steps as in Section 4.1.1, taking into account that both wavefields radiate through the actual medium, yields

$$H(x_3 - x'_3)\hat{p}^{\rm tot}(\mathbf{x};\mathbf{x}^{\rm S}) = \left\langle \hat{G}^{q}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\rm tot}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b} + \left\langle \hat{G}_3^{f}(\mathbf{x};\mathbf{x}'), \hat{p}^{\rm tot}(\mathbf{x}';\mathbf{x}^{\rm S})\right\rangle_{\rm b}, \tag{50}$$

At $x_3 = x'_3$ the integral is a Cauchy principal value integral.

4.2 Dirichlet-to-Neumann operators

We proceed by defining the following *single-layer potential* boundary integral operator,

$$\hat{\mathcal{S}}^{\mathrm{b}} f(\mathbf{x}_{\mathrm{T}}, x_3) \stackrel{\mathrm{def}}{=} 2\big\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x}_{\mathrm{T}}, x_3; \mathbf{x}'_{\mathrm{T}}, x_3),\ f(\mathbf{x}'_{\mathrm{T}}, x_3)\big\rangle_{\mathrm{b}}, \tag{51}$$

and the *double-layer potential* boundary integral operator,

$$\hat{\mathcal{D}}^{\mathrm{b}} g(\mathbf{x}_{\mathrm{T}}, x_3) \stackrel{\mathrm{def}}{=} 2\big\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x}_{\mathrm{T}}, x_3; \mathbf{x}'_{\mathrm{T}}, x_3), g(\mathbf{x}'_{\mathrm{T}}, x_3)\big\rangle_{\mathrm{b}}, \tag{52}$$

(Colton & Kress 1983), in which the functions $f, g \in [L^2(\mathbb{R}^2)]^1$, that is, elements of a Hilbert space of scalar-valued ($[\,]^1$) functions on $\mathbb{R}^2$(x_3 is regarded as a parameter of the bilinear forms). The integrals in these last two definitions are Cauchy principal value integrals. Evaluating the incident wavefield representation (48) and the scattered wavefield representation (49), at $x_3 = x'_3$, using the operators (51) and (52), yields

$$(\mathcal{I} - \hat{\mathcal{D}}^{\mathrm{b}})\hat{p}^{\mathrm{inc}} - \hat{\mathcal{S}}^{\mathrm{b}}\hat{v}_3^{\mathrm{inc}} = 0, \tag{53}$$

and

$$(\mathcal{I} + \hat{\mathcal{D}}^{\mathrm{b}})\hat{p}^{\mathrm{sct}} + \hat{\mathcal{S}}^{\mathrm{b}}\hat{v}_3^{\mathrm{sct}} = 0. \tag{54}$$

With the vertical direction as longitudinal, these last two equations represent *down-going* and *up-going* wavefield conditions, respectively (Weston 1988). The directionality is given with respect to the background medium, that is, down- and up-going are understood to be global properties, in contrast to local wavefield splitting, e.g. de Hoop (1992, 1996) and Wapenaar (1996a), for which directionality is defined for a constant x_3. Assuming the existence of the inverse of $\hat{\mathcal{S}}^{\mathrm{b}}$, we arrive at the following operators, at any level surface $x_3^{\mathrm{S}} < x_3 < x_3^{\mathrm{sct}}$,

$$\hat{v}_3^{\mathrm{inc}} = \hat{\mathcal{Y}}^{\mathrm{d}}\hat{p}^{\mathrm{inc}}, \tag{55}$$

and

$$\hat{v}_3^{\mathrm{sct}} = \hat{\mathcal{Y}}^{\mathrm{u}}\hat{p}^{\mathrm{sct}}, \tag{56}$$

with

$$\hat{\mathcal{Y}}^{\mathrm{d}} = (\hat{\mathcal{S}}^{\mathrm{b}})^{-1}(\mathcal{I} - \hat{\mathcal{D}}^{\mathrm{b}}), \tag{57}$$

and

$$\hat{\mathcal{Y}}^{\mathrm{u}} = -(\hat{\mathcal{S}}^{\mathrm{b}})^{-1}(\mathcal{I} + \hat{\mathcal{D}}^{\mathrm{b}}). \tag{58}$$

The operators $\hat{\mathcal{Y}}^{\mathrm{d}}$ and $\hat{\mathcal{Y}}^{\mathrm{u}}$ are D-to-N operators that map the pressure functions, $\hat{p}^{\mathrm{inc}}$ and $\hat{p}^{\mathrm{sct}}$, to the longitudinal component of the particle velocity functions, $\hat{v}_3^{\mathrm{inc}}$ and $\hat{v}_3^{\mathrm{sct}}$, respectively.

4.3 Decomposition operator

At any level surface, $x_3^{\mathrm{S}} < x_3 < x_3^{\mathrm{sct}}$, eqs (16), (55) and (56) yield the following *wavefield composition* operation

$$\hat{\mathbf{F}}^{\mathrm{tot}} = \hat{\mathbf{T}}^{\mathrm{b}}\hat{\mathbf{P}}^{\mathrm{b}}, \tag{59}$$

with the wavefield vector of wavefield components and the composition matrix operator given by

$$\hat{\mathbf{P}}^{\mathrm{b}} = \begin{pmatrix} \hat{p}^{\mathrm{inc}} \\ \hat{p}^{\mathrm{sct}} \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{T}}^{\mathrm{b}} = \begin{pmatrix} \mathcal{I} & \mathcal{I} \\ \hat{\mathcal{Y}}^{\mathrm{d}} & \hat{\mathcal{Y}}^{\mathrm{u}} \end{pmatrix}, \tag{60}$$

respectively. The inverse operation, *wavefield decomposition*, is given by

$$\hat{\mathbf{P}}^{\mathrm{b}} = \hat{\mathbf{L}}^{\mathrm{b}}\hat{\mathbf{F}}^{\mathrm{tot}}, \tag{61}$$

with

$$\hat{\mathbf{L}}^{\mathrm{b}} = (\hat{\mathcal{Y}}^{\mathrm{d}} - \hat{\mathcal{Y}}^{\mathrm{u}})^{-1}\begin{pmatrix} -\hat{\mathcal{Y}}^{\mathrm{u}} & \mathcal{I} \\ \hat{\mathcal{Y}}^{\mathrm{d}} & -\mathcal{I} \end{pmatrix}. \tag{62}$$

Observe that using eqs (57) and (58),

$$\hat{\mathcal{Y}}^{\mathrm{d}} - \hat{\mathcal{Y}}^{\mathrm{u}} = 2(\hat{\mathcal{S}}^{\mathrm{b}})^{-1}, \tag{63}$$

and, following Weston (1988), in terms of single- and double-layer potentials, we have

$$\hat{\mathbf{L}}^{\mathrm{b}} = \frac{1}{2}\begin{pmatrix} \mathcal{I} + \hat{\mathcal{D}}^{\mathrm{b}} & \mathcal{S}^{\mathrm{b}} \\ \mathcal{I} - \hat{\mathcal{D}}^{\mathrm{b}} & -\mathcal{S}^{\mathrm{b}} \end{pmatrix}. \tag{64}$$

In Haines & de Hoop (1996) a similar decomposition is implemented, in terms of curvilinear coordinates, for the case of internal wavefields inside a scattering domain.

4.4 Reflection operator

We define the single-layer potential boundary integral operator

$$\hat{\mathcal{S}} f(\mathbf{x}_\mathrm{T}, x_3) \stackrel{\mathrm{def}}{=} 2\left\langle \hat{G}^q(\mathbf{x}_\mathrm{T}, x_3; \mathbf{x}'_\mathrm{T}, x_3),\ f(\mathbf{x}'_\mathrm{T}, x_3)\right\rangle_\mathrm{b}, \tag{65}$$

and the double-layer potential boundary integral operator,

$$\hat{\mathcal{D}} g(\mathbf{x}_\mathrm{T}, x_3) \stackrel{\mathrm{def}}{=} 2\left\langle \hat{G}^f_3(\mathbf{x}_\mathrm{T}, x_3; \mathbf{x}'_\mathrm{T}, x_3), g(\mathbf{x}'_\mathrm{T}, x_3)\right\rangle_\mathrm{b}, \tag{66}$$

in which the functions $f, g \in [L^2(\mathbb{R}^2)]^1$. These last singular integral operators are defined with respect to the actual medium in contrast to the operators of eqs (51) and (52), which are defined with respect to the background medium. Evaluating the total wavefield representation (50), at $x_3 = x'_3$, using the operators (65) and (66), yields

$$(\mathcal{I} - \hat{\mathcal{D}})\hat{p}^\mathrm{tot} - \hat{\mathcal{S}}\hat{v}_3^\mathrm{tot} = 0. \tag{67}$$

This last equation represents the down-going wavefield condition for the total wavefield with respect to the actual medium. Assuming the existence of the inverse of $\hat{\mathcal{S}}$, we obtain

$$\hat{v}_3^\mathrm{tot} = \hat{\mathcal{Y}}\hat{p}^\mathrm{tot}, \tag{68}$$

with

$$\hat{\mathcal{Y}} = \hat{\mathcal{S}}^{-1}(\mathcal{I} - \hat{\mathcal{D}}). \tag{69}$$

The operator $\hat{\mathcal{Y}}$ constitutes the D-to-N map of the total wavefield. Combining the down-going wavefield condition of the incident wavefield of eq. (53), with the up-going wavefield condition of the scattered wavefield of eq. (54), together with the wavefield composition of eq. (16), yields

$$(\mathcal{I} + \hat{\mathcal{D}}^\mathrm{b})\hat{p}^\mathrm{tot} + \hat{\mathcal{S}}^\mathrm{b}\hat{v}_3^\mathrm{tot} = 2\hat{p}^\mathrm{inc}. \tag{70}$$

Using eqs (58), (63) and (68) we obtain

$$\hat{p}^\mathrm{tot} = \hat{\mathcal{T}}\hat{p}^\mathrm{inc}, \tag{71}$$

in which the operator $\hat{\mathcal{T}}$ is given by

$$\hat{\mathcal{T}} = (\hat{\mathcal{Y}} - \hat{\mathcal{Y}}^\mathrm{u})^{-1}(\hat{\mathcal{Y}}^\mathrm{d} - \hat{\mathcal{Y}}^\mathrm{u}). \tag{72}$$

The *reflection operator* $\hat{\mathcal{R}}$ is taken as

$$\hat{p}^\mathrm{sct} = \hat{\mathcal{R}}\hat{p}^\mathrm{inc}. \tag{73}$$

From the wavefield composition of eq. (16) we obtain

$$\hat{\mathcal{R}} = \hat{\mathcal{T}} - \mathcal{I}. \tag{74}$$

Hence,

$$\hat{\mathcal{R}} = (\hat{\mathcal{Y}} - \hat{\mathcal{Y}}^\mathrm{u})^{-1}(\hat{\mathcal{Y}}^\mathrm{d} - \hat{\mathcal{Y}}). \tag{75}$$

The reflection operator $\hat{\mathcal{R}}$ quantifies the spatial contrasts between the background medium and the actual medium, in terms of the D-t-N operators of the incident, scattered and total wavefields. It is a global operator encompassing all contrasts for $x_3 > x_3^\mathrm{sct}$ (Fishman 1994; Lu & McLaughlin 1996; Fishman *et al.* 1997, 1998). In these last references also the transmission problem is considered with the associated global transmission operator, and its relation with the global reflection operator. In this paper we do not consider the transmission problem. Our operator $\hat{\mathcal{T}}$ of eq. (72) is defined in $\mathbb{D}^{\mathrm{sct}'}$, operating at the same level surface as the reflection operator (Fig. 2).

5 SYMMETRY OF THE D-T-N OPERATORS

D-t-N operations for the monopole and dipole Green's wavefield are derived. Using these D-t-N operations it is shown that the causality boundary conditions, applied to arrive at the wavefield representations in Section 4.1, yield symmetry for the D-t-N operators of Sections 4.2 and 4.4.

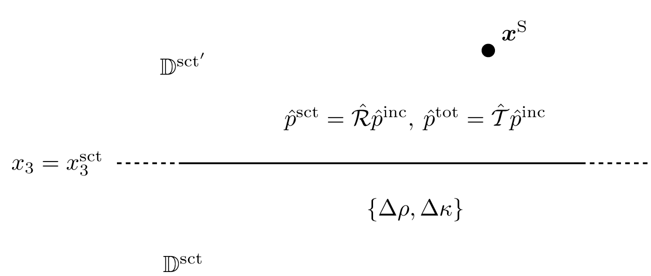

Figure 2. The reflection operator $\hat{\mathcal{R}}$ and the operator $\hat{\mathcal{T}}$ are defined in $\mathbb{D}^{\mathrm{sct}'}$. These operators account for the entire perturbation $\{\Delta\rho, \Delta\kappa\}$ given in $\mathbb{D}^\mathrm{sct}$.

5.1 D-t-N operators of the Green's functions

Because the incident and Green's wavefields of eqs (17) and (20), respectively, are linearly related to their source, we have, comparing these equations

$$\hat{\mathbf{F}}^{\rm inc} = \hat{q}\hat{\mathbf{G}}^{q,\rm b}. \tag{76}$$

In accordance with the incident wavefield, the volume injection Green's wavefield $\{\hat{G}^{q,\rm b}, \hat{\Gamma}_3^{q,\rm b}\}$ is a down-going wavefield, obeying the down-going wavefield condition of eq. (53), when its source position $\mathbf{x}$ is *above* its evaluation position $\mathbf{x}'$, that is, $x_3 < x_3'$. Using eqs (76) and (55) we obtain

$$\hat{\Gamma}_3^{q,\rm b}(\mathbf{x}';\mathbf{x}) = \hat{\mathcal{Y}}^{\rm d}\hat{G}^{q,\rm b}(\mathbf{x}';\mathbf{x}), \quad x_3 < x_3'. \tag{77}$$

Substituting the Source–receiver reciprocity relation (44), yields

$$\hat{G}_3^{f,\rm b}(\mathbf{x};\mathbf{x}') = -\hat{\mathcal{Y}}^{\rm d}\hat{G}^{q,\rm b}(\mathbf{x}';\mathbf{x}), \quad x_3 < x_3'. \tag{78}$$

Analogously, one can show that the volume-injection Green's wavefield $\{\hat{G}^{q,\rm b}, \hat{\Gamma}_3^{q,\rm b}\}$ is an up-going wavefield, obeying the up-going wavefield condition of eq. (54), for a source position $\mathbf{x}$ *below* its evaluation position $\mathbf{x}'$,

$$\hat{\Gamma}_3^{q,\rm b}(\mathbf{x}';\mathbf{x}) = \hat{\mathcal{Y}}^{\rm u}\hat{G}^{q,\rm b}(\mathbf{x}';\mathbf{x}), \quad x_3 > x_3'. \tag{79}$$

Again, substituting (44), yields

$$\hat{G}_3^{f,\rm b}(\mathbf{x};\mathbf{x}') = -\hat{\mathcal{Y}}^{\rm u}\hat{G}^{q,\rm b}(\mathbf{x}';\mathbf{x}), \quad x_3 > x_3'. \tag{80}$$

Eqs (78) and (80) constitute the D-t-N operations for the Green's wavefields, as these appear in the wavefield representations of the incident and scattered wavefields of eqs (48) and (49).

5.2 Symmetry

In Section 4.1.1, a radiation condition is applied to arrive at a representation for the incident wavefield. Omitting the source coordinate $\mathbf{x}^{\rm S}$ this radiation conditions can be expressed as

$$\left\langle \hat{G}^{q,\rm b}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\rm inc}(\mathbf{x}')\right\rangle_{\rm b} + \left\langle \hat{G}_3^{f,\rm b}(\mathbf{x};\mathbf{x}'), \hat{p}^{\rm inc}(\mathbf{x}')\right\rangle_{\rm b} = 0, \tag{81}$$

with $x_3 < x_3'$. Substituting the D-t-N operations of eq. (55) and eq. (78) yields

$$\langle \hat{G}^{q,\rm b}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\rm d}\hat{p}^{\rm inc}(\mathbf{x}')\rangle_{\rm b} - \langle \hat{\mathcal{Y}}^{\rm d}\hat{G}^{q,\rm b}(\mathbf{x}';\mathbf{x}), \hat{p}^{\rm inc}(\mathbf{x}')\rangle_{\rm b} = 0. \tag{82}$$

Using Source–receiver reciprocity (43) for $\hat{G}^{q,\rm b}$ in the first term of the left-hand side of eq. (82), and generalizing to an appropriately chosen subspace of functions $f, g \in [L^2(\mathbb{R}^2)]^1$, we obtain, according to eq. (A4), the symmetry,

$$\hat{\mathcal{Y}}^{\rm d} = (\hat{\mathcal{Y}}^{\rm d})^{\rm t}, \tag{83}$$

with respect to the chosen function space. Consequently, we can regard $\hat{\mathcal{Y}}^{\rm d}$ in the first bilinear form on the left-hand side of eq. (82), as either operating to the right on $\hat{p}^{\rm inc}$ or operating to the left on $\hat{G}^{q,\rm b}$. In both cases $\hat{\mathcal{Y}}^{\rm d}$ operates on the transverse coordinate $\mathbf{x}_{\rm T}'$, representing the transverse evaluation coordinate of $\hat{p}^{\rm inc}$ and the transverse source coordinate of $\hat{G}^{q,\rm b}$.

The radiation condition applied in Section 4.1.2, to obtain the representation of the scattered wavefield, is

$$\left\langle \hat{G}^{q,\rm b}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\rm sct}(\mathbf{x}')\right\rangle_{\rm b} + \left\langle \hat{G}_3^{f,\rm b}(\mathbf{x};\mathbf{x}'), \hat{p}^{\rm sct}(\mathbf{x}')\right\rangle_{\rm b} = 0, \tag{84}$$

with $x_3 > x_3'$. Substituting the D-t-N operations of eqs (56) and (80) yields

$$\langle \hat{G}^{q,\rm b}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\rm u}\hat{p}^{\rm sct}(\mathbf{x}')\rangle_{\rm b} - \langle \hat{\mathcal{Y}}^{\rm u}\hat{G}^{q,\rm b}(\mathbf{x}';\mathbf{x}), \hat{p}^{\rm sct}(\mathbf{x}')\rangle_{\rm b} = 0. \tag{85}$$

Using Source–receiver reciprocity (43) for $\hat{G}^{q,\rm b}$ in the first term of the integrand of eq. (85), and generalizing to an appropriately chosen subspace of functions $f, g \in [L^2(\mathbb{R}^2)]^1$, gives the symmetry

$$\hat{\mathcal{Y}}^{\rm u} = (\hat{\mathcal{Y}}^{\rm u})^{\rm t}, \tag{86}$$

with respect to the chosen function space.

The radiation condition applied in Section 4.1.3, in the derivation of the representation of the total wavefield, is expressed as

$$\left\langle \hat{G}^{q}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\rm tot}(\mathbf{x}')\right\rangle_{\rm b} + \left\langle \hat{G}_3^{f}(\mathbf{x};\mathbf{x}'), \hat{p}^{\rm tot}(\mathbf{x}')\right\rangle_{\rm b} = 0, \tag{87}$$

with $x_3 < x_3'$. Substitute the D-t-N operations of eqs (68) and (78), the latter taken with respect to the actual medium instead of the background medium. Following similar steps as the ones that lead to eq. (83), we obtain the symmetry

$$\hat{\mathcal{Y}} = \hat{\mathcal{Y}}^{\rm t}, \tag{88}$$

with respect to the chosen function space. The symmetry of the D-t-N operators is a consequence of the applicability of the causality radiation condition.

6 FORWARD PROPAGATION

The symmetry of the D-t-N operators is used to derive down- and up-going forward propagators for the incident and scattered wavefield, respectively. In terms of these operators the scattered wavefield is represented by Berkhout's WRW model, in which the reflectivity operator is a global operator with respect to the x_3-coordinate. Employing reciprocity, a symmetry relation can be derived between the down-going and the up-going propagators.

6.1 Incident wavefield

To arrive at a propagator for the incident wavefield, take in eq. (48) $x_3 > x_3'$, and substitute the D-t-N operations of eqs (55) and (80). We obtain, omitting the source coordinate $\mathbf{x}^{\mathrm{S}}$,

$$\hat{p}^{\mathrm{inc}}(\mathbf{x}) = \langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{d}}\hat{p}^{\mathrm{inc}}(\mathbf{x}')\rangle_{\mathrm{b}} - \langle \hat{\mathcal{Y}}^{\mathrm{u}}\hat{G}^{q,\mathrm{b}}(\mathbf{x}';\mathbf{x}), \hat{p}^{\mathrm{inc}}(\mathbf{x}')\rangle_{\mathrm{b}}. \tag{89}$$

Using symmetry (86) and eq. (63), defining the inverse of the single-layer potential operator as,

$$\hat{\mathcal{Y}}^{\mathrm{b}} \overset{\mathrm{def}}{=} (\hat{\mathcal{S}}^{\mathrm{b}})^{-1}, \tag{90}$$

yields

$$\hat{p}^{\mathrm{inc}}(\mathbf{x}) = 2\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{b}}\hat{p}^{\mathrm{inc}}(\mathbf{x}')\rangle_{\mathrm{b}}. \tag{91}$$

Observe that, using eqs (83) and (86), the right-hand side of eq. (63) is symmetric, and hence eq. (90) gives

$$\hat{\mathcal{Y}}^{\mathrm{b}} = (\hat{\mathcal{Y}}^{\mathrm{b}})^{\mathrm{t}}. \tag{92}$$

Defining the propagator for the function $f \in [L^2(\mathbb{R}^2)]^1$ as

$$\hat{\mathcal{W}}^{\mathrm{d}}(x_3;x_3')f(\mathbf{x}_{\mathrm{T}},x_3') \overset{\mathrm{def}}{=} 2\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x}_{\mathrm{T}},x_3;\mathbf{x}_{\mathrm{T}}',x_3'), \hat{\mathcal{Y}}^{\mathrm{b}} f(\mathbf{x}_{\mathrm{T}}',x_3')\right\rangle_{\mathrm{b}}, \quad x_3 > x_3', \tag{93}$$

we can write eq. (91) as

$$\hat{p}^{\mathrm{inc}}(\mathbf{x}_{\mathrm{T}},x_3) = \hat{\mathcal{W}}^{\mathrm{d}}(x_3;x_3')\hat{p}^{\mathrm{inc}}(\mathbf{x}_{\mathrm{T}},x_3'). \tag{94}$$

This last equation represents a forward propagation of the incident wavefield from the level surface at x_3' to x_3. The propagator $\hat{\mathcal{W}}^{\mathrm{d}}$ constitutes a one-parameter operator because it only depends on x_3, whereas x_3' is furnished by the function it operates on. The x_3' argument is included in the propagator's definition in order to be able to consider it, inappropriately, isolated from its associated function.

6.2 Scattered wavefield

Take in eq. (49) $x_3 < x_3'$, and substitute the D-t-N operations of eqs (56) and (78). We obtain, omitting the source coordinate $\mathbf{x}^{\mathrm{S}}$,

$$\hat{p}^{\mathrm{sct}}(\mathbf{x}) = -\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{u}}\hat{p}^{\mathrm{sct}}(\mathbf{x}')\rangle_{\mathrm{b}} + \langle \hat{\mathcal{Y}}^{\mathrm{d}}\hat{G}^{q,\mathrm{b}}(\mathbf{x}';\mathbf{x}), \hat{p}^{\mathrm{sct}}(\mathbf{x}')\rangle_{\mathrm{b}}. \tag{95}$$

Using symmetry (83), using eqs (63) and (90), and defining the propagator for $g \in [L^2(\mathbb{R}^2)]^1$,

$$\hat{\mathcal{W}}^{\mathrm{u}}(x_3;x_3')g(\mathbf{x}_{\mathrm{T}},x_3') \overset{\mathrm{def}}{=} 2\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x}_{\mathrm{T}},x_3;\mathbf{x}_{\mathrm{T}}',x_3'), \hat{\mathcal{Y}}^{\mathrm{b}} g(\mathbf{x}_{\mathrm{T}}',x_3')\right\rangle_{\mathrm{b}}, \quad \text{for} \quad x_3 < x_3', \tag{96}$$

yields

$$\hat{p}^{\mathrm{sct}}(\mathbf{x}_{\mathrm{T}},x_3) = \hat{\mathcal{W}}^{\mathrm{u}}(x_3;x_3')\hat{p}^{\mathrm{sct}}(\mathbf{x}_{\mathrm{T}},x_3'). \tag{97}$$

In this last equation, the scattered wavefield is forward propagated from level surface x_3' to level surface x_3.

6.3 Semi-group

Taking the limit $x_3' \uparrow x_3$ in eq. (93) and taking the limit $x_3' \downarrow x_3$ in eq. (96), using the same limiting operations for the single-layer potential that led to the Cauchy principal values in eqs (48) and (49), using eqs (51) and (90), we obtain the useful identity operator according to

$$\mathcal{I} f(\mathbf{x}_{\mathrm{T}},x_3) = 2\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x}_{\mathrm{T}},x_3;\mathbf{x}_{\mathrm{T}}',x_3), \hat{\mathcal{Y}}^{\mathrm{b}} f(\mathbf{x}_{\mathrm{T}}',x_3)\right\rangle_{\mathrm{b}}, \tag{98}$$

and

$$\hat{\mathcal{W}}^{\mathrm{d}}(x_3;x_3) = \hat{\mathcal{W}}^{\mathrm{u}}(x_3;x_3) = \mathcal{I}. \tag{99}$$

For $x_3 < x_3' < x_3''$, one can show that

$$\hat{\mathcal{W}}^{\mathrm{d}}(x_3'';x_3) = \hat{\mathcal{W}}^{\mathrm{d}}(x_3'';x_3')\hat{\mathcal{W}}^{\mathrm{d}}(x_3';x_3), \tag{100}$$

$$\hat{\mathcal{W}}^{\mathrm{u}}(x_3;x_3'') = \hat{\mathcal{W}}^{\mathrm{u}}(x_3;x_3')\hat{\mathcal{W}}^{\mathrm{u}}(x_3';x_3''). \tag{101}$$

Because the one-parameter families of operators $\hat{\mathcal{W}}^{\mathrm{d}}$ and $\hat{\mathcal{W}}^{\mathrm{u}}$ satisfy the identity property of eq. (99) and the transitivity property of eqs (100) and (101), they constitute semigroups of operators (Pazy 1983; Goldstein 1985).

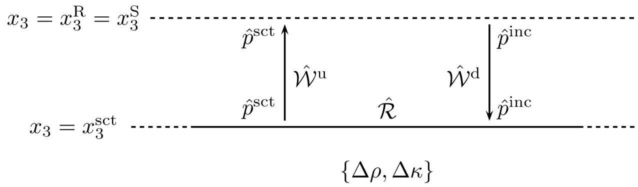

Figure 3. Depiction of the WRW model in which, from the right to the left, the operator $\hat{\mathcal{W}}^{\rm d}$ propagates the incident wavefield from the source level surface, $x_3 = x_3^{\rm S}$, to the level of the scattering medium surface, $x_3 = x_3^{\rm sct}$, the reflection operator $\hat{\mathcal{R}}$ transforms, at the scattering medium surface, the incident wavefield to the scattered wavefield, and the operator $\hat{\mathcal{W}}^{\rm u}$ propagates the scattered wavefield to the recording level surface, $x_3 = x_3^{\rm R}$, which in this figure is equal to the source level surface.

6.4 WRW model

Using the reflection operator of eq. (73), and using eqs (94) and (97) can be expressed, in terms of the 'WRW model' of Berkhout (1985), as

$$\hat{p}^{\rm sct}\big(\mathbf{x}_{\rm T}, x_3^{\rm R}; \mathbf{x}^{\rm S}\big) = \hat{\mathcal{W}}^{\rm u}\big(x_3^{\rm R}; x_3^{\rm sct}\big)\hat{\mathcal{R}}\hat{\mathcal{W}}^{\rm d}\big(x_3^{\rm sct}; x_3^{\rm S}\big)\,\hat{p}^{\rm inc}\big(\mathbf{x}_{\rm T}, x_3^{\rm S}; \mathbf{x}^{\rm S}\big). \tag{102}$$

The radiation in eq. (102) is downward from the source level at $x_3^{\rm S}$ to the boundary of the scattering domain at $x_3^{\rm sct}$, and subsequently, after reflection, upward from $x_3^{\rm sct}$ to the receiver surface at $x_3^{\rm R}$ (Fig. 3). In order to obtain the initial condition for the incident wavefield we consider, using eq. (76),

$$\hat{p}^{\rm inc}\big(\mathbf{x}_{\rm T}, x_3; \mathbf{x}_{\rm T}^{\rm S}, x_3^{\rm S}\big) = \hat{q}\big\langle \hat{G}^{q,{\rm b}}\big(\mathbf{x}_{\rm T}, x_3; \mathbf{x}'_{\rm T}, x_3^{\rm S}\big), \delta\big(\mathbf{x}'_{\rm T} - \mathbf{x}_{\rm T}^{\rm S}\big)\big\rangle_{\rm b}. \tag{103}$$

Taking the limit $x_3 \to x_3^{\rm S}$, using the single-layer potential of eqs (51) and (90), we obtain the incident wavefield at its source level surface,

$$\hat{p}^{\rm inc}\big(\mathbf{x}_{\rm T}, x_3^{\rm S}; \mathbf{x}_{\rm T}^{\rm S}, x_3^{\rm S}\big) = \frac{1}{2}\,\hat{q}\,(\hat{\mathcal{Y}}^{\rm b})^{-1}\delta\big(\mathbf{x}_{\rm T} - \mathbf{x}_{\rm T}^{\rm S}\big). \tag{104}$$

The reflection operator $\hat{\mathcal{R}}$ in eq. (102) contains all the interactions of the incident wavefield with the scattering domain. It is therefore a global operator with respect to the x_3-coordinate. On the contrary, the reflection operator in the WRW model of Berkhout (1985) is local in x_3, and the total scattering is represented by an integral with respect to the x_3-coordinate.

6.5 Propagator reciprocity

Including the source coordinate of the incident wavefield in eq. (91), we write

$$\hat{p}^{\rm inc}(\mathbf{x}; \mathbf{x}^{\rm S}) = 2\langle \hat{G}^{q,{\rm b}}(\mathbf{x}; \mathbf{x}'), \hat{\mathcal{Y}}^{\rm b}\hat{p}^{\rm inc}(\mathbf{x}'; \mathbf{x}^{\rm S})\rangle_{\rm b}, \quad \text{for} \quad x_3 > x'_3. \tag{105}$$

Using symmetry (92) in this last equation yields,

$$\hat{p}^{\rm inc}(\mathbf{x}; \mathbf{x}^{\rm S}) = 2\langle \hat{\mathcal{Y}}^{\rm b}\hat{G}^{q,{\rm b}}(\mathbf{x}'; \mathbf{x}), \hat{p}^{\rm inc}(\mathbf{x}'; \mathbf{x}^{\rm S})\rangle_{\rm b}. \tag{106}$$

Using Source–receiver reciprocity (eq. 43) for the incident pressure wavefields on the left- and right-hand sides of this last equation, and changing the order of the bilinear form, using its symmetry, yields

$$\hat{p}^{\rm inc}(\mathbf{x}^{\rm S}; \mathbf{x}) = 2\big\langle \hat{p}^{\rm inc}(\mathbf{x}^{\rm S}; \mathbf{x}'), \hat{\mathcal{Y}}^{\rm b}\hat{G}^{q,{\rm b}}(\mathbf{x}'; \mathbf{x})\big\rangle_{\rm b}. \tag{107}$$

Similar to eq. (105), for which we identified a propagator in Section 6.1, which acts on the evaluation depth coordinate, we can devise a propagator from eq. (107), which operates on the source depth coordinate. In eq. (105) $\hat{\mathcal{Y}}^{\rm b}$ is operating from the left on $\hat{p}^{\rm inc}$ resulting in the left-propagator in eq. (94). In eq. (107), with $\hat{\mathcal{Y}}^{\rm b}$ acting from the right on $\hat{p}^{\rm inc}$ (see Section 5.2), we obtain the propagator which operates on the source coordinate from the right,

$$\hat{p}^{\rm inc}(\mathbf{x}^{\rm S}; \mathbf{x}_{\rm T}, x_3) = \hat{p}^{\rm inc}\big(\mathbf{x}^{\rm S}; \mathbf{x}_{\rm T}, x'_3\big)\hat{\mathcal{W}}^{\rm u}\,\big(x'_3; x_3\big)\,. \tag{108}$$

Because $x_3 > x'_3$, the propagator in this last equation is the propagator $\hat{\mathcal{W}}^{\rm u}$ of eq. (96), in contrast to eq. (105), for which we identified the propagator $\hat{\mathcal{W}}^{\rm d}$ of eq. (93). Using eqs (94) and (108), and Source–receiver reciprocity, yields

$$\hat{\mathcal{W}}^{\rm d}(x_3; x'_3)\hat{p}^{\rm inc}\big(\mathbf{x}_{\rm T}, x'_3; \mathbf{x}^{\rm S}\big) = \hat{p}^{\rm inc}\big(\mathbf{x}^{\rm S}; \mathbf{x}_{\rm T}, x'_3\big)\hat{\mathcal{W}}^{\rm u}(x'_3; x_3), \quad \text{for} \quad x_3 > x'_3. \tag{109}$$

Hence, downward propagation of the receivers of the down-going incident wavefield from x'_3 to x_3 equals downward propagation of the sources of the reciprocal up-going incident wavefield from x'_3 to x_3. One can easily show that, using eq. (A3), this operator reciprocity is formalized by the symmetry

$$\big[\hat{\mathcal{W}}^{\rm d}(x_3; x'_3)\big]^{\rm t} = \hat{\mathcal{W}}^{\rm u}(x'_3; x_3). \tag{110}$$

Similarly, one can derive

$$\hat{p}^{\rm sct}\big(\mathbf{x}^{\rm S}; \mathbf{x}_{\rm T}, x_3\big) = \hat{p}^{\rm sct}\big(\mathbf{x}^{\rm S}; \mathbf{x}_{\rm T}, x'_3\big)\hat{\mathcal{W}}^{\rm d}(x'_3; x_3), \tag{111}$$

and

$$\hat{\mathcal{W}}^{\rm u}(x_3; x'_3)\hat{p}^{\rm sct}\big(\mathbf{x}_{\rm T}, x'_3; \mathbf{x}^{\rm S}\big) = \hat{p}^{\rm sct}\big(\mathbf{x}^{\rm S}; \mathbf{x}_{\rm T}, x'_3\big)\hat{\mathcal{W}}^{\rm d}(x'_3; x_3), \quad \text{for} \quad x_3 < x'_3. \tag{112}$$

The propagation of the sources as well as the receivers enables to propagate an entire experiment from one level to another by deploying both right- and left-operators, respectively, on a wavefield. This procedure is commonly referred to as redatuming.

7 REFLECTION KERNEL

In the following analysis, the kernel of the reflection operator is derived. Using the identity operator of eq. (98) the scattered wavefields representation is given by the following singular bilinear form,

$$\hat{p}^{\mathrm{sct}}(\mathbf{x};\mathbf{x}^{\mathrm{S}}) = 2\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{b}}\hat{p}^{\mathrm{sct}}(\mathbf{x}';\mathbf{x}^{\mathrm{S}})\right\rangle_{\mathrm{b}}, \quad \text{for} \quad x_3 = x_3'. \tag{113}$$

Using Source–receiver reciprocity for the scattered wavefield on the right-hand side of this last equation yields

$$\hat{p}^{\mathrm{sct}}(\mathbf{x};\mathbf{x}^{\mathrm{S}}) = 2\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x}^{\mathrm{S}};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{b}}\hat{p}^{\mathrm{sct}}(\mathbf{x}';\mathbf{x})\right\rangle_{\mathrm{b}}. \tag{114}$$

Using symmetry (92), interchanging the order of the bilinear form, and using Source–receiver reciprocity for the scattered wavefield inside the bilinear form, yields

$$\hat{p}^{\mathrm{sct}}(\mathbf{x};\mathbf{x}^{\mathrm{S}}) = 2\langle \hat{p}^{\mathrm{sct}}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{b}}\hat{G}^{q,\mathrm{b}}(\mathbf{x}';\mathbf{x}^{\mathrm{S}})\rangle_{\mathrm{b}}. \tag{115}$$

Using eqs (76) and (73) gives for the reflection operator

$$\hat{\mathcal{R}}\hat{p}^{\mathrm{inc}}(\mathbf{x}_{\mathrm{T}},x_3;\mathbf{x}^{\mathrm{S}}) = 2\hat{q}^{-1}\left\langle \hat{p}^{\mathrm{sct}}(\mathbf{x}_{\mathrm{T}},x_3;\mathbf{x}_{\mathrm{T}}',x_3), \hat{\mathcal{Y}}^{\mathrm{b}}\hat{p}^{\mathrm{inc}}\left(\mathbf{x}_{\mathrm{T}}',x_3;\mathbf{x}^{\mathrm{S}}\right)\right\rangle_{\mathrm{b}}. \tag{116}$$

Writing $\hat{\mathcal{Y}}^{\mathrm{b}}$ as a left operator on $\hat{p}^{\mathrm{sct}}$, using symmetry (92), the kernel of $\hat{\mathcal{R}}$ is given by

$$\hat{R}(\mathbf{x}_{\mathrm{T}},x_3;\mathbf{x}_{\mathrm{T}}',x_3) = 2\hat{q}^{-1}\hat{\mathcal{Y}}^{\mathrm{b}}\hat{p}^{\mathrm{sct}}(\mathbf{x}_{\mathrm{T}}',x_3;\mathbf{x}_{\mathrm{T}},x_3). \tag{117}$$

The kernel of the reflection operator is represented by the scattered wavefield at $(\mathbf{x}_{\mathrm{T}}', x_3)$, due to an incident wavefield source at $(\mathbf{x}_{\mathrm{T}}, x_3)$. This scattered wavefield is deconvolved with the source function, and normalized by the inverse single-layer potential operator (eq. 90), which makes the reflection kernel effectively a dipole wavefield. We can think of the scattered wavefield associated with the reflection kernel, as given by eq. (117), as related to a virtual experiment. The real experiment is then a redatumed version of the virtual experiment, where source and receiver have been propagated upward from the scatterer boundary. Using eqs (97) and (111) this redatuming procedure is given by

$$\hat{p}^{\mathrm{sct}}\left(\mathbf{x}_{\mathrm{T}},x_3^{\mathrm{R}};\mathbf{x}_{\mathrm{T}}',x_3^{\mathrm{S}}\right) = \hat{\mathcal{W}}^{\mathrm{u}}\left(x_3^{\mathrm{R}};x_3^{\mathrm{sct}}\right)\hat{p}^{\mathrm{sct}}\left(\mathbf{x}_{\mathrm{T}},x_3^{\mathrm{sct}};\mathbf{x}_{\mathrm{T}}',x_3^{\mathrm{sct}}\right)\hat{\mathcal{W}}^{\mathrm{d}}\left(x_3^{\mathrm{sct}};x_3^{\mathrm{S}}\right). \tag{118}$$

In imaging one exploits the causal relation between the incident and the scattered wavefield by inverting this last equation, in order to obtain the reflection kernel at the boundary of the scattering domain at $x_3 = x_3^{\mathrm{sct}}$. For this we need the inverse propagators that are derived in the next section. Invoking causality, a reflectivity measure is obtained from the inverse propagated wavefield, by collecting the wavefield at zero time in the time domain (Claerbout 1985). In de Bruin *et al.* (1990) non-coincident Source–receiver pairs yield an angle-dependent reflectivity, by application of a Radon transform with respect to transverse receiver and source coordinates.

8 INVERSE PROPAGATION

To recover the acoustic reflectivity, represented by the kernel of the reflection operator, we need to propagate the sources and receivers of the scattered wavefield back towards the reflector. Backward propagation is accomplished by inverse propagators in terms of sesquilinear forms. Backward radiation from one level surface to another is an approximate operation requiring an imposed radiation condition. With this radiation condition we associate with the inverse propagator the adjoint of the forward propagator.

8.1 Wavefield decomposition

To derive a boundary integral representation of the scattered wavefield in terms of a sesquilinear form (Appendix B) we proceed by considering the domain of application as $\mathbb{D}^{\mathrm{a}} = \{(\mathbf{x}_{\mathrm{T}}', x_3') | \, \mathbf{x}_{\mathrm{T}}' \in \mathbb{R}^2, x_3^0 < x_3' < x_3^1\}$. Taking the same states as in Section 4.1.2, application of the reciprocity theorem of the time-correlation type of eq. (41), yields

$$\hat{p}^{\mathrm{sct}} = \hat{p}^{\mathrm{sct,d}} + \hat{p}^{\mathrm{sct,u}}, \tag{119}$$

with

$$\hat{p}^{\mathrm{sct,d}}(\mathbf{x}) = \left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct}}(\mathbf{x}')\right\rangle_{\mathrm{s}}\left(x_3' = x_3^1\right) - \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct}}(\mathbf{x}')\right\rangle_{\mathrm{s}}\left(x_3' = x_3^1\right), \tag{120}$$

and

$$\hat{p}^{\mathrm{sct,u}}(\mathbf{x}) = -\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct}}(\mathbf{x}')\right\rangle_{\mathrm{s}}\left(x_3' = x_3^0\right) + \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct}}(\mathbf{x}')\right\rangle_{\mathrm{s}}\left(x_3' = x_3^0\right), \tag{121}$$

in which $\mathbf{x} \in \mathbb{D}^{\mathrm{a}}$. Eq. (119) constitutes a wavefields decomposition. The integral representations of the wavefield components contain anticausal Green's functions, verified by the property that complex conjugation of a Fourier transformed real-valued wavefield amounts to a time reversal in the time domain. The wavefield decomposition is not a decomposition in terms of down- and up-going components, as defined in Section 4.2, that is, the components of eqs (120) and (121) do not satisfy eqs (53) and (54), or eqs (55) and (56), respectively. These components satisfy other wavefields conditions, corresponding to other D-t-N operations, as will be shown later. Because of the anticausality of the Green's wavefields causality radiation conditions do not apply at infinity in the case of time-correlation representations (Bojarski 1983). Taking in this last equation the limiting operations,

$$\lim_{x_3^1 \downarrow x_3} \hat{p}^{\mathrm{sct,d}} \quad \text{and} \quad \lim_{x_3^0 \uparrow x_3} \hat{p}^{\mathrm{sct,u}}, \tag{122}$$

of the single- and double-layer potentials, taking into account the $\frac{1}{2}\,\hat{p}^{\mathrm{sct}}$ contribution of the latter (Colton & Kress 1983; Fokkema & van den Berg 1993), taking the complex conjugate, and using the operators of eqs (51) and (52), yields

$$\begin{pmatrix} \hat{p}^{\mathrm{sct,d}\,*} \\ \hat{p}^{\mathrm{sct,u}\,*} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{I} - \hat{\mathcal{D}}^{\mathrm{b}} & \hat{\mathcal{S}}^{\mathrm{b}} \\ \mathcal{I} + \hat{\mathcal{D}}^{\mathrm{b}} & -\hat{\mathcal{S}}^{\mathrm{b}} \end{pmatrix} \begin{pmatrix} \hat{p}^{\mathrm{sct}\,*} \\ \hat{v}_3^{\mathrm{sct}\,*} \end{pmatrix}. \tag{123}$$

Inversion gives, using the D-t-N operators of eqs (57) and (58),

$$\begin{pmatrix} \hat{p}^{\mathrm{sct}\,*} \\ \hat{v}_3^{\mathrm{sct}\,*} \end{pmatrix} = \begin{pmatrix} \mathcal{I} & \mathcal{I} \\ -\hat{\mathcal{Y}}^{\mathrm{u}} & -\hat{\mathcal{Y}}^{\mathrm{d}} \end{pmatrix} \begin{pmatrix} \hat{p}^{\mathrm{sct,d}\,*} \\ \hat{p}^{\mathrm{sct,u}\,*} \end{pmatrix}. \tag{124}$$

Writing

$$\hat{v}_3^{\mathrm{sct}} = \hat{v}_3^{\mathrm{sct,d}} + \hat{v}_3^{\mathrm{sct,u}}, \tag{125}$$

we take

$$\hat{v}_3^{\mathrm{sct,d}\,*} = -\hat{\mathcal{Y}}^{\mathrm{u}}\, \hat{p}^{\mathrm{sct,d}\,*}, \tag{126}$$

and

$$\hat{v}_3^{\mathrm{sct,u}\,*} = -\hat{\mathcal{Y}}^{\mathrm{d}}\, \hat{p}^{\mathrm{sct,u}\,*}. \tag{127}$$

Using the symmetries of eqs (83) and (86), and (B10), we obtain the D-t-N operations for the wavefield components with respect to the time-correlation type reciprocity theorem.

$$\hat{v}_3^{\mathrm{sct,d}} = -\,\hat{\mathcal{Y}}^{\mathrm{u}\,\dagger}\, \hat{p}^{\mathrm{sct,d}}, \tag{128}$$

and

$$\hat{v}_3^{\mathrm{sct,u}} = -\,\hat{\mathcal{Y}}^{\mathrm{d}\,\dagger}\, \hat{p}^{\mathrm{sct,u}}. \tag{129}$$

8.2 Boundary integral representations

Substituting the wavefield decomposition of eqs (119) and (125) into eq. (120), using sesquilinearity, yields

$$\hat{p}^{\mathrm{sct,d}}(\mathbf{x}) = \left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct,d}}(\mathbf{x}') \right\rangle_{\mathrm{s}} - \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct,d}}(\mathbf{x}') \right\rangle_{\mathrm{s}} + \left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct,u}}(\mathbf{x}') \right\rangle_{\mathrm{s}} - \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct,u}}(\mathbf{x}') \right\rangle_{\mathrm{s}}, \quad \text{for} \quad x_3 < x_3'. \tag{130}$$

Substituting the D-t-N operations of eqs (78) and (129) into the last two sesquilinear forms on the right-hand side of this last equation, gives, using eq. (B5),

$$\langle \hat{G}^{q,\mathrm{b}}, -\,\hat{\mathcal{Y}}^{\mathrm{d}\,\dagger}\, \hat{p}^{\mathrm{sct,u}} \rangle_{\mathrm{s}} - \langle -\hat{\mathcal{Y}}^{\mathrm{d}} \hat{G}^{q,\mathrm{b}}, \hat{p}^{\mathrm{sct,u}} \rangle_{\mathrm{s}} = 0. \tag{131}$$

Hence, we obtain the radiation condition

$$\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct,u}}(\mathbf{x}') \right\rangle_{\mathrm{s}} - \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct,u}}(\mathbf{x}') \right\rangle_{\mathrm{s}} = 0, \qquad \text{for} \quad x_3 < x_3', \tag{132}$$

Similarly, substituting the wavefield decomposition of eqs (119) and (125) into eq. (121), using sesquilinearity, yields

$$\hat{p}^{\mathrm{sct,u}}(\mathbf{x}) = -\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct,d}}(\mathbf{x}') \right\rangle_{\mathrm{s}} + \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct,d}}(\mathbf{x}') \right\rangle_{\mathrm{s}} - \left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct,u}}(\mathbf{x}') \right\rangle_{\mathrm{s}} + \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct,u}}(\mathbf{x}') \right\rangle_{\mathrm{s}}, \quad \text{for} \quad x_3 > x_3'. \tag{133}$$

Substituting the D-t-N operations of eqs (80) and (128) into the first two sesquilinear forms on the right-hand side of this last equation, gives

$$-\left\langle \hat{G}^{q,\mathrm{b}}, -\hat{\mathcal{Y}}^{\mathrm{u}\dagger}\, \hat{p}^{\mathrm{sct,d}} \right\rangle_{\mathrm{s}} + \left\langle -\,\hat{\mathcal{Y}}^{\mathrm{u}} \hat{G}^{q,\mathrm{b}}, \hat{p}^{\mathrm{sct,d}} \right\rangle_{\mathrm{s}} = 0. \tag{134}$$

Hence, we obtain the radiation condition

$$\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct,d}}(\mathbf{x}') \right\rangle_{\mathrm{s}} - \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct,d}}(\mathbf{x}') \right\rangle_{\mathrm{s}} = 0, \quad \text{for} \quad x_3 > x_3'. \tag{135}$$

The radiation conditions of eqs (132) and (135) follow from the symmetry of the D-t-N operators $\hat{\mathcal{Y}}^{\mathrm{d}}$ and $\hat{\mathcal{Y}}^{\mathrm{u}}$, as these are used to derive the D-t-N operations of eqs (128) and (129).

Substituting the radiation conditions of eqs (132) and (135), and the first limiting operation of eq. (122), into eq. (130), yields the representation for $\hat{p}^{\mathrm{sct,d}}$, as

$$H(x_3' - x_3)\hat{p}^{\mathrm{sct,d}}(\mathbf{x}) = \left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct,d}}(\mathbf{x}') \right\rangle_{\mathrm{s}} - \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct,d}}(\mathbf{x}') \right\rangle_{\mathrm{s}}. \tag{136}$$

Again, substituting the radiation conditions of eqs (132) and (135), and the second limiting operation of eq. (122), into eq. (133), yields the representation for $\hat{p}^{\mathrm{sct,u}}$, as

$$H(x_3 - x_3')\hat{p}^{\mathrm{sct,u}}(\mathbf{x}) = -\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct,u}}(\mathbf{x}')\right\rangle_{\mathrm{s}} + \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct,u}}(\mathbf{x}')\right\rangle_{\mathrm{s}}. \tag{137}$$

8.3 Propagators

We proceed by substituting the D-t-N operations of eqs (78) and (128), in eq. (136), with $x_3 < x_3'$,

$$\hat{p}^{\mathrm{sct,d}}(\mathbf{x}) = \left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), -\hat{\mathcal{Y}}^{\mathrm{u}\dagger}\hat{p}^{\mathrm{sct,d}}(\mathbf{x}')\right\rangle_{\mathrm{s}} + \left\langle \hat{\mathcal{Y}}^{\mathrm{d}}\hat{G}^{q,\mathrm{b}}(\mathbf{x}';\mathbf{x}), \hat{p}^{\mathrm{sct,d}}(\mathbf{x}')\right\rangle_{\mathrm{s}}, \quad \text{for} \quad x_3 < x_3'. \tag{138}$$

Using eqs (B5), (63) and (90), we obtain

$$\hat{p}^{\mathrm{sct,d}}(\mathbf{x}) = 2\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{b}\dagger}\hat{p}^{\mathrm{sct,d}}(\mathbf{x}')\right\rangle_{\mathrm{s}}, \tag{139}$$

Observe that eq. (139) has a similar form as eq. (91). Taking the complex conjugate of eq. (139), using symmetry (92) and eq. (B10), yields, in terms of a bilinear form

$$\hat{p}^{\mathrm{sct,d}\,*}(\mathbf{x}) = 2\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{b}}\hat{p}^{\mathrm{sct,d}*}(\mathbf{x}')\right\rangle_{\mathrm{b}}. \tag{140}$$

Hence, using eq. (96), we have

$$\hat{p}^{\mathrm{sct,d}\,*}(\mathbf{x}_{\mathrm{T}}, x_3) = \hat{\mathcal{W}}^{\mathrm{u}}(x_3; x_3')\hat{p}^{\mathrm{sct,d}*}(\mathbf{x}_{\mathrm{T}}, x_3'). \tag{141}$$

Again, taking the complex conjugate, using eqs (B10) and (110), we arrive at

$$\hat{p}^{\mathrm{sct,d}}(\mathbf{x}_{\mathrm{T}}, x_3) = [\hat{\mathcal{W}}^{\mathrm{d}}(x_3'; x_3)]^{\dagger}\hat{p}^{\mathrm{sct,d}}(\mathbf{x}_{\mathrm{T}}, x_3'), \quad \text{for} \quad x_3 < x_3'. \tag{142}$$

In eq. (142) the wavefield $\hat{p}^{\mathrm{sct,d}}$ is backward propagated from level surface x_3' to level surface x_3, using the adjoint of the forward propagator $\hat{\mathcal{W}}^{\mathrm{d}}$. The kernel of the adjoint propagator is given by eq. (139), which, when evaluated in the time domain, is given in terms of an anticausal Green's wavefield.

Similarly, as above, substituting the D-t-N operations of eqs (80) and (129), in eq. (137), with $x_3 > x_3'$, using eqs (B5), (63) and (90), we obtain

$$\hat{p}^{\mathrm{sct,u}}(\mathbf{x}) = 2\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{\mathcal{Y}}^{\mathrm{b}\,\dagger}\hat{p}^{\mathrm{sct,u}}(\mathbf{x}')\rangle_{\mathrm{s}}. \tag{143}$$

Following the same steps as above one can easily show that

$$\hat{p}^{\mathrm{sct,u}}(\mathbf{x}_{\mathrm{T}}, x_3) = \left[\hat{\mathcal{W}}^{\mathrm{u}}\left(x_3'; x_3\right)\right]^{\dagger}\hat{p}^{\mathrm{sct,u}}(\mathbf{x}_{\mathrm{T}}, x_3'), \quad \text{for} \quad x_3 > x_3'. \tag{144}$$

In this last equation the wavefield $\hat{p}^{\mathrm{sct,u}}$ is backward propagated from level surface x_3' to level surface x_3, using the adjoint of the forward propagator $\hat{\mathcal{W}}^{\mathrm{u}}$.

8.4 Focussing

In order to make the scattered wavefield one-directional with respect to a sesquilinear form we assume in eq. (119) the approximation,

$$\left\{\hat{p}^{\mathrm{sct}}, \hat{v}_3^{\mathrm{sct}}\right\} \approx \left\{\hat{p}^{\mathrm{sct,u}}, \hat{v}_3^{\mathrm{sct,u}}\right\}. \tag{145}$$

Hence, we enforce on the scattered wavefield the radiation condition,

$$\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct}}(\mathbf{x}')\right\rangle_{\mathrm{s}} - \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct}}(\mathbf{x}')\right\rangle_{\mathrm{s}} = 0, \quad \text{for} \quad x_3 < x_3'. \tag{146}$$

Given approximation (145), we obtain from eq. (137) the integral representation for the scattered wavefield with respect to the reciprocity theorem of the time-correlation type,

$$H(x_3 - x_3')\hat{p}^{\mathrm{sct}}(\mathbf{x}) = -\left\langle \hat{G}^{q,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{v}_3^{\mathrm{sct}}(\mathbf{x}')\right\rangle_{\mathrm{s}} + \left\langle \hat{G}_3^{f,\mathrm{b}}(\mathbf{x};\mathbf{x}'), \hat{p}^{\mathrm{sct}}(\mathbf{x}')\right\rangle_{\mathrm{s}}. \tag{147}$$

Approximation (145) yields the D-t-N operation

$$\hat{v}_3^{\mathrm{sct}} = -\,\hat{\mathcal{Y}}^{\mathrm{d}\,\dagger}\,\hat{p}^{\mathrm{sct}}. \tag{148}$$

Also, eq. (144) gives,

$$\hat{p}^{\mathrm{sct}}(\mathbf{x}_{\mathrm{T}}, x_3) = \left[\hat{\mathcal{W}}^{\mathrm{u}}(x_3'; x_3)\right]^{\dagger}\hat{p}^{\mathrm{sct}}(\mathbf{x}_{\mathrm{T}}, x_3'), \quad \text{for} \quad x_3 > x_3'. \tag{149}$$

Hence, using eq. (97) we obtain

$$\left[\hat{\mathcal{W}}^{\mathrm{u}}(x_3'; x_3)\right]^{-1} = \left[\hat{\mathcal{W}}^{\mathrm{u}}(x_3'; x_3)\right]^{\dagger}; \tag{150}$$

I.e. $\hat{\mathcal{W}}^{\mathrm{u}}$ is an *unitary* operator, meaning that the inverse propagator is equal to the adjoint of the associated forward propagator. By neglecting the contribution from the sesquilinear form (120) in eq. (119) we obtain a correlation type representation of the scattered wavefield in terms of a single integral. In this way, uni-directionality of the scattered wavefield is enforced with respect to the background medium. In general, the two contributions of eq. (119) make it meaningless to assign a direction to the scattered wavefield with respect to the background medium. This means that Huygens' principle, which states that an infinitesimal change in a wavefield can be constructed from infinitesimal contributions from secondary sources along a single surface, is not valid for these representations. Wapenaar (1992) associates the negligence of one surface integral with the erroneous handling of the evanescent wavefield, in the case of an homogeneous medium. In Berkhout (1985), eq. (150) is called the matched filter approach to inverse propagation. Using reciprocity, following the same analysis as in Section 6.5 with respect to a sesquilinear form instead of a bilinear form, yields, under approximation (145), the adjoint propagator which operates on the source coordinates from the right, for $x_3 > x_3'$,

$$\hat{p}^{\text{sct}}\left(\mathbf{x}^{\text{S}}; \mathbf{x}_{\text{T}}, x_3\right) = \hat{p}^{\text{sct}}\left(\mathbf{x}^{\text{S}}; \mathbf{x}_{\text{T}}, x_3'\right)\left[\hat{\mathcal{W}}^{\text{d}}\left(x_3; x_3'\right)\right]^{\dagger}, \tag{151}$$

This last equation and eq. (111) gives the unitary property

$$\left[\hat{\mathcal{W}}^{\text{d}}\left(x_3; x_3'\right)\right]^{-1} = \left[\hat{\mathcal{W}}^{\text{d}}\left(x_3; x_3'\right)\right]^{\dagger}. \tag{152}$$

Using eqs (150) and (152), and eq. (118), focussing is achieved, under approximation (145), according to

$$\hat{p}^{\text{sct}}\left(\mathbf{x}_{\text{T}}, x_3^{\text{sct}}; \mathbf{x}_{\text{T}}', x_3^{\text{sct}}\right) = \left[\hat{\mathcal{W}}^{\text{u}}\left(x_3^{\text{R}}; x_3^{\text{sct}}\right)\right]^{\dagger} \hat{p}^{\text{sct}}\left(\mathbf{x}_{\text{T}}, x_3^{\text{R}}; \mathbf{x}_{\text{T}}', x_3^{\text{S}}\right)\left[\hat{\mathcal{W}}^{\text{d}}\left(x_3^{\text{sct}}; x_3^{\text{S}}\right)\right]^{\dagger}. \tag{153}$$

In this last equation the source of the incident wavefield which gives rise to the scattered wavefield is back-propagated to the scatterer surface, while the scattered wavefield is evaluated at the same level. As a consequence, the scattered wavefield focuses on its sources located at the scatterer boundary. Using eq. (117) we obtain the kernel of the reflection operator, which is a function of the perturbations of eq. (15).

9 FUNDAMENTAL SOLUTION

To extrapolate the incident and scattered wavefields, using eqs (94) and (97), respectively, we need to, according to eqs (93) and (96), compute the Green's function $\hat{G}^{q,\text{b}}$ for $x_3 > x_3'$ and $x_3 < x_3'$, respectively. We will follow the fundamental solution approach (Pazy 1983; Krueger & Ochs 1989; Haines & de Hoop 1996; Wapenaar 1996a; Fishman *et al.* 1997; Grimbergen *et al.* 1998; Fishman 2004). The fundamental solutions yield product integrals for the propagators. The associated D-t-N operators are solutions of an operator Riccati equation (Haines & de Hoop 1996; Lu & McLaughlin 1996; Fishman *et al.* 1997, 1998).

9.1 Product integral

Substituting the D-t-N operators of eqs (77) and (79) into eq. (20), for vanishing sources, and defining,

$$\hat{\mathcal{P}}^{\text{d}} \overset{\text{def}}{=} i\omega\rho^{\text{b}}\hat{\mathcal{Y}}^{\text{d}}, \tag{154}$$

$$\hat{\mathcal{P}}^{\text{u}} \overset{\text{def}}{=} i\omega\rho^{\text{b}}\hat{\mathcal{Y}}^{\text{u}}, \tag{155}$$

yields for $\hat{G}^{q,\text{b}} = \hat{G}^{q,\text{b}}(\mathbf{x}_{\text{T}}, x_3; \mathbf{x}_{\text{T}}', x_3')$ the following evolution equations,

$$\partial_3\hat{G}^{q,\text{b}} + \hat{\mathcal{P}}^{\text{d}}\hat{G}^{q,\text{b}} = 0, \quad x_3 > x_3', \tag{156}$$

$$\partial_3\hat{G}^{q,\text{b}} + \hat{\mathcal{P}}^{\text{u}}\hat{G}^{q,\text{b}} = 0, \quad x_3 < x_3', \tag{157}$$

respectively. The initial condition at $x_3 = x_3'$, is given by, using eqs (76) and (104),

$$\hat{G}^{q,\text{b}}\left(\mathbf{x}_{\text{T}}, x_3'; \mathbf{x}_{\text{T}}', x_3'\right) = (2\hat{\mathcal{Y}}^{\text{b}})^{-1}\delta(\mathbf{x}_{\text{T}} - \mathbf{x}_{\text{T}}'). \tag{158}$$

In order to solve eq. (156), from the source level x_3^{S} to the scattering boundary level x_3^{sct}, we follow Dollard & Friedman (1979) and Goldstein (1985), omitting any discussion on required norms in a Banach space. Following Goldstein (1985), consider the partition π of the interval $[x_3^{\text{S}}, x_3^{\text{sct}}]$, and select a value z^j in each mutually disjunct partition interval, according to

$$\pi : x_3^{\text{S}} = x_3^0 < x_3^1 < \ldots < x_3^n = x_3^{\text{sct}}, \quad z^j \in \left(x_3^{j-1}, x_3^j\right], \qquad j = 1, \ldots, n. \tag{159}$$

Approximate, in eq. (156), $\hat{G}^{q,\text{b}}$ by $\hat{G}_n^{q,\text{b}}$,

$$\partial_3\hat{G}_n^{q,\text{b}} + \hat{\mathcal{P}}_\pi^{\text{d,j}}\hat{G}_n^{q,\text{b}} = 0, \quad \text{in} \quad \left(x_3^{j-1}, x_3^j\right], \quad j = 1, \ldots, n, \tag{160}$$

with initial condition given by eq. (158), for $x_3' = x_3^0$,

$$\hat{G}_n^{q,\text{b}}\left(\mathbf{x}_{\text{T}}, x_3^0; \mathbf{x}_{\text{T}}', x_3^0\right) = \hat{G}^{q,\text{b}}\left(\mathbf{x}_{\text{T}}, x_3^0; \mathbf{x}_{\text{T}}', x_3^0\right). \tag{161}$$

The 'step operator' $\hat{\mathcal{P}}_\pi^{\text{d,j}}$ is defined, through the kernel of $\hat{P}^{\text{d}}$, by the constant value z^j in each partition interval $(x_3^{j-1}, x^j{}_3]$;

$$\hat{\mathcal{P}}_\pi^{\text{d,j}} f(\mathbf{x}_{\text{T}}, x_3) \overset{\text{def}}{=} \left\langle \hat{P}^{\text{d}}(\mathbf{x}_{\text{T}}, z^j; \mathbf{x}_{\text{T}}', z^j), f(\mathbf{x}_{\text{T}}', x_3)\right\rangle_{\text{b}}, \quad x_3 \in \left(x_3^{j-1}, x_3^j\right], \quad j = 1, \ldots, n. \tag{162}$$

Hence, $\hat{\mathcal{P}}_\pi^{\text{d,j}}$ is x_3-invariant within a partition interval. Exploiting this invariance, taking $\Delta x_3^j = x_3^j - x_3^{j-1}$, $j = 1, \ldots, n$, eq. (160) is solved for the first interval of the partition π, using eq. (161), as

$$\hat{G}_n^{q,\text{b}}\left(\mathbf{x}_{\text{T}}, x_3^1; \mathbf{x}_{\text{T}}', x_3^0\right) = \exp(-\Delta x_3^1\hat{\mathcal{P}}_\pi^{\text{d},1})(2\hat{\mathcal{Y}}^{\text{b}})^{-1}\delta(\mathbf{x}_{\text{T}} - \mathbf{x}_{\text{T}}'). \tag{163}$$

Using this last equation as the initial value, we can solve eq. (160) for the second interval of π,

$$\hat{G}^{q,\mathrm{b}}_n\left(\mathbf{x}_\mathrm{T}, x_3^2; \mathbf{x}'_\mathrm{T}, x_3^1\right) = \exp\left(-\Delta x_3^2 \hat{\mathcal{P}}^{\mathrm{d},2}_\pi\right)\hat{G}^{q,\mathrm{b}}_n\left(\mathbf{x}_\mathrm{T}, x_3^1; \mathbf{x}'_\mathrm{T}, x_3^0\right). \tag{164}$$

Continuing we obtain, as a solution for eq. (160), the ordered product

$$\hat{G}^{q,\mathrm{b}}_n\left(\mathbf{x}_\mathrm{T}, x_3^n; \mathbf{x}'_\mathrm{T}, x_3^0\right) = \prod_{j=1}^{n} \exp\left(-\Delta x_3^j \hat{\mathcal{P}}^{\mathrm{d},\mathrm{j}}_\pi\right)(2\hat{\mathcal{Y}}^\mathrm{b})^{-1}\delta(\mathbf{x}_\mathrm{T} - \mathbf{x}'_\mathrm{T}), \tag{165}$$

in which

$$\prod_{j=1}^{n} \exp\left(-\Delta x_3^j \hat{\mathcal{P}}^{\mathrm{d},\mathrm{j}}_\pi\right) = \exp\left(-\Delta x_3^n \hat{\mathcal{P}}^{\mathrm{d},\mathrm{n}}_\pi\right)\ldots\exp\left(-\Delta x_3^1 \hat{\mathcal{P}}^{\mathrm{d},1}_\pi\right). \tag{166}$$

The function $\hat{G}^{q,\mathrm{b}}_n$ is called a Peano polygonal approximation (Goldstein 1985). One can show that (Dollard & Friedman 1979; DeWitt-Morette *et al.* 1979; Goldstein 1985) when the length of the longest subinterval of the partition, $m(\pi) = \max_j(\Delta x_3^j)$, goes to zero, while n goes to infinity, we obtain, for $x_3^0 = x_3^\mathrm{S}$ and $x_3^n = x_3^\mathrm{sct}$,

$$\hat{G}^{q,\mathrm{b}}\left(\mathbf{x}_\mathrm{T}, x_3^\mathrm{sct}; \mathbf{x}'_\mathrm{T}, x_3^\mathrm{S}\right) = \prod_{x_3^\mathrm{S}}^{x_3^\mathrm{sct}} \exp\left(-\mathrm{d}x_3 \hat{\mathcal{P}}^\mathrm{d}\right)(2\hat{\mathcal{Y}}^\mathrm{b})^{-1}\delta(\mathbf{x}_\mathrm{T} - \mathbf{x}'_\mathrm{T}), \tag{167}$$

with the product integral defined as

$$\prod_{x_3^0}^{x_3^n} \exp\left(-\mathrm{d}x_3 \hat{\mathcal{P}}^\mathrm{d}\right) \overset{\mathrm{def}}{=} \lim_{m(\pi)\to 0} \prod_{j=1}^{n} \exp\left(-\Delta x_3^j \hat{\mathcal{P}}^{\mathrm{d},\mathrm{j}}_\pi\right). \tag{168}$$

Hence, using eq. (167) in eq. (93), the propagator for the down-going wavefield is given by

$$\hat{\mathcal{W}}^\mathrm{d}\left(x_3^\mathrm{sct}; x_3^\mathrm{S}\right) = \prod_{x_3^\mathrm{S}}^{x_3^\mathrm{sct}} \exp\left(-\mathrm{d}x_3 \hat{\mathcal{P}}^\mathrm{d}\right). \tag{169}$$

Likewise, starting with eq. (157), one can derive for the propagator of eq. (96) the product integral from the scattering boundary to the receiver level,

$$\hat{\mathcal{W}}^\mathrm{u}\left(x_3^\mathrm{R}; x_3^\mathrm{sct}\right) = \prod_{x_3^\mathrm{sct}}^{x_3^\mathrm{R}} \exp\left(-\mathrm{d}x_3 \hat{\mathcal{P}}^\mathrm{u}\right). \tag{170}$$

eqs (169) and (170) show that the WRW model of eq. (102) can be given as the product of a product integral, a reflection operator and another product integral.

9.2 Riccati equation

To compute $\hat{\mathcal{P}}^\mathrm{d}$ or $\hat{\mathcal{P}}^\mathrm{u}$ we eliminate $\hat{\Gamma}_3^{q,\mathrm{b}}$ from eq. (20) yielding,

$$\partial_3^2 \hat{G}^{q,\mathrm{b}} - \frac{1}{\rho^\mathrm{b}}\left(\partial_3 \rho^\mathrm{b}\right)\partial_3 \hat{G}^{q,\mathrm{b}} + \omega^2 \rho^\mathrm{b} \hat{\mathcal{K}}^\mathrm{b} \hat{G}^{q,\mathrm{b}} = 0. \tag{171}$$

Substituting eqs (156) and (157), one can show that the operators $\hat{\mathcal{P}}' = \hat{\mathcal{P}}^\mathrm{d}$ and $\hat{\mathcal{P}}' = \hat{\mathcal{P}}^\mathrm{u}$ are solutions of the non-linear Riccati equation,

$$\partial_3 \hat{\mathcal{P}}' - (\hat{\mathcal{P}}')^2 - \frac{1}{\rho^\mathrm{b}}\left(\partial_3 \rho^\mathrm{b}\right)\hat{\mathcal{P}}' - \omega^2 \rho^\mathrm{b} \hat{\mathcal{K}}^\mathrm{b} = \mathcal{O}. \tag{172}$$

Substituting eqs (154) and (155) in this last equation gives the Riccati equation,

$$\frac{i}{\omega}\partial_3 \hat{\mathcal{Y}}' + \hat{\mathcal{Y}}' \rho^\mathrm{b} \hat{\mathcal{Y}}' - \hat{\mathcal{K}}^\mathrm{b} = \mathcal{O}, \tag{173}$$

with D-t-N operator solutions $\hat{\mathcal{Y}}' = \hat{\mathcal{Y}}^\mathrm{d}$ and $\hat{\mathcal{Y}}' = \hat{\mathcal{Y}}^\mathrm{u}$. The initial conditions of $\hat{\mathcal{Y}}^\mathrm{d}$ and $\hat{\mathcal{Y}}^\mathrm{u}$, are furnished by the radiation conditions of the down-going and up-going wavefields, towards plus and minus infinity, respectively, where the medium is longitudinal-invariant. Implementing longitudinal-invariance in the Riccati eq. (173),

$$(\rho^\mathrm{b} \hat{\mathcal{Y}}')^2 - \rho^\mathrm{b} \hat{\mathcal{K}}^\mathrm{b} = \mathcal{O}, \tag{174}$$

in which $\rho^\mathrm{b} \hat{\mathcal{K}}^\mathrm{b}$ represents the Helmholtz operator, yields the initial values, $\hat{\mathcal{Y}}' = \hat{\mathcal{Y}}_0^\mathrm{d}$ and $\hat{\mathcal{Y}}' = \hat{\mathcal{Y}}_0^\mathrm{u}$, as

$$\hat{\mathcal{Y}}_0^\mathrm{d} = -\hat{\mathcal{Y}}_0^\mathrm{u} = (\rho^\mathrm{b})^{-1}(\rho^\mathrm{b} \hat{\mathcal{K}}^\mathrm{b})^{\frac{1}{2}}. \tag{175}$$

The Riccati eq. (173) and its initial condition in (175) have been obtained in many contexts, in particular, by Haines & de Hoop (1996), Fishman *et al.* (1997, 1998), and Lu & McLaughlin (1996). The operator $\hat{\mathcal{Y}}^\mathrm{d}$ is solved upward, whereas $\hat{\mathcal{Y}}^\mathrm{u}$ is solved downward. Using eqs (154) and (155), $\hat{\mathcal{W}}^\mathrm{d}$ and $\hat{\mathcal{W}}^\mathrm{u}$ are solved downward and upward, respectively, in the reverse direction of the associated D-t-N operators (Haines & de Hoop 1996).

10 CONCLUSIONS

Implementing bilinear and sesquilinear forms allowed us to describe the forward- and inverse-scattering problem, in terms of the WRW model, analogously. Using time-domain causality and reciprocity theorems, operator symmetries are derived from radiation conditions, which enable to express this model in a concise way. Scattering from an acoustic contrast is represented by a reflection operator, and left- and right-operating propagators, in terms of the D-t-N operator of the total wavefield and the D-t-N operators of its two-way wavefield components. By solving for the D-t-N operators of the wavefields components, which are solutions of an operator Riccati equation, the inverse-scattering problem can be resolved. Hence, the D-t-N operators are central to the two-way WRW model.

The family of propagators, forming a semi-group, are recursive in the propagation direction, which, in the limit, is represented by a product integral. Because of the semi-group property, the propagators have no inverse that collapse a wavefield back on its source, due to the inapplicability of a radiation condition for anticausal wavefield solutions to the Helmholtz equation. Therefore, wavefields representations with respect to a sesquilinear form can not be marched from one level surface to another according to Huygens' principle. By applying a wavefield decomposition in wavefield components that can be marched with respect to a single surface an approximate inverse propagator is derived by equating the wavefield with one of its wavefield components. For this we need to neglect one boundary integral which is equivalent to enforcing a radiation condition. The propagator becomes an unitary operator, meaning that the inverse propagator is equal to the adjoint of the forward propagator. In this way Huygen's principle, as a means to march a wavefield forward becomes applicable to the inverse problem, where a wavefield component is marched backward, hence, transforming an initial-value problem to a final-value problem.

According to our model imaging represents evaluating the scattered wavefield at the scattering boundary, while moving the illuminating source also to it. This redatuming procedure is then accomplished by adjoint propagators operating from the left and right on the back-scattered wavefield.

ACKNOWLEDGMENTS

The authors would like to thank the editor John Haines for his encouraging comments and for his suggestions to enhance the readability of this paper. We appreciate the help of Louis Fishman for his corrections and for his suggestions to include by reference and discussion previous and current work on relevant operator characterization and approximation results. This paper is contribution number 1064 from the Geoscience Department at the University of Texas at Dallas.

REFERENCES

Bellman, R. & Vasudevan R., 1986. *Wave propagation, an invariant embedding approach*, Reidel, Dordrecht, Netherlands.

Berkhout, A.J., 1985. *Seismic migration: imaging of acoustic energy by wave field extrapolation*, Elsevier, Amsterdam, Netherlands.

Bojarski, N.N., 1983. Generalized reaction principles and reciprocity theorems for the wave equations, and the relationship between the time-advanced and time-retarded fields, *J. acoust. Soc. Am.,* **74**(1), 281–285.

de Bruin, C.G.M., Wapenaar, C.P.A. & Berkhout A.J., 1990. Angle-dependent reflectivity by means of pre-stack migration, *Geophysics,* **55,** 1223–1234.

Claerbout, J., 1985. *Imaging the Earth's interior,* Blackwell Scientific Publications, Oxford, UK.

Colton, D. & Kress, R., 1983. *Integral equation methods in scattering theory,* John Wiley & Sons, New York, USA.

de Hoop, M.V., 1992. Directional decomposition of transient acoustic waves, *PhD thesis*, Delft University of Technology.

de Hoop, A.T., 1995. *Handbook of radiation and scattering of waves,* Academic Press Limited, New York, USA.

de Hoop, M.V., 1996. Generalization of the Bremmer coupling series, *J. Math. Physics,* **37,** 3246–3282.

de Hoop, M.V. & Gautesen, A.K., 2000. Uniform asymptotic expansion of the generalized Bremmer series, *SIAM J. Math. Physics,* **60,** 1302–1329.

de Hoop, M.V. & Gautesen, A.K., 2003. Uniform asymptotic expansion of the square-root Helmholtz operator and the one-way wave propagator, *SIAM J. Math. Physics,* **63,** 777–800.

DeWitt-Morette, C., Maheshwari, A. & Nelson, B., 1979. Path integration in non-relativistic quantum mechanics, *Physics Report,* **50**(5), 255–372.

Dollard, J.D. & Friedman, C.N., 1979. *Product integration with applications to differential equations,* Addison-Wesley Publishing Company, MA, USA.

Fishman, L. & McCoy, J.J., 1984a. Derivation and application of extended parabolic wave theories, Part I, The factorized Helmholtz equation, *J. Math. Physics,* **25,** 285–296.

Fishman, L. & McCoy, J.J., 1984b. Derivation and application of extended parabolic wave theories, Part II, Path integral representations, *J. Math. Physics,* **25,** 297–308.

Fishman, L., 1992. Exact and operator rational approximate solutions of the Helmholtz, Weyl composition equation in underwater acoustics—the quadratic profile, *J. Math. Physics,* **33,** 1887–1914.

Fishman, L., 1994. Exact solutions for reflection and Dirichlet-to-Neumann operator symbols in direct and inverse wave propagation modeling, in *Inverse optics III, SPIE,* Vol. 2241, pp. 16–27, ed. Fiddy, M.A., SPIE, Bellingham.

Fishman, L., 2002. Applications of directional wave field decomposition, phase space, and path integral methods to seismic wave propagation and inversion, *Pure appl. Geophys.,* **159,** 1637–1679.

Fishman, L., 2004. One-way wave equation modeling in two-way wave propagation problems, in *Mathematical modelling of wave phenomena 2002, Series: Mathematical modelling in physics, engineering and cognitive sciences,* pp. 91–111, eds Nilsson, B. & Fishman, L., v.7, Växjö University Press, Växjö, Sweden.

Fishman, L., Gautesen, A.K. & Sun, Z., 1997. Uniform high-frequency approximations of the square-root Helmholtz operator symbol, *Wave Motion,* **26,** 127–161.

Fishman, L., Gautesen, A.K. & Sun, Z., 1998. An exact, well-posed, one-way reformulation of the Helmholtz equation with application to direct and inverse wave propagation modeling, in *New perspectives on problems in classical and quantum physics, Part II, Acoustic propagation and scattering, electromagnetic scattering,* pp. 75–97, eds Delsanto, P.P. & Saenz, A.W., Gordon and Breach Science Publishers, Amsterdam, the Netherlands.

Fishman, L., de Hoop, M.V. & van Stralen, M.J.N., 2000. Exact constructions of square-root Helmholtz operator symbols: The focusing quadratic profile, *J. Math. Physics,* **41,** 4881–4938.

Fokkema, J.T. & van den Berg, P.M., 1993. *Seismic applications of acoustic reciprocity,* Elsevier, Amsterdam, the Netherlands.

Goldstein, J.A., 1985. *Semigroups of linear operators and applications,* Oxford University Press, New York, USA.

Grimbergen, J.T.L., Dessing, F.J. & Wapenaar, C.P.A., 1998. Modal expansion of one-way operators in laterally varying media, *Geophysics,* **63,** 995–1005.

Grubb, G., 1996. *Functional calculus of pseudodifferential boundary problems,* 2nd edn, Birkhäuser, Germany.

Haines, A.J. & de Hoop, M.V., 1996. An invariant embedding analysis of general wave scattering problems, *J. Math. Physics,* **37**(8), 3854–3881.

Krueger, R.J. & Ochs, R.L., 1989. A Green's function approach to the determination of internal fields, *Wave Motion,* **11,** 525–543.

Lang, S., 1993. *Algebra,* 3rd edn, Addison-Wesley Publishing Company, MA, USA.

Lu, Y.Y. & McLaughlin, J.R., 1996. The Riccati method for the Helmholtz equation, *J. acoust. Soc. Am.,* **100**(3), 1432.

Pazy, A., 1983. *Semigroups of linear operators and applications to partial differential equations,* Springer-Verlag, New York, USA.

Le Rousseau, J.H. & de Hoop, M.V., 2001. Modeling and imaging with the scalar generalized-screen algorithms in isotropic media, *Geophysics,* **66**(5), 1551–1568.

Wapenaar, C.P.A., 1992. The infinite aperture paradox, *Journal of Seismic Exploration,* **1,** 325–336.

Wapenaar, C.P.A., 1996a. One-way representations of seismic data, *Geophys. J. Int.,* **127,** 178–188.

Wapenaar, C.P.A., 1996b. Reciprocity theorems for two-way and one-way wave vectors: a comparison, *J. acoust. Soc. Am.,* **100**(6), 3508–3518.

Wapenaar, C.P.A. & Grimbergen, J.T.L., 1996. Reciprocity theorems for one-way wave fields, *Geophys. J. Int.,* **127,** 169–177.

Weston, V.H., 1988. Factorization of the wave equation in a nonplanar stratified medium, *J. Math. Physics,* **29**(1), 36–45.

APPENDIX A: BILINEAR FORMS

Consider the *bilinear form* $f : [L^2(\mathbb{R}^2)]^\alpha \times [L^2(\mathbb{R}^2)]^\alpha \to \mathbb{C}$, associated with the linear map $\mathbf{B} : [L^2(\mathbb{R}^2)]^\alpha \to [L^2(\mathbb{R}^2)]^\alpha$,

$$f(\mathbf{F}, \mathbf{G}) \stackrel{\text{def}}{=} \int_{\mathbf{x}_\mathrm{T} \in \mathbb{R}^2} \mathbf{F}^\mathrm{t}(\mathbf{x}_\mathrm{T})\, \mathbf{B}\mathbf{G}(\mathbf{x}_\mathrm{T})\, \mathrm{d}\mathbf{x}_\mathrm{T}. \tag{A1}$$

The bilinear form f constitutes a Cartesian product of two Hilbert spaces into the complex plane $\mathbb{C}$. Each Hilbert space $[L^2(\mathbb{R}^2)]^\alpha$ is a set of square-integrable (L^2) functions $\mathbf{F}, \mathbf{G} \in [L^2(\mathbb{R}^2)]^\alpha$, defined on the transverse coordinate space $\mathbb{R}^2$. For $\alpha = 1$ the functions are scalars, whereas for $\alpha = 2$ the functions are vectors with two components. The superscript $^\mathrm{t}$ denotes transposition of the vector **F**. Bilinearity means that we have linearity in the first variable **F** and linearity in the second variable **G** (Lang 1993). The bilinear form f associated with the operator $\mathbf{B}$ is symbolized as

$$f(\mathbf{F}, \mathbf{G}) = \langle \mathbf{F}, \mathbf{B}\mathbf{G} \rangle_\mathrm{b}\,. \tag{A2}$$

Given the bilinear form f and its associated linear operator $\mathbf{B}$ there exists a unique linear map $\mathbf{B}^\mathrm{t} : [L^2(\mathbb{R}^2)]^\alpha \to [L^2(\mathbb{R}^2)]^\alpha$, the *transpose* of $\mathbf{B}$, such that

$$\langle \mathbf{F}, \mathbf{B}\mathbf{G} \rangle_\mathrm{b} = \langle \mathbf{B}^\mathrm{t}\mathbf{F}, \mathbf{G} \rangle_\mathrm{b}. \tag{A3}$$

Whenever,

$$\mathbf{B} = \mathbf{B}^\mathrm{t}, \tag{A4}$$

$\mathbf{B}$ is said to be *symmetric* with respect to f. If

$$\mathbf{B} = -\mathbf{B}^\mathrm{t} \tag{A5}$$

$\mathbf{B}$ is said to be *skew-symmetric* or *alternating* with respect to f. For skew-symmetric $\mathbf{B}$ we have

$$\langle \mathbf{F}, \mathbf{B}\mathbf{F} \rangle_\mathrm{b} = 0, \quad \forall\, \mathbf{F} \in [L^2(\mathbb{R}^2)]^\alpha, \tag{A6}$$

and

$$\langle \mathbf{F}, \mathbf{B}\mathbf{G} \rangle_\mathrm{b} = -\langle \mathbf{G}, \mathbf{B}\mathbf{F} \rangle_\mathrm{b}\,, \quad \forall\, \mathbf{F}, \mathbf{G} \in [L^2(\mathbb{R}^2)]^\alpha. \tag{A7}$$

APPENDIX B: SESQUILINEAR FORMS

Consider the *sesquilinear form* $g : [L^2(\mathbb{R}^2)]^\alpha \times [L^2(\mathbb{R}^2)]^\alpha \to \mathbb{C}$, associated with the linear map $\mathbf{B} : [L^2(\mathbb{R}^2)]^\alpha \to [L^2(\mathbb{R}^2)]^\alpha$,

$$g(\mathbf{F}, \mathbf{G}) \stackrel{\text{def}}{=} \int_{\mathbf{x}_\mathrm{T} \in R^2} \mathbf{F}^\dagger(\mathbf{x}_\mathrm{T})\, \mathbf{B}\mathbf{G}(\mathbf{x}_\mathrm{T})\, \mathrm{d}\mathbf{x}_\mathrm{T}. \tag{B1}$$

The superscript † denotes the product operation of transposition, signified by $^\mathrm{t}$, and complex conjugation, signified by *. Sesquilinearity means that, $\forall\, \mathbf{F}, \mathbf{F}_1, \mathbf{F}_2, \mathbf{G} \in [L^2(\mathbb{R}^2)]^\alpha$ and $\forall a \in \mathbb{C}$, we have *antilinearity* in the first variable, that is,

$$\begin{aligned} g(\mathbf{F}_1 + \mathbf{F}_2, \mathbf{G}) &= g(\mathbf{F}_1, \mathbf{G}) + g(\mathbf{F}_2, \mathbf{G})\,, \\ g(a\mathbf{F}, \mathbf{G}) &= a^* g(\mathbf{F}, \mathbf{G})\,, \end{aligned} \tag{B2}$$

and linearity in the second variable (Lang 1993). The sesquilinear form g associated with $\mathbf{B}$ is symbolized as

$$g(\mathbf{F}, \mathbf{G}) = \langle \mathbf{F}, \mathbf{B}\mathbf{G} \rangle_\mathrm{s}\,. \tag{B3}$$

Observe that a sesquilinear form can be expressed as a bilinear form according to

$$\langle \mathbf{F}, \mathbf{B}\mathbf{G}\rangle_{\mathrm{s}} = \langle \mathbf{F}^{*}, \mathbf{B}\mathbf{G}\rangle_{\mathrm{b}} . \tag{B4}$$

Given the linear operator $\mathbf{B}$ and its associated sesquilinear form g there exists a unique linear map $\mathbf{B}^{\dagger} : [L^{2}(\mathbb{R}^{2})]^{\alpha} \to [L^{2}(\mathbb{R}^{2})]^{\alpha}$, the *adjoint* of $\mathbf{B}$, such that

$$\langle \mathbf{F}, \mathbf{B}\mathbf{G}\rangle_{\mathrm{s}} = \langle \mathbf{B}^{\dagger}\mathbf{F}, \mathbf{G}\rangle_{\mathrm{s}}. \tag{B5}$$

Whenever,

$$\mathbf{B} = \mathbf{B}^{\dagger}, \tag{B6}$$

$\mathbf{B}$ is said to be *self-adjoint* or *hermitian* with respect to g. For self-adjoint $\mathbf{B}$ we have

$$\langle \mathbf{F}, \mathbf{B}\mathbf{F}\rangle_{\mathrm{s}} \text{ is real}, \quad \forall\, \mathbf{F} \in [L^{2}(\mathbb{R}^{2})]^{\alpha}, \tag{B7}$$

and

$$\langle \mathbf{F}, \mathbf{B}\mathbf{G}\rangle_{\mathrm{s}} = \langle \mathbf{G}, \mathbf{B}\mathbf{F}\rangle_{\mathrm{s}}^{*}, \quad \forall\, \mathbf{F}, \mathbf{G} \in [L^{2}(\mathbb{R}^{2})]^{\alpha}. \tag{B8}$$

Using eqs (A3), (B4) and (A5), we have

$$\langle \mathbf{B}^{\mathrm{t}}\mathbf{F}^{*}, \mathbf{G}\rangle_{\mathrm{b}} = \langle \mathbf{F}^{*}, \mathbf{B}\mathbf{G}\rangle_{\mathrm{b}} = \langle \mathbf{F}, \mathbf{B}\mathbf{G}\rangle_{\mathrm{s}} = \langle \mathbf{B}^{\dagger}\mathbf{F}, \mathbf{G}\rangle_{\mathrm{s}} = \langle (\mathbf{B}^{\dagger}\mathbf{F})^{*}, \mathbf{G}\rangle_{\mathrm{b}}. \tag{B9}$$

Hence,

$$\mathbf{B}^{\dagger}\mathbf{F} = (\mathbf{B}^{\mathrm{t}}\mathbf{F}^{*})^{*}, \tag{B10}$$

gives a relation between the transpose and the adjoint of an operator.